CS 161: Design and Analysis of Algorithms
Midterm

• Wednesday, July 25\textsuperscript{th} in class 2:15 – 3:30
• Covers material through today
• No bluebooks needed
• SCPD students:
  – Can take exam on campus, let us know by Monday
  – Otherwise, must take at scheduled time with exam proctor
Divide & Conquer II: Sorting/Median Finding

- Merge Sort
- Quick Sort
- Sorting Lower Bound
- Median Finding
Merge Sort

• Want to sort a list of n elements
• Divide and conquer approach:
  – Split list into two sublists of size n/2
  – Recursively sort each sublist
  – Construct sorted list by merging sorted sublists
Merge Sort
Merge Sort
Merge Sort
Merge Sort
Merge Sort

• Splitting the list: Easy! O(n)
• Two recursive calls: Easy!
• Merging two sorted lists?
  – Lowest element in merged list is the lowest element of one of the lists
  – Pick smaller of the first elements of the two lists, remove it, and add it to the final list.
Merge Sort
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[Diagram of Merge Sort process]
Merge Sort
Merge Sort
Merge Sort
Merge Sort
Merge Sort
Merge Sort

• Splitting the list: Easy! O(n)
• Two recursive calls: Easy!
• Merging two sorted lists?
  – Pick smaller of the first elements of the two lists, remove it, and add it to the final list.
  – Every iteration, length of final list grows
    • Can only iterate O(n) times
  – O(n) for merge
Merge Sort

• Running time: $T(n) = 2 \cdot T(n/2) + O(n)$
• Master Method:
  – $a = 2$, $b = 2$, $d = 1$
  – $a = b^d$, so $O(n^d \log n) = O(n \log n)$
QuickSort

• What if instead of merging at end, we make sure all the elements in one list are less than all the elements in the other.

• Then we just concatenate the two lists, and are done

• To accomplish, take an element from the list, called the **pivot**, and make left list all elements less than it, right list all elements greater than it
QuickSort
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QuickSort
QuickSort Running Time

• O(n) work before collision to split lists
• Let p be the pivot, k the number of elements less than p
• One recursive call of size k, one of size n-1-k
• T(n) = T(k) + T(n-1-k) + O(n)
QuickSort Running Time

• Best case: $k = n/2$
• $T(n) = 2T(n/2) + O(n) \rightarrow T(n) = O(n \log n)$
• Worst case: $k = 0$ (i.e. elements are already in order)
• $T(n) = T(n-1) + T(0) + O(n) = T(n-1) + O(n)$
  – $T(n) = O(n^2)$
QuickSort Average Case

• What if input is in random order?
  – k is a random value between 0 and n-1
  – Expected running time?

\[
T(n) \leq O(n) + \frac{1}{n} \sum_{k=0}^{n-1} (T(k) + T(n - k - 1))
\leq O(n) + \frac{2}{n} \sum_{k=0}^{n-1} T(k)
\]
QuickSort Average Case

\[ T(n) \leq cn + \frac{2^{n-1}}{n} \sum_{k=0}^{n-1} T(k) \]

• Claim: there is a constant d such that \( T(n) \leq dn \log n \)
QuickSort Average Case

• Proof: Assume $T(k) \leq dk \log k$ for $k < n$

$$T(n) \leq cn + \frac{2}{n} \sum_{k=0}^{n-1} (dk \log k)$$

$$= cn + \frac{2d}{n} \left( \sum_{k=0}^{\lfloor(n-1)/2\rfloor} k \log k + \sum_{k=\lfloor(n-1)/2\rfloor+1}^{n-1} k \log k \right)$$

$$\leq cn + \frac{2d}{n} \left( \log \frac{n}{2} \left( \sum_{k=0}^{\lfloor(n-1)/2\rfloor} k \right) + \log n \left( \sum_{k=\lfloor(n-1)/2\rfloor+1}^{n-1} k \right) \right)$$
QuickSort Average Case

\[ T(n) \leq cn + \frac{2d}{n} \left( \log \frac{n}{2} \left( \sum_{k=0}^{\left\lceil (n-1)/2 \right\rceil} k \right) + \log n \left( \sum_{k=\left\lceil (n-1)/2 \right\rceil+1}^{n-1} k \right) \right) \]

\[ = cn + \frac{2d}{n} \left( \log n \left( \sum_{k=0}^{n-1} k \right) - \left( \sum_{k=0}^{\left\lceil (n-1)/2 \right\rceil} k \right) \right) \]

\[ = cn + \frac{2d}{n} \left( \frac{n(n-1)}{2} \log n - \frac{(n-1)/2\left(\left\lceil (n-1)/2 \right\rceil - 1\right)}{2} \right) \]
QuickSort Average Case

\[ T(n) \leq cn + \frac{2d}{n} \left( \frac{n(n-1)}{2} \log n - \frac{(n-1)/2}{2} \left( \frac{(n-1)/2}{2} - 1 \right) \right) \]

\[ = cn + d(n - 1) \log n - \frac{d}{8} \left( n - \frac{1}{n} \right) \]

\[ \leq dn \log n - \left( \frac{d}{8} - c \right)n + d \left( \frac{1}{8} - \log n \right) \]

Thus, we can set \( d = 8c \), and the desired inequality holds for \( \log n \geq 1/8 \), which holds for \( n \geq 2 \)
QuickSort Randomized

• What if we really want worst-case bounds?
• Instead of picking pivot to be the first element, pick pivot at random
• \( k \), the number of elements below the pivot, is still a random integer form 0 to n-1
• Expected running time:

\[
T(n) \leq O(n) + \frac{1}{n} \sum_{k=0}^{n-1} (T(k) + T(n-k-1))
\]
Comparison-Based Sorting

• Heap sort, Merge sort, and QuickSort all have running time $O(n \log n)$. Why?

• Theorem: Any sorting algorithm that only makes questions of the form “is $x < y$?” must make $\Omega(n \log n)$ comparisons.
Decision-Tree Model

• Any algorithm that only asks questions about the input of the form “is x < y?” can be represented as a tree.
Decision-Tree Model

- Label leaves with permutations $p$ of $[1,\ldots,n]$
  - $[p(1), p(2), \ldots, p(n)]$
  - Corresponds to ordering where $x_{p(1)} < x_{p(2)} < \ldots < x_{p(n)}$
- Permutation must be consistent with answers to questions
  - Let $r_i$ and $r_j$ be the integers such that $i = p(r_i)$ and $j = p(r_j)$
  - If $x_i < x_j$ was answered yes, then $r_i < r_j$
Decision-Tree Model

• All possible permutations must be present
  – What if permutation is missing, and we give algorithm an input with the corresponding ordering?
  – The algorithm will think we are in a different ordering, and produce the wrong output
Decision-Tree Model

• Number of permutations?
  – First pick $p(1)$: $n$ choices
  – Then pick $p(2)$: $n-1$ choices
  – …
  – Pick $p(n)$: 1 choice
  – Total number of choices: $n!$
Decision-Tree Model

• Number of permutations: $n!$
• Number of leaves: $\geq n!$
• Depth of tree: $\geq \log n!$
• Number of comparisons in algorithm: $\geq \log n!$
• Need to asymptotically bound $\log n!$
Bounding $\log n!$

- $\log n! = O(n \log n)$
  - $\log n! = \log n + \log (n-1) + \ldots + \log 1$
  - $< n \log n$

- $\log n! = \Omega(n \log n)$
  - $\log n! = \log n + \log (n-1) + \ldots + \log 1$
  - $> \log n + \log (n-1) + \ldots + \log n/2$
  - $> (n/2) \log (n/2) = \Omega(n \log n)$
Comparison-Based Sorting

- Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons
- One of the very few non-trivial lower bounds that we know of
- What about linear sorting algorithms?
  - Not comparison-based
n!

• We can actually do better for bounding n! using Stirling’s Approximation:

\[ n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \]
Finding the Median

• Finding the smallest value of a list is easy
  – Go through the list, keeping track of the smallest element. $O(n)$

• Finding the kth smallest value of a list, for constant $k$, is easy
  – Go through the list, keeping track of the $k$ smallest elements. $O(n)$ if $k$ is constant

• What about $k = n/2$?
  – Sort, pick out middle element. $O(n \log n)$
  – Is there any way to get $O(n)$?
Select

• Find the kth smallest element in a list
• Divide and conquer approach?
  – Pick a pivot p
  – Create two lists, l1 with all elements less than p, and l2 with all elements greater than p
  – If k is at most |l1|, then recursively call on l1
  – If k = |l1|+1, return p
  – Otherwise, call on l2 with k’ = k - |l1|-1
Select

• Running Time?
  – Best case: \( p \) happens to be the median
    • \( T(n) = T(n/2) + O(n) \) \( \Rightarrow \) \( T(n) = O(n) \)
  – Worst case: \( p \) is the smallest or largest element
    • \( T(n) = T(n-1) + O(n) \) \( \Rightarrow \) \( T(n) = O(n^2) \)
Select Expected Running Time

• If we choose pivot randomly, the number of elements smaller than it will be a random integer from 0 to n-1

• Can write recurrence for expected run time:
  – $T(n) = T(3n/4) + g \ O(n)$
  – $g =$ expected number of recursive calls until the list has size $3n/4$
Select Expected Running Time

- How many splits until pivot between \( n/4 \) and \( 3n/4 \)? \( g = 2 \)
Select Expected Running Time

- T(n) = T(3n/4) + O(n)
  - a = 1, b = 4/3, d = 1
  - a < b^d, so T(n) = O(n^d) = O(n)
Worst Case Linear Time?

• How can we get a linear time worst case select?
• Idea: want to pick pivot close to the median
  – Can use select to pick good pivot
Median-of-Medians

- Group elements off arbitrarily into $n/5$ groups of 5
- Find median of each group
- Find and output median of medians
Median-of-Medians

• Finding median of 5 elements: $O(1)$ since a fixed number of comparisons
• Finding medians of all $n/5$ groups: $O(n)$
• Finding median of $n/5$ medians: $T(n/5)$
Median-of-Medians

• How good is the median-of-medians?
  – The median of each group is larger than 2 elements
  – The median-of-medians is larger than \( \frac{n}{5}/2 = \frac{n}{10} \)
    group medians, as well as the elements these medians are larger than
  – Median-of-medians is larger than \( \frac{3n}{10} \)
  – Also smaller than \( \frac{7n}{10} \)
  – Therefore, next recursive call has size at most \( \frac{7n}{10} \)
Worst-case Linear Select

• Select(l,k) =
  – Arbitrarily group elements into groups of five
  – Construct l1, the list of medians of each group
  – Let p = Select(l1, |l1|/2)
  – Construct l2 and l3, the lists of elements smaller and greater than p
  – If |l2| ≤ k, call Select(l2,k)
  – If |l2| = k+1, return p
  – Otherwise, call Select(l3,k-|l2|-1)
Worst-case Linear Select

- Running Time:
  - $O(n)$ for grouping and constructing list of medians
  - $T(n/5)$ for computing pivot
  - $O(n)$ for constructing $l_2$ and $l_3$
  - At most $T(7n/10)$ for recursive call to Select
  - $T(n) = T(n/5) + T(7n/10) + O(n)$
Akra-Bazzi Method

\[ T(n) = \sum_{i} a_i T\left(\frac{n}{b_i}\right) + O(n^d) \]

- Let \( f \) be the solution to \( \sum_{i} \frac{a_i}{b_i^f} = 1 \)

- Then:
  - If \( f < d \), \( T(n) = O(n^d) \)
  - If \( f > d \), \( T(n) = O(n^f) \)
  - If \( f = d \), \( T(n) = O(n^d \log n) \)
Worst-case Linear Select

• $T(n) = T(n/5) + T(7n/10) + O(n)$
  
  $-$ \( a_1 = a_2 = 1 \)
  
  $-$ \( d = 1 \)
  
  $-$ \( b_1 = 5, \ b_2 = 10/7 \)
  
  $-$ Solve

\[
1 = \sum_{i} \frac{a_i}{b_i^f} = \left(\frac{1}{5}\right)^f + \left(\frac{7}{10}\right)^f
\]

$-$ \( f \approx 0.84, \) do \( d > f \)

$-$ Therefore, \( T(n) = O(n^d) = O(n) \)