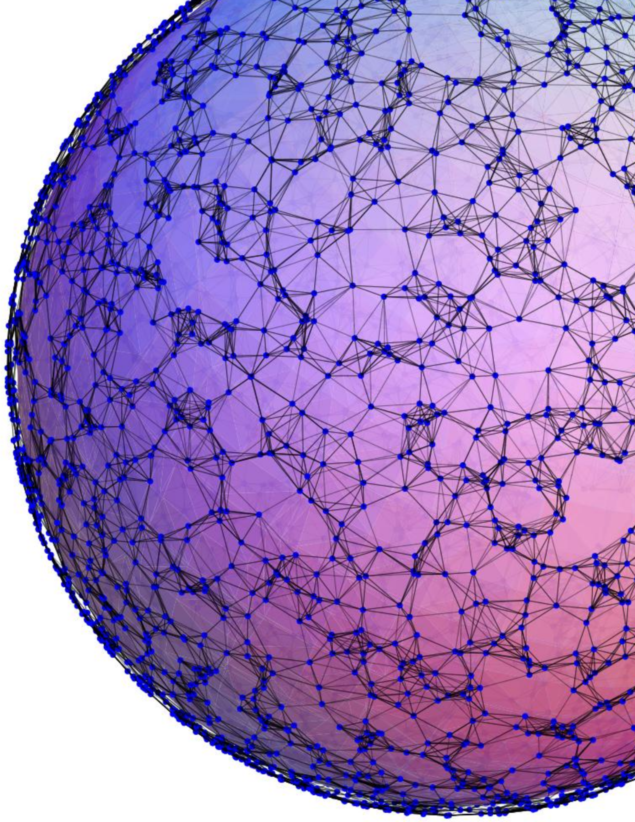
Subsampling Mathematical Relaxations and Average-Case Complexity

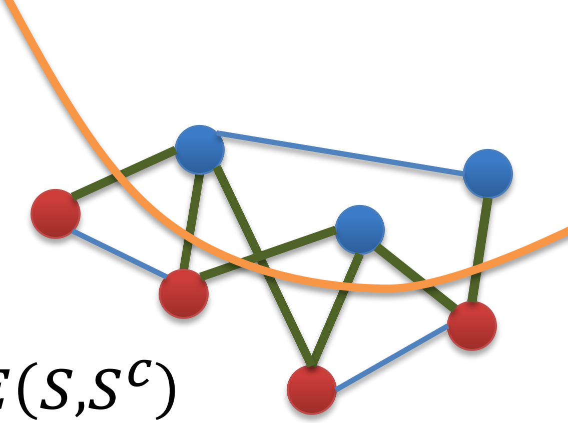


Joint work with **Boaz Barak** (Microsoft Research), **Thomas Holenstein** (ETH Zurich) and **David Steurer** (Microsoft Research)

Speaker: **Moritz Hardt** (Princeton)

SODA 2011

Max-Cut

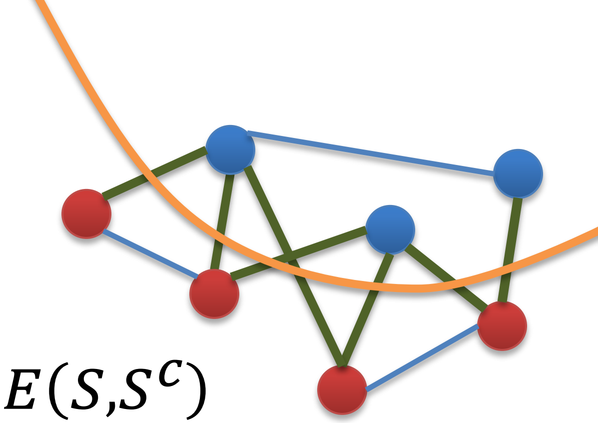


Given graph $G = (V, E)$, find

$$\text{Max-Cut}(G) = \max \frac{E(S, S^C)}{|E|}$$

- Fundamental algorithmic problem
- Testbed for algorithms & hardness

Max-Cut



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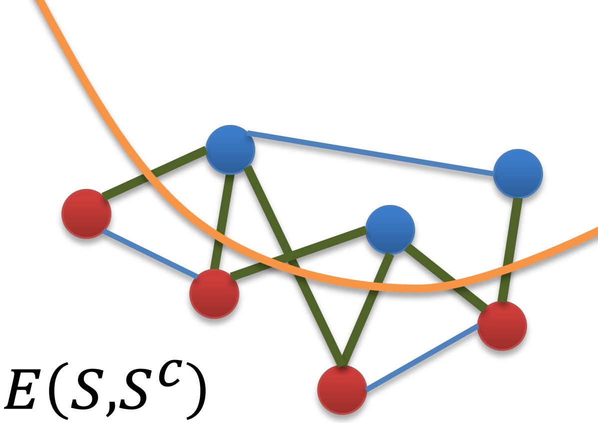

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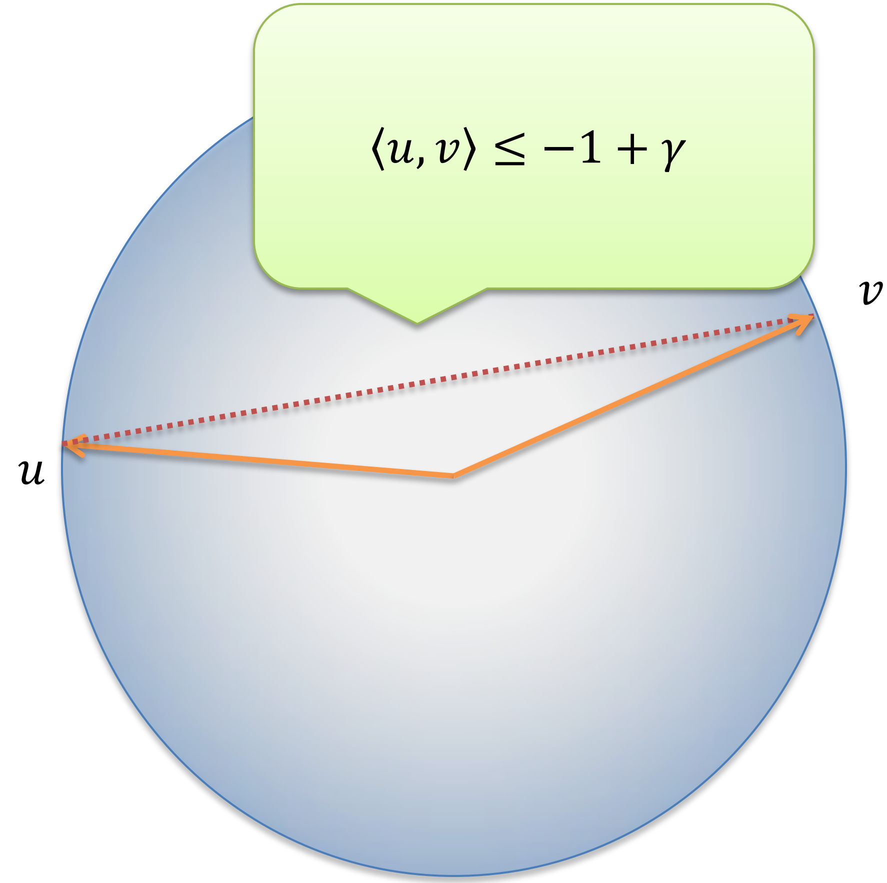
[KKMO'04, MOO'05] Optimal assuming UGC

[FS'02] Analysis tight for sphere graph

Max-Cut on the Sphere

Continuous graph G_γ



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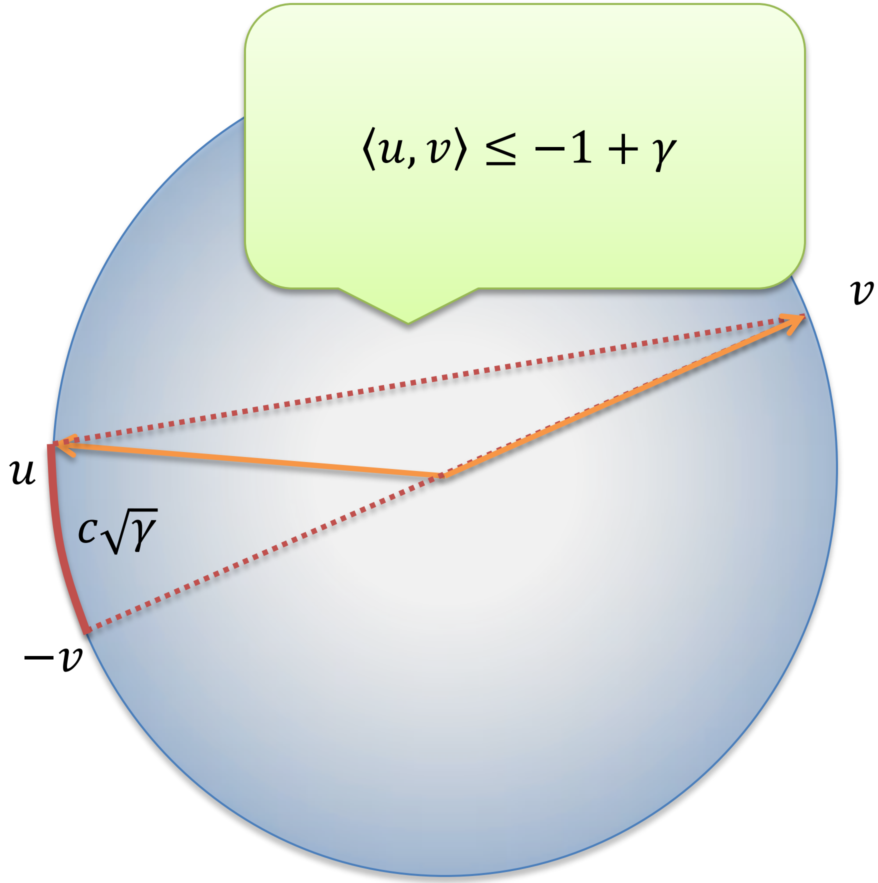

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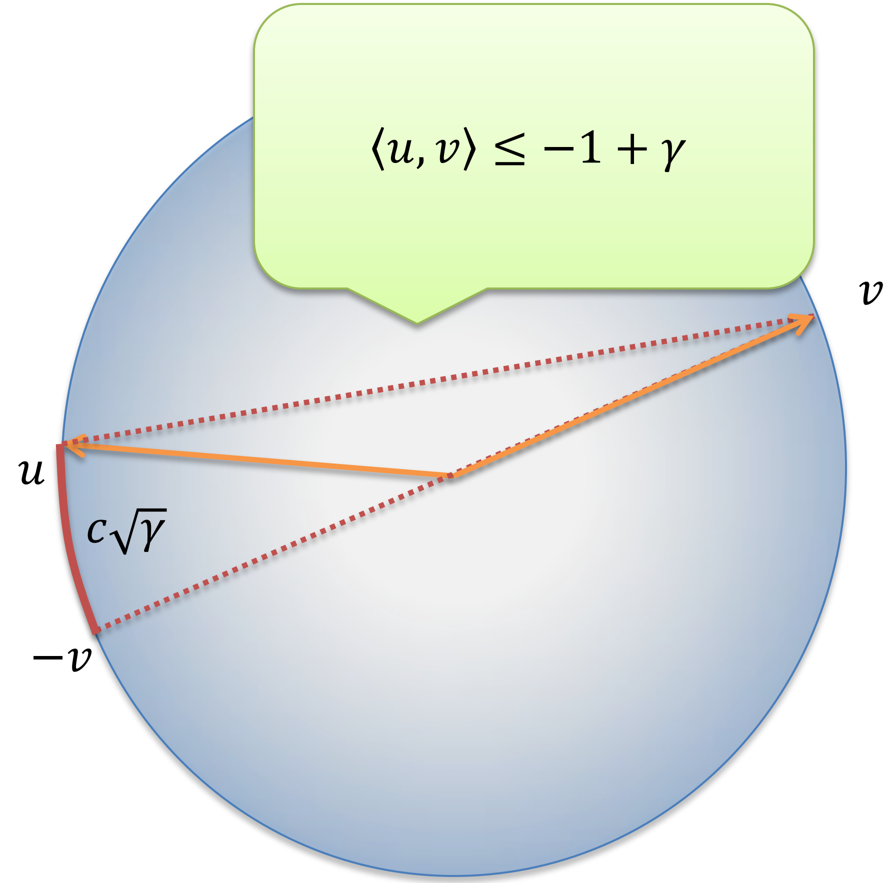
Random subsample of G_γ
(random geometric graphs)



Max-Cut on the Sphere

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Random subsample of G_γ
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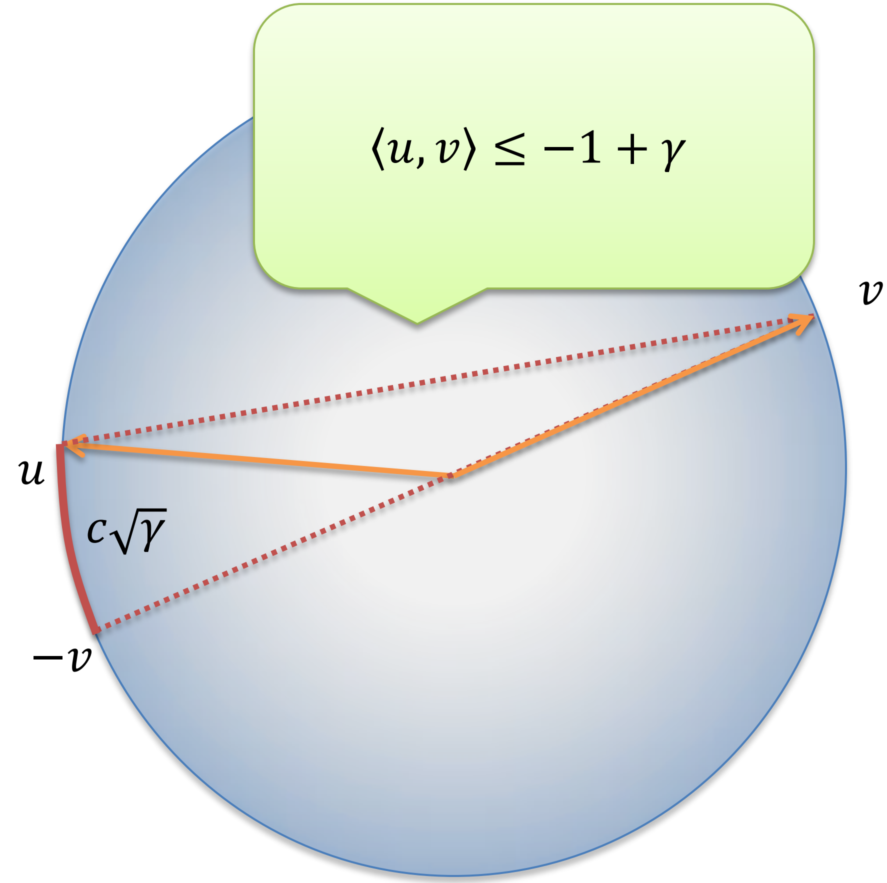
Fact: $\mathrm{sdp}(G_\gamma) \geq 1 - \gamma$

Theorem [Feige-Schechtman'02]:
Max-Cut$(G_\gamma) \leq 1 - c\sqrt{\gamma}$



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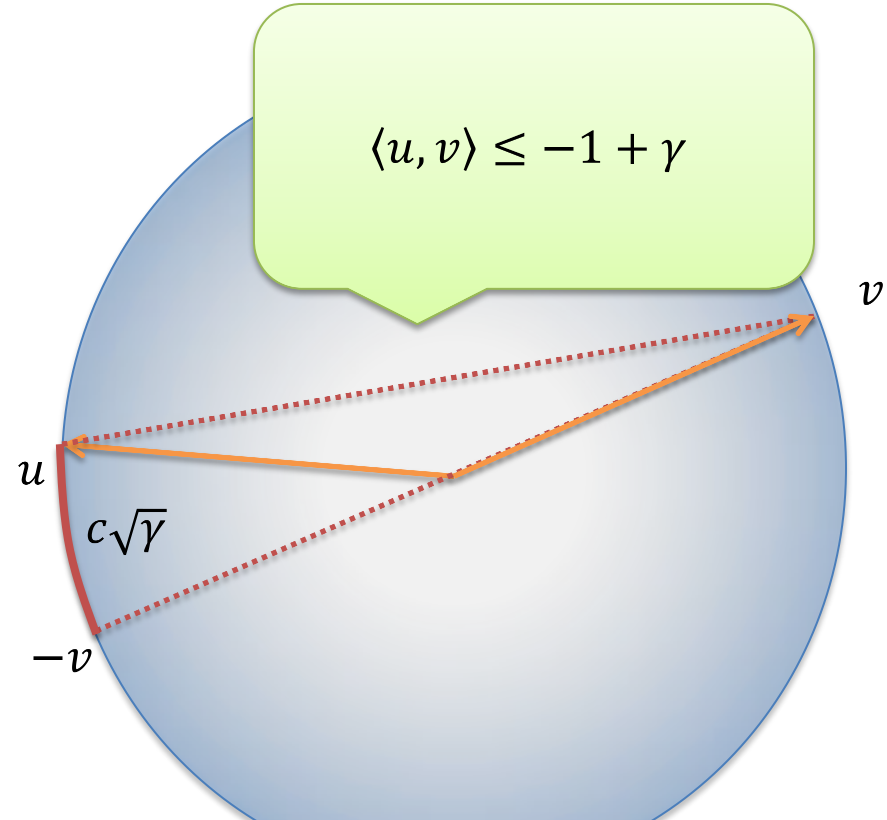
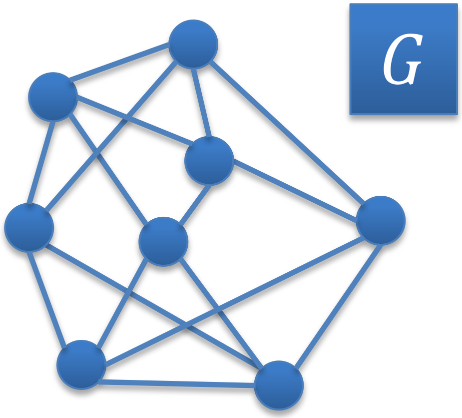


Question:
Is there a polynomial time algorithm certifying $\text{Max-Cut}(G_\gamma) \leq 1 - c\sqrt{\gamma}$?

Subsampling Mathematical Relaxations



Let G be a d-regular graph, n vertices

$\phi(G) \in [0,1]$ some parameter

E.g., $\phi(G) = 1$ - sdp(G)

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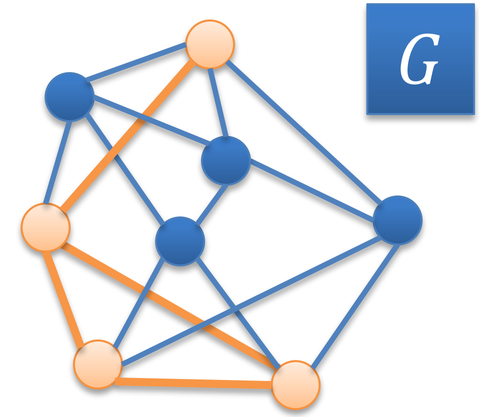

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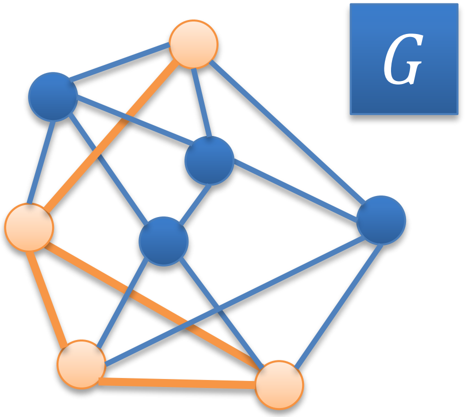

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When is $\phi(G) \approx \phi(G')$?

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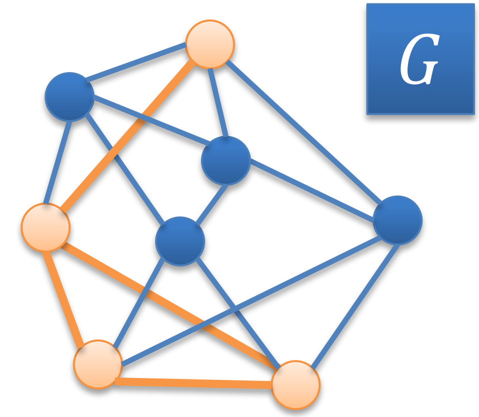

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True for $|S| = n$
False for $|S| \ll \frac{n}{d}$
We consider
$|S| = C\frac{n}{d}$

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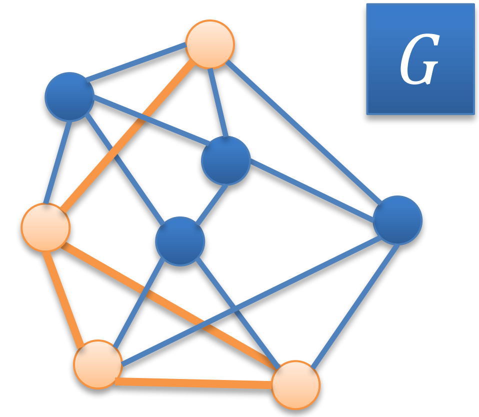

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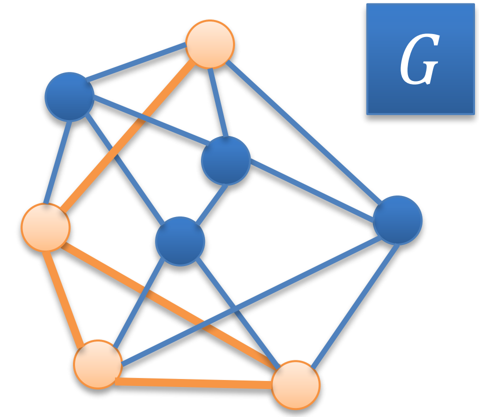

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More general case: G constraint satisfaction problem (CSP)

Our results

G CSP $\phi(G) = \gamma$

Minimization problem

$G' = G[S]$ for random $S \subseteq V$, $|S| = C\frac{n}{d}$

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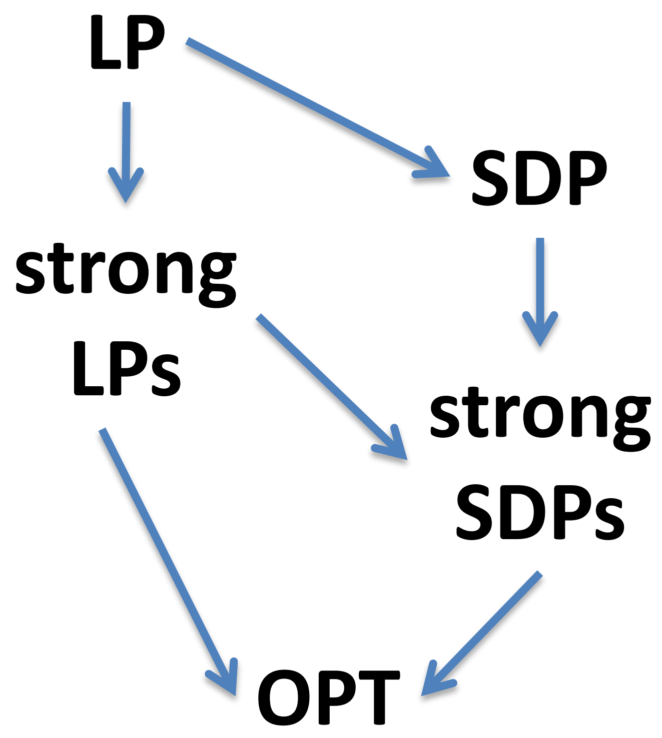

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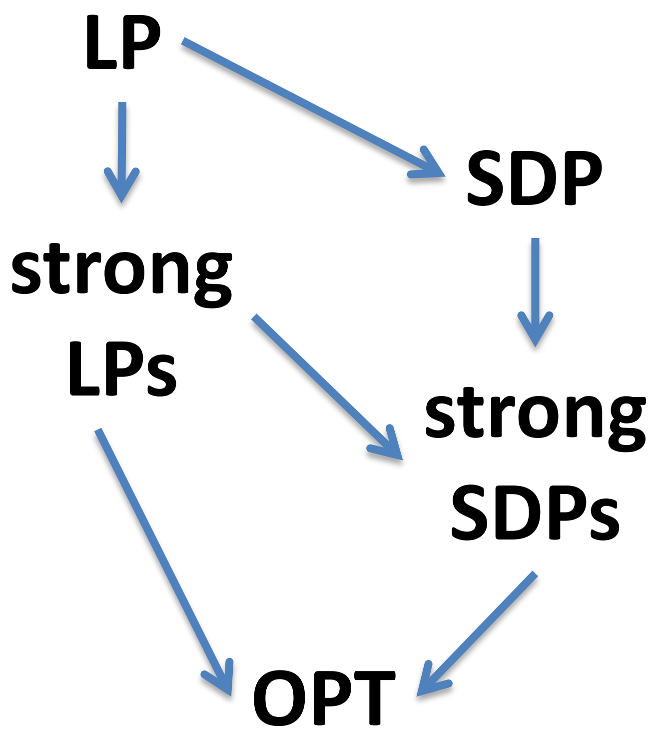

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Graphs: GGR'98, FS'00

CSPs: AdlVKK'02

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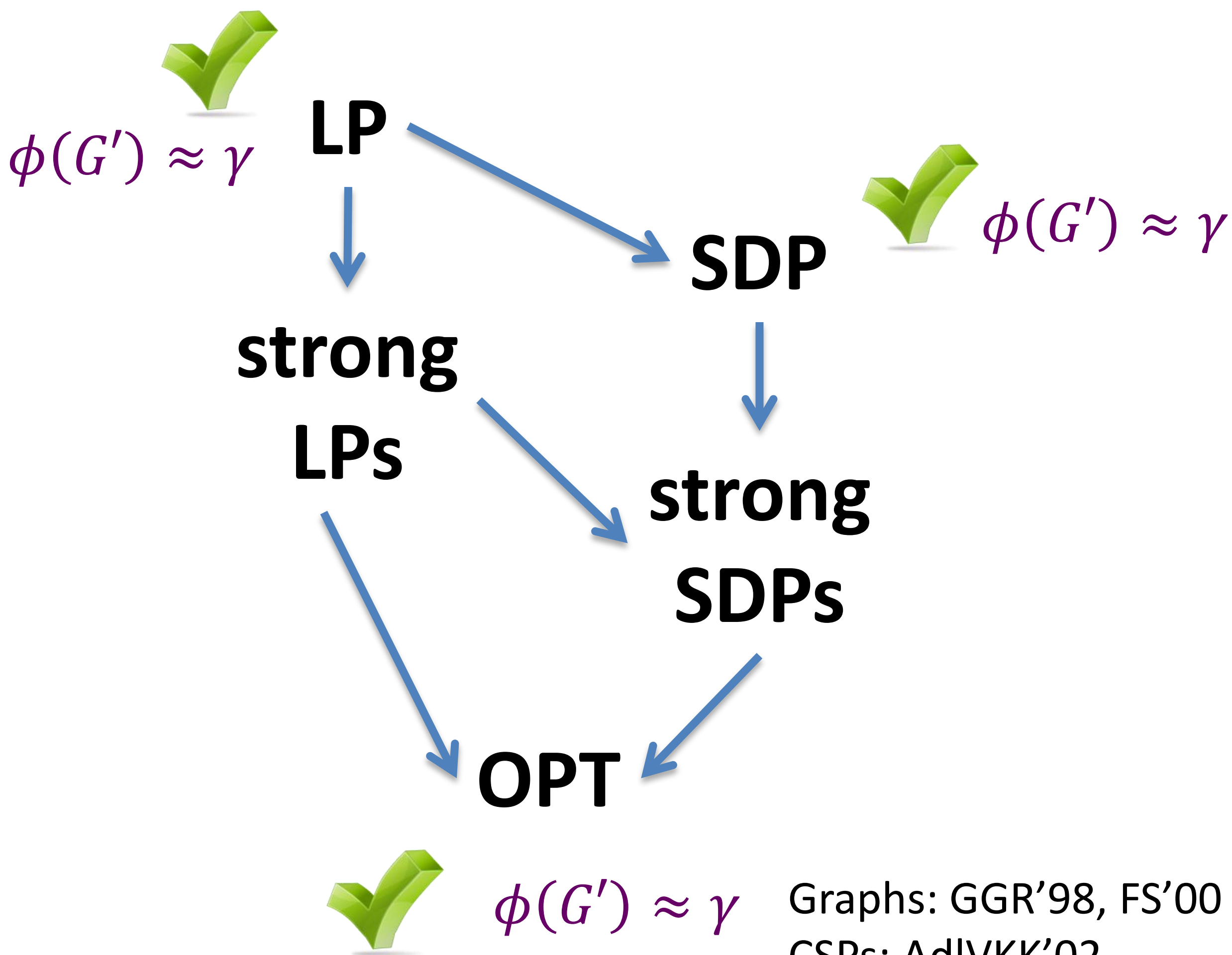

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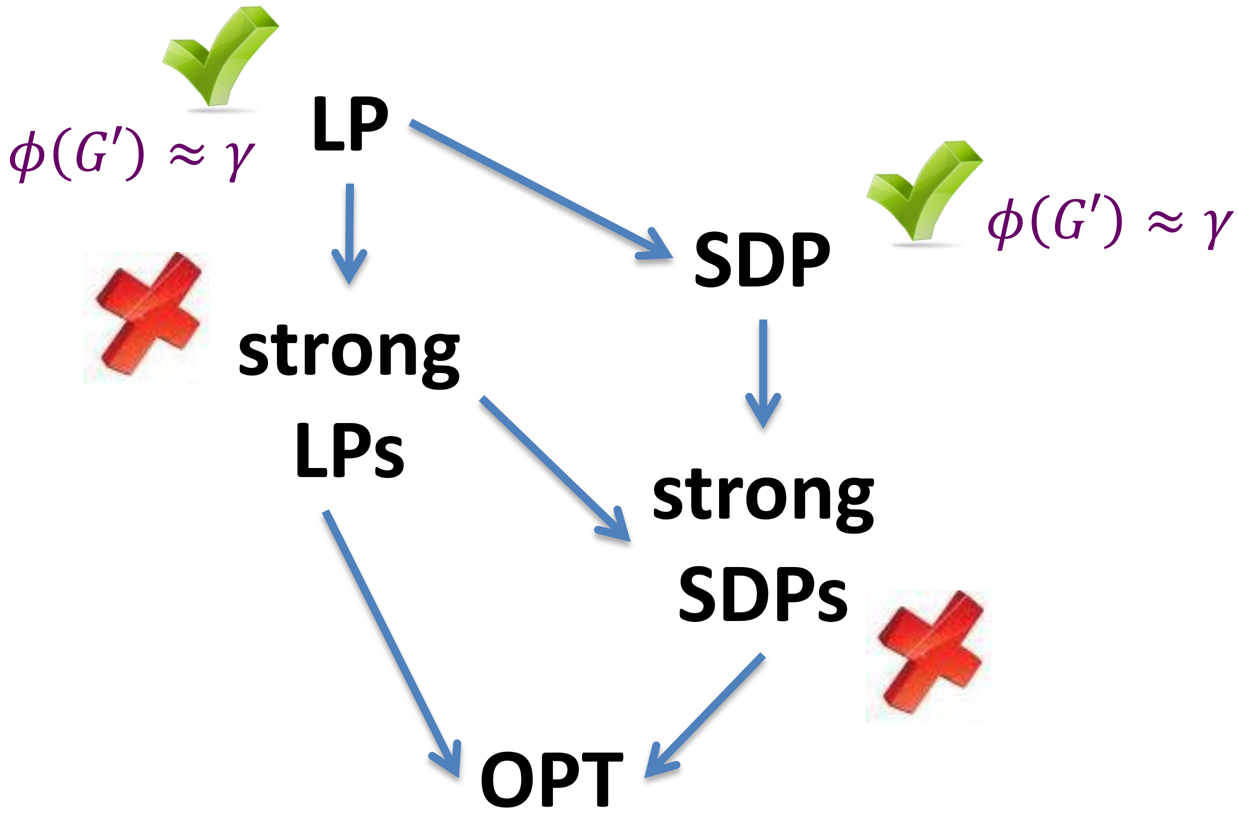

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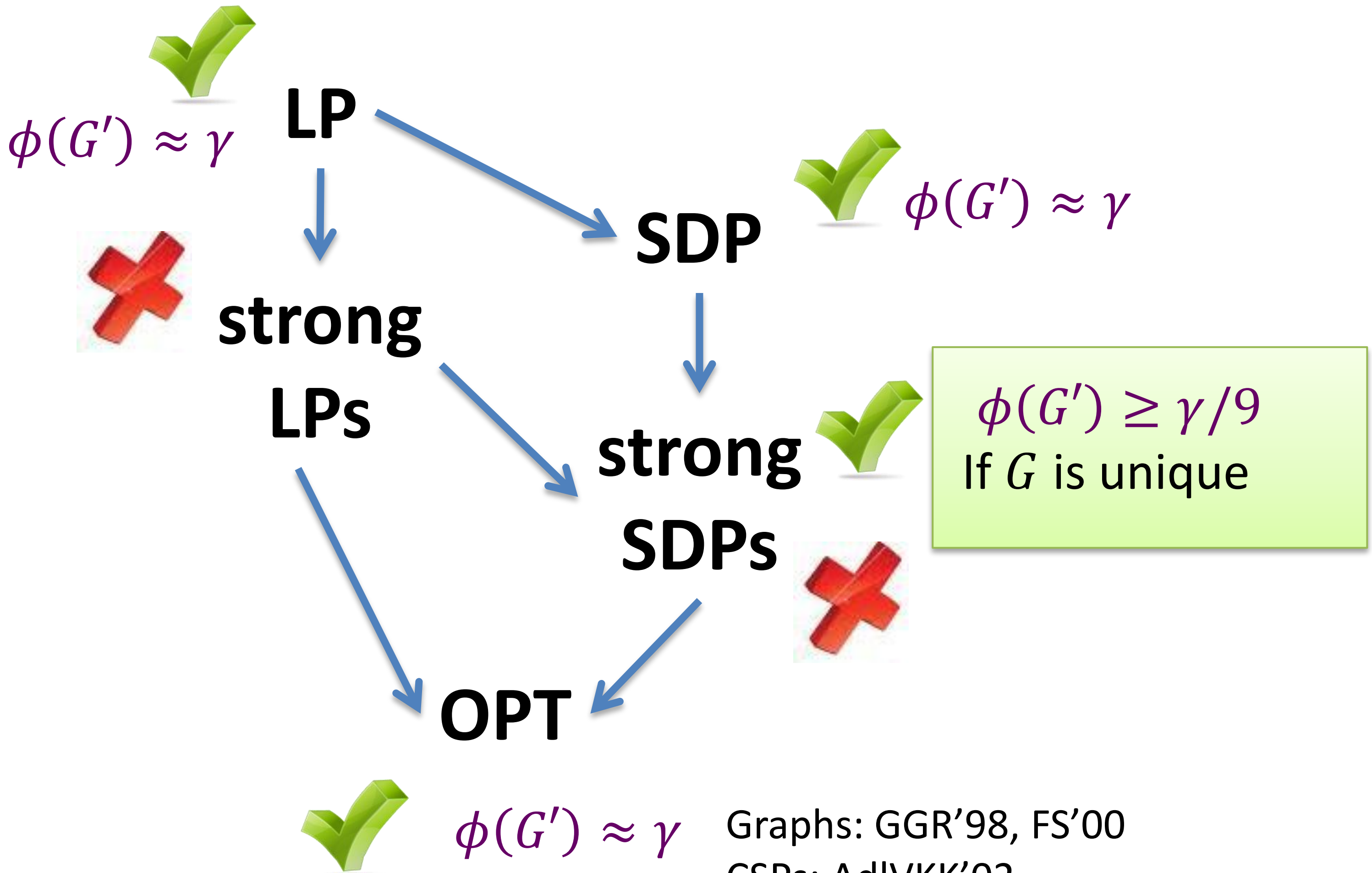

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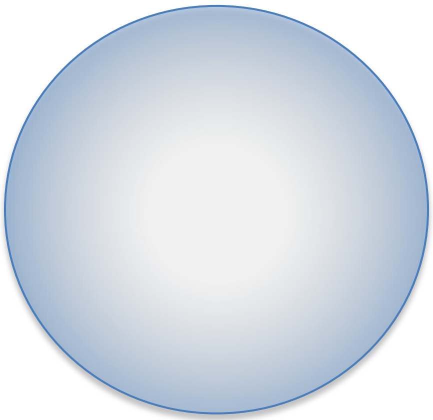
Connecting the two questions



Show that
Sdp$_3(G_\gamma) \leq 1 - c\sqrt{\gamma}$

sdp with ℓ_2^2-triangle inequalities

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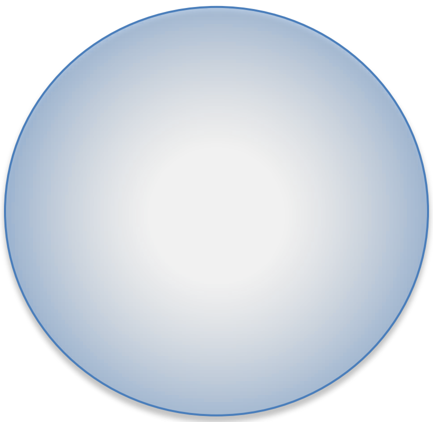
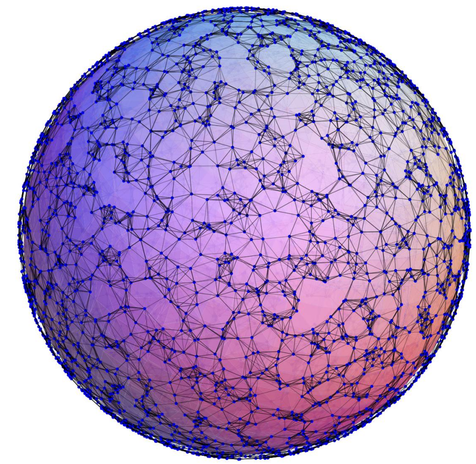


Main subsampling theorem for strong SDPs



Show that
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$\mathrm{Sdp}_3(G_\gamma{}') \leq 1 - c\sqrt{\gamma}/9$

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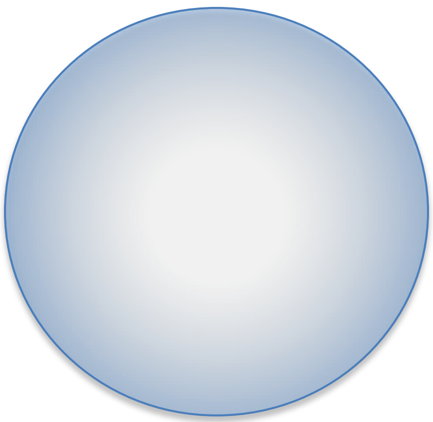

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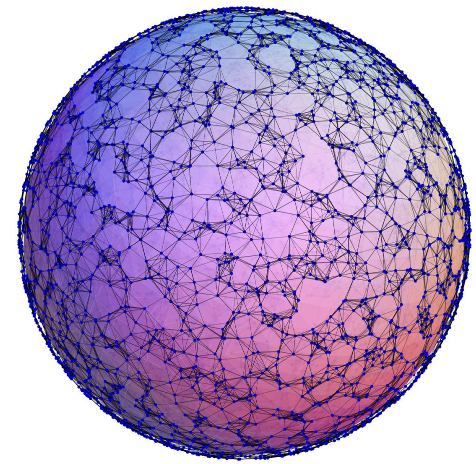

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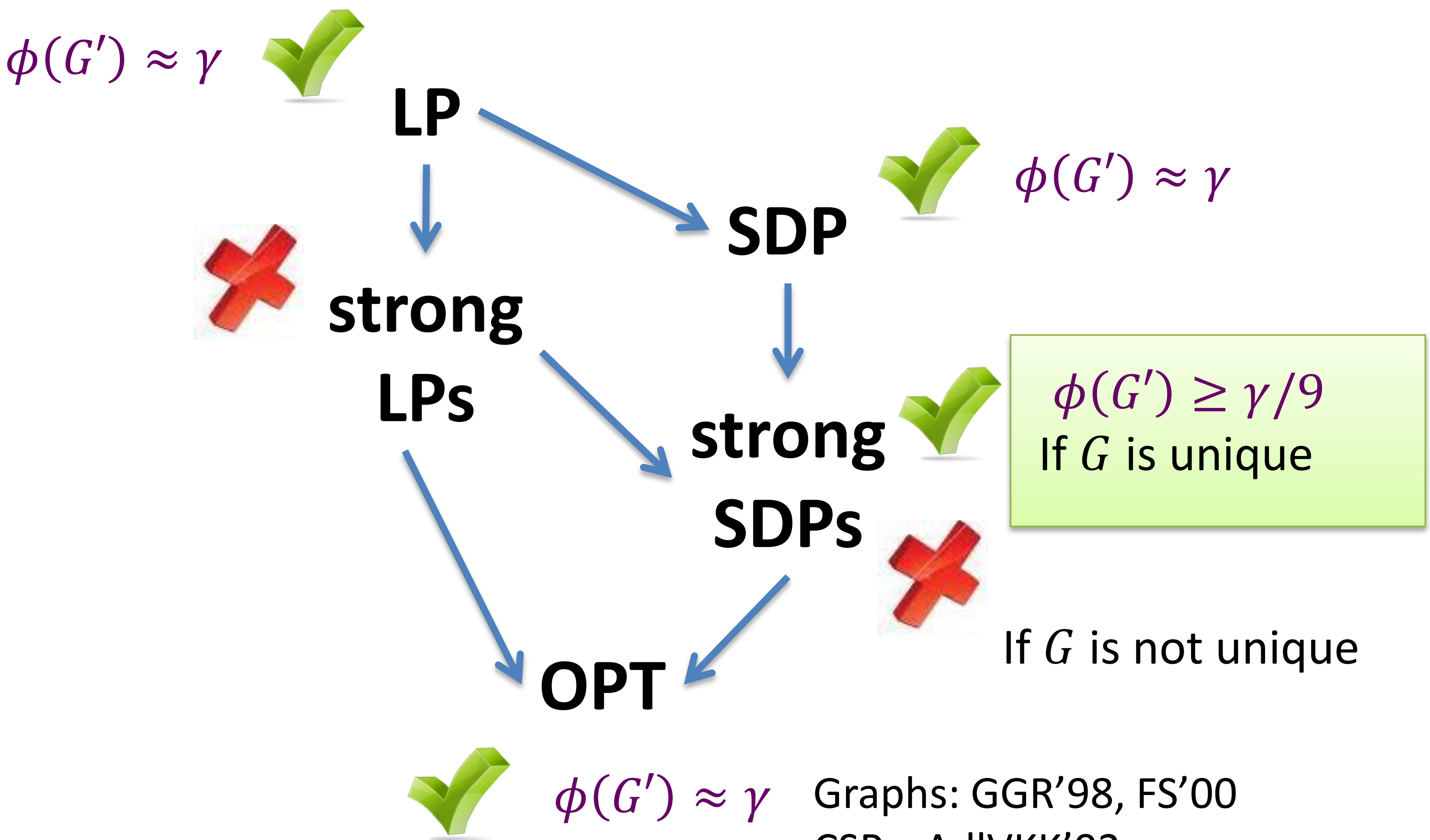
Certifying algorithm for Max-Cut

Proof overview



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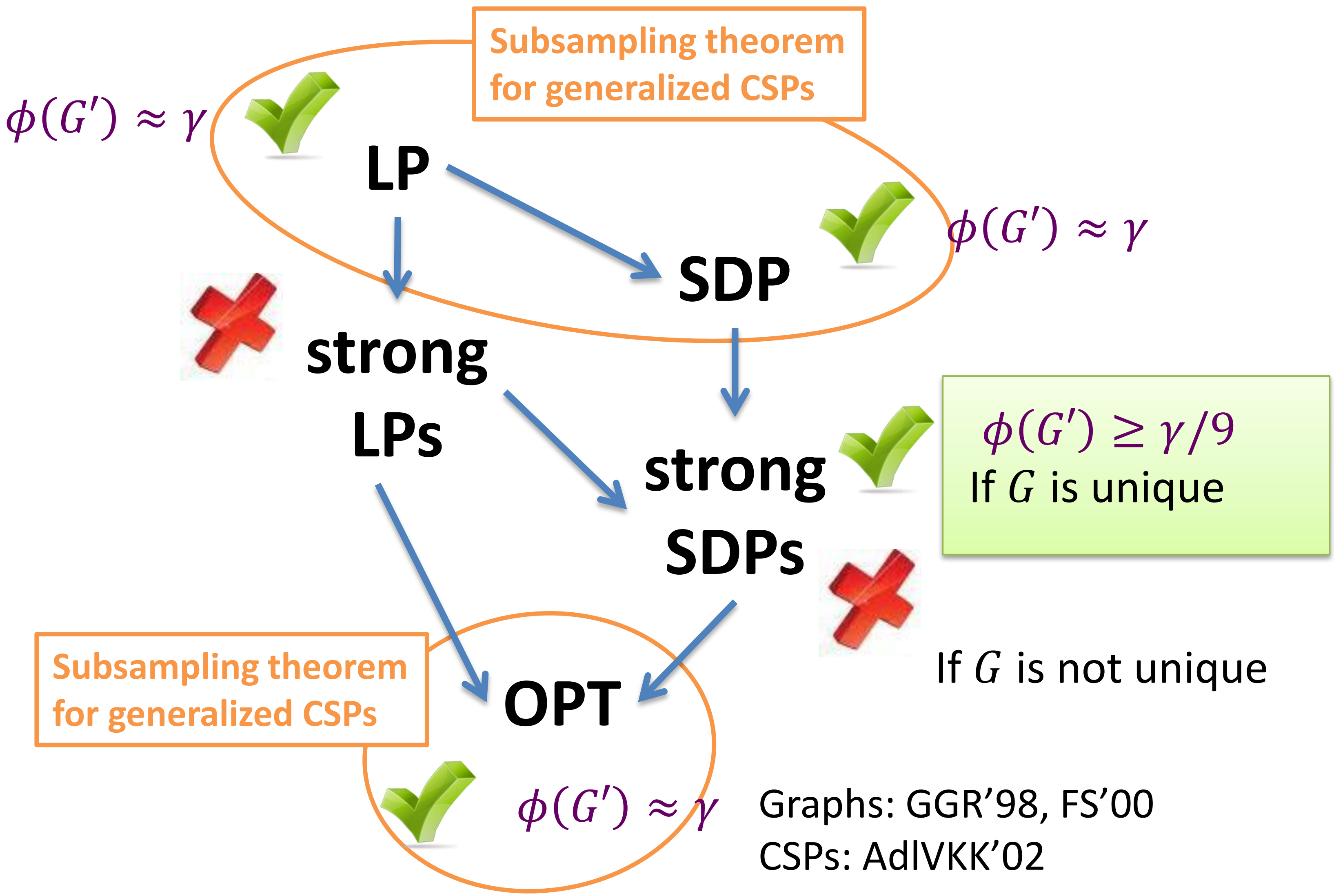

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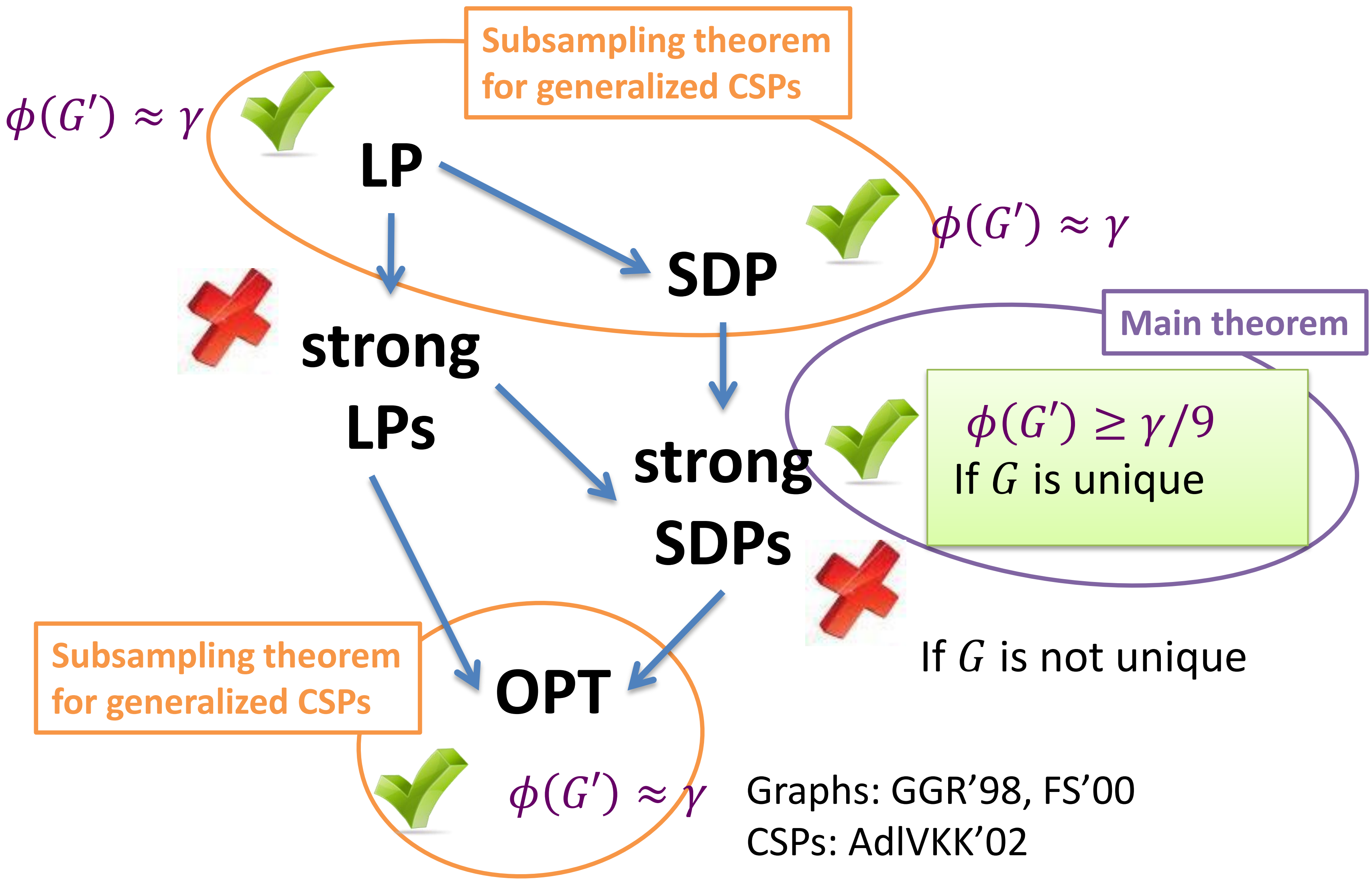

Subsampling theorem

Theorem: Let G be an n-vertex d-regular graph, constraints $f_e : \Sigma^2 \to [0,1]$ where $\Sigma \subseteq R^{O(1)}$ and f_e is Lipschitz.

Let
$$\phi(G) = \min_{x_1,\ldots,x_n \in \Sigma} \mathbf{E}_{(u,v)\in E} f_{(u,v)}(x_u, x_v)$$

Then, whp,

$$|\phi(G) - \phi(G')| \le \epsilon$$

where $G' = G[S]$, random S of size $|S| = \text{poly}\left(\frac{1}{\epsilon}\right) \cdot \frac{n}{d}$

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<u>Previous work</u>:

Graphs: Goldreich-Goldwasser-Ron'98 (degree $\Omega(n)$)

Feige-Schechtman'00 (degree $\Omega(\log n)$)

CSPs: Alon-de la Vega-Kannan-Karpinski'02 ("dense" CSPs)

Main Theorem

Theorem: Let G be any unique CSP. Then, w.h.p.

$$\frac{1}{9} sdp_C(G) - \epsilon \leq sdp_C(G') \leq sdp_C(G) + \epsilon$$

where $sdp_C(G)$ denotes any "reasonable" SDP (e.g. sdp+triangle Inequalities, Lasserre hierarchy)

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Tricky part

Easy part

Proxy graph idea

Goal: $$\frac{1}{9} sdp_C(G) - \epsilon \leq sdp_C(G')$$

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(3) $sdp_C(G) - \epsilon \leq sdp_C(H')$

Taking the 3rd power

We'll take: $H = G^3$

(3) $$sdp_C(G) - \epsilon \leq sdp_C((G^3)')$$

Decode solution to $(G^3)'$ to a solution for G!

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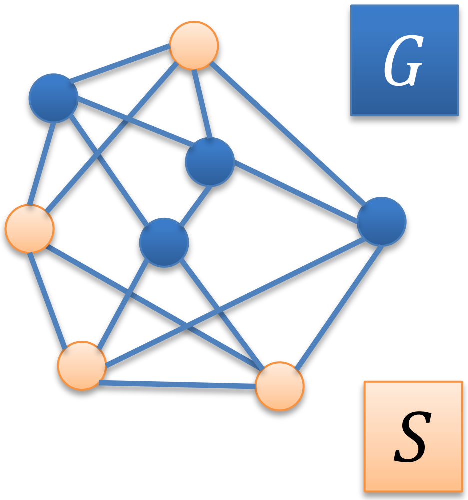

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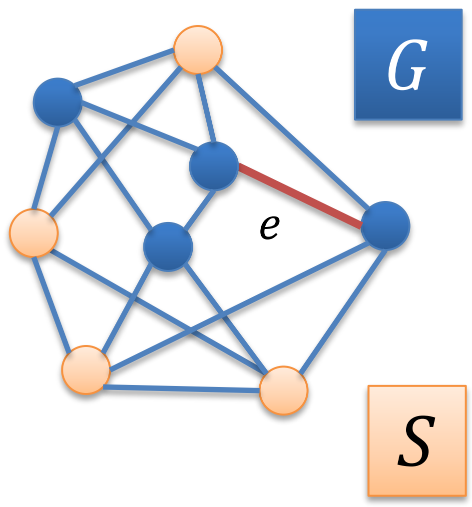

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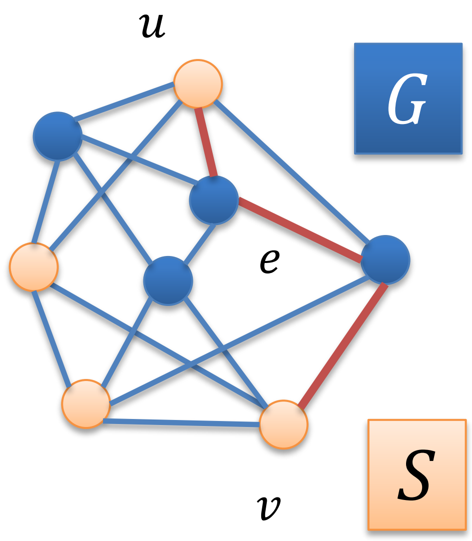
2. Random nbrs u, v in S



Conclusion

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  - Answer is more subtle than for integral value
- General **recipe for average-case algorithms**:
  - Show that relaxation works on a dense graph → algorithm on random subsamples
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