Bilateral and Multilateral Exchanges for Peer-Assisted Content Distribution

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***Abstract*—Users of the BitTorrent file-sharing protocol and its variants are incentivized to contribute their upload capacity in a bilateral manner: downloading is possible in return for uploading to the same user. An alternative is to use multilateral exchange to match user demand for content to available supply at other users in the system. We provide a formal comparison of peer-to-peer system designs based on bilateral exchange with those that enable multilateral exchange via a price-based market mechanism to match supply and demand.**

First, we compare the two types of exchange in terms of the equilibria that arise. A multilateral equilibrium allocation is Pareto efficient, while we demonstrate that bilateral equilibrium allocations are not Pareto efficient in general. We show that Pareto efficiency represents the "gap" between bilateral and multilateral equilibria: a bilateral equilibrium allocation corresponds to a multilateral equilibrium allocation if and only if it is Pareto efficient. Our proof exploits the fact that Pareto efficiency implies reversibility of an appropriately constructed Markov chain.

Second, we compare the two types of exchange through the expected percentage of users that can trade in a large system, assuming a fixed file popularity distribution. Our theoretical results as well as analysis of a BitTorrent dataset provide quantitative insight into regimes where bilateral exchange may perform quite well even though it does not always give rise to Pareto-efficient equilibrium allocations.

***Index Terms*—Peer-to-peer systems, Markov chains, Economics**

I. Introduction

Early peer-to-peer systems did not provide any incentives for participation, leading to extensive *free riding*: many peers were using the resources of other peers without contributing their own [2, 16]. The peer-to-peer community responded with mechanisms to prevent free riding by incentivizing sharing on a *bilateral exchange* basis, as used by BitTorrent [10] and its variants [28, 34].

According to the BitTorrent protocol, each user splits his available upload rate among users from which he gets the highest download rates. As a result, an increase in the upload rate to one user may increase the download rate from that particular user; however, it does not increase the download rate from other users. Thus, two users are incentivized to exchange only if each has content the other wants. This results in a significant drawback of bilateral exchange: it breaks down between users that do not have reciprocally desired files.

The difficulties of bilateral exchange (or barter) in an economy have been long known, the most important being the improbability of coincidence between persons wanting and possessing [19]. As discussed in [19], "there may be many people wanting, and many possessing those things wanted; but to allow of an act of barter, there must be a double coincidence, which will rarely happen." In modern economies, the aforementioned difficulty is eliminated by the use of *money*. Money can enable multilateral exchange by serving as a medium of exchange and a common measure of value. Even though modern societies take the use of money for granted, the same is not the case in peer-to-peer systems.

Peer-to-peer systems could potentially also use *market-based multilateral exchange* to match user demand for content to available supply at other users in the system. This can be done by using virtual currency and assigning a budget to each user that decreases when downloading and increases when uploading. Monetary incentives with a virtual currency have been previously proposed to encourage contribution in peer-to-peer systems [15, 33, 31, 6, 5]; however, such designs are usually more complex than bilateral protocols and are not widespread. Thus, there is a significant tradeoff: *bilateral exchange without money is simple, while multilateral exchange allows more users to trade*. In this paper, we provide a formal comparison of two peer-to-peer system designs: bilateral barter systems such as BitTorrent, and a market-based exchange of content enabled by a price mechanism to match supply and demand. Our main goal is to identify precisely what benefits a currency-based system might offer, and whether these benefits are sufficient to actually warrant all the complexity of implementation presented by such systems.

We start in Section II with a fundamental abstraction of content exchange in systems like BitTorrent: *exchange ratios*. The exchange ratio from one user to another gives the download rate received per unit upload rate. Exchange ratios are a useful formal tool because they allow us to define and study the equilibria of bilateral exchange. In the model of bilateral exchange we consider, each user optimizes with respect to exchange ratios. In Section III, we define *bilateral equilibrium* as a rate vector and a vector of exchange ratios, where all users have simultaneously optimized given exchange ratios. We also define *multilateral equilibrium*, where users optimize with respect to prices; our definition of multilateral equilibrium is the same as competitive equilibrium in economics [22]. In a multilateral equilibrium, the presence of money enables a potentially wider set of exchanges than is possible in bilateral equilibrium.

In Section IV, we compare bilateral and multilateral peer-to-peer systems through the allocations that arise at equilibria. A multilateral equilibrium allocation is always Pareto efficient, while bilateral equilibria may be inefficient. Our main result in this section is that a bilateral equilibrium allocation is Pareto efficient if and only if it is a multilateral equilibrium allocation—in other words, efficient bilateral equilibria must effectively yield "supporting prices" as in a multilateral equilibrium. This result provides formal justification of the efficiency benefits of multilateral equilibria. The proof exploits an interesting connection between equilibria and Markov chains:

an important step of the proof is to show that Pareto efficiency of a bilateral equilibrium rate allocation implies reversibility of an appropriately defined Markov chain, and that this chain has an invariant distribution that corresponds to a price vector of a multilateral equilibrium.

From a practical standpoint, however, the preceding insight is somewhat unsatisfying, because it does not *quantify* the benefits of multilateral exchange. Although all efficient equilibria are multilateral equilibria, if the potential loss of efficiency in bilateral equilibrium is small, then it may be an acceptable tradeoff in return for a significantly simpler system design. In Section V, we perform a quantitative comparison of bilateral and multilateral exchange. We quantify how rarely a double coincidence of wants occurs under different assumptions on the popularity of files in the system.

We first perform an asymptotic analysis assuming that file popularity follows a power law. We find that asymptotically all users are able to trade bilaterally when the file popularity is very concentrated (*i.e.*, when the popularity distribution has a relatively light tail). On the other hand, multilateral exchange performs significantly better than bilateral exchange when the file popularity is not concentrated (*i.e.*, when the distribution has a heavy tail). We complement our theoretical analysis by studying file popularity from a large BitTorrent dataset [29]. Although bilateral equilibria may in general be inefficient, the gap between bilateral and multilateral exchange can be narrowed significantly if each user shares a sufficient number of files: for example, for systems of the size in the dataset, over 96% of users can trade if each user shares at least 10 files. The last result is informative: it suggests that taking small steps to increase the number of bilateral matches possible can actually significantly eliminate almost all the advantage of multilateral exchange.

Section VI discusses the related literature and Section VII concludes. The proofs are included in the Appendix.

II. Exchange Ratios in Bilateral Protocols

Many peer-to-peer protocols enable exchange on a *bilateral* basis between users: a user i uploads to a user j if and only if user j uploads to user i in return. Of course, such an exchange is only possible if each user has something the other wants; this is known as "double coincidence of wants" in economics. The foremost examples of such a protocol are BitTorrent and its variants. While such protocols are traditionally studied solely through the rates that users obtain, this section provides an interpretation of these protocols through *exchange ratios*. As exchange ratios can be interpreted in terms of prices, these ratios allow us to compare bilateral barter-based peer-to-peer systems with multilateral price-based peer-to-peer systems.

Let r_{ij} denote the rate sent from user i to user j at a given point in time in a bilateral peer-to-peer protocol. We define the *exchange ratio* between user i and user j as the ratio $\gamma_{ij} = r_{ji}/r_{ij}$; this is the download rate received by i from j, per unit of rate uploaded to j. By definition, $\gamma_{ij} = 1/\gamma_{ji}$. Clearly, a rational user i would prefer to download from users with which he has higher exchange ratios.

The exchange ratio has a natural interpretation in terms of prices. In particular, assume that users charge each other for content in a common monetary unit, but that all transactions are *settlement-free*, *i.e.*, no money ever changes hands. In this case, if user i charged user j a price p_{ij} per unit rate, the exchange of content between users i and j must satisfy:

$$p_{ij} r_{ij} = p_{ji} r_{ji}$$

We refer to p_{ij} as the *bilateral price* from i to j. Note that the preceding condition thus shows the exchange ratio is equivalent to the ratio of bilateral prices: $\gamma_{ij} = p_{ij}/p_{ji}$ (as long as the prices and rates are nonzero).

What is the exchange ratio for BitTorrent? A user splits his upload capacity equally among those users in his active set from which he gets the highest download rates. Let α be the size of the active set. Suppose all rates r_{kj} that user j receives from users $k \neq i$ are fixed and let R_j^α be the α-th highest rate that j receives. Let B_j be the upload capacity of user j. Then, r_{ji} depends on r_{ij}. In particular,

$$r_{ji} = \begin{cases} B_j/\alpha & \text{if } r_{ij} > R_j^\alpha \\ 0 & \text{otherwise} \end{cases}$$

Thus for BitTorrent, the exchange ratio is $\gamma_{ij} = B_j/(\alpha \cdot r_{ij})$ if user i is in the active set, and zero otherwise. Note that the exchange ratios $\gamma_{i_1,j}$ and $\gamma_{i_2,j}$ may be different for two users i_1, i_2 in j's active set.

The exchange ratio γ_{ij} *decreases* with r_{ij} as long as user i is in user j's active set (in which case r_{ji} is constant). Hence, a strategic user i would prefer to choose r_{ij} as small as possible while remaining in j's active set. This behavior is exactly the approach taken by the BitTyrant [28] variation on BitTorrent. In fact, if all users follow this policy, then $r_{ij} = R_j^\alpha$ for all users i in j's active set. Note that in this case, $\gamma_{ij} = B_j/(\alpha \cdot R_i^\alpha)$. Thus, user j has the same exchange ratio to all users i with which he bilaterally exchanges content.

The preceding discussion highlights the fact that the rates in a bilateral peer-to-peer system can be interpreted via exchange ratios. Thus far we have assumed that *transfer rates* are given, and exchange ratios are computed from these rates. In the next section, we turn this relationship around: we explicitly consider an abstraction of bilateral peer-to-peer systems where users react to given exchange ratios, and we compare the resulting outcomes to price-based multilateral exchange.

III. Bilateral and Multilateral Equilibria

In this section, we define bilateral equilibrium (BE) and multilateral equilibrium (ME), *i.e.*, the market equilibria corresponding to bilateral and multilateral exchange. In the formal model we consider, a set of users U shares a set of files F. User i has a subset of the files $S_i \subseteq F$ and is interested in downloading files in $T_i \subseteq F - S_i$. Throughout, we use r_{ijf} to denote the rate at which user i uploads file f to user j. We then let $x_{if} = \sum_j r_{jif}$ be the rate at which user i downloads file f. We denote the vector of download rates for user i by $\boldsymbol{x}_i = (x_{if}, f \in T_i)$. Finally, let $y_i = \sum_{j,f} r_{ijf}$ be the total upload rate of user i. We measure the desirability of a rate vector to user i by a *utility function* $v_i(\boldsymbol{x}_i, y_i)$, according to the following assumption. *This assumption remains in force throughout the paper.*

Assumption 1 *The preference relation of a user on the set of feasible rate vectors is represented by a continuous strictly concave utility function $v_i(\boldsymbol{x}_i, y_i)$, which is strictly increasing*

Bilateral Peer Optimization:

$$\begin{aligned}
&\text{maximize} && v_i(\boldsymbol{x}_i, y_i) \\
&\text{subject to} && x_{if} = \textstyle\sum_j r_{jif} \text{ for all } f \\
& && y_i = \textstyle\sum_{j,f} r_{ijf} \\
& && \boldsymbol{r} \in \mathcal{X} \\
& && \textstyle\sum_f r_{jif} = \gamma_{ij} \sum_f r_{ijf} \text{ for all } j
\end{aligned}$$

Multilateral Peer Optimization:

$$\begin{aligned}
&\text{maximize} && v_i(\boldsymbol{x}_i, y_i) \\
&\text{subject to} && x_{if} = \textstyle\sum_j r_{jif} \text{ for all } f \in T_i \\
& && y_i = \textstyle\sum_{j,f} r_{ijf} \\
& && \boldsymbol{r} \in \mathcal{X} \\
& && \textstyle\sum_{j,f} p_j r_{jif} = p_i \sum_{j,f} r_{ijf}
\end{aligned}$$

Fig. 1. Optimization problems for price-based exchange.

in each download rate x_{if} for all $f \in T_i$ and strictly decreasing in the upload rate y_i.

Note that utility functions in our model depend on instantaneous transfer rates, rather than the number of bytes exchanged. This is consistent with the "snapshot" view that our model of peer-to-peer file-sharing adopts: informally, it is as if we are analyzing efficiency of the system at a fixed moment in time. This is also why our model keeps fixed both the set of files available for upload and the set of files desired for download at a given peer; these sets remain constant on the timescale we are considering. An important open direction related to this work concerns the analysis of *dynamic* efficiency, where these sets might change over time.

Each user is assumed to have a constraint on the available upload rate; let B_i denote this upper bound for user i. We assume that users do not face any constraint on their download rate; this is consistent with most end-users' asymmetric connections today, where upload capacity is far exceeded by download capacity. Further, for the purposes of this paper, we also assume that there are no constraints in the middle of the network, though our prior work suggests a natural approach for including such constraints [5].

Let

$$\mathcal{X} = \{\boldsymbol{r} : \boldsymbol{r} \geq 0;\ r_{kjf} = 0 \text{ if } f \notin S_k; \\ \sum_{j,f} r_{ijf} \leq B_i\ \forall i \in U\}$$

be the set of feasible rate vectors. In particular, this ensures that (i) all rates are nonnegative, (ii) users only upload files they possess, and (iii) each user does not violate his bandwidth constraint.

In the next two subsections, we look at two different models of equilibrium, corresponding to exchange in bilateral and multilateral systems, respectively. In bilateral exchange, we assume that users maximize their utility given the exchange ratios they see to other users. Thus, in a bilateral equilibrium, the exchange ratios must be chosen to exactly balance the rates each user considers optimal—this is informally the condition that "supply equals demand." In multilateral exchange, on the other hand, we assume that users can earn currency by uploading to others, and they can spend that currency in downloading from any users they wish. In this model, users maximize their utility given the current prices in the system, and the prices must be set to exactly balance the rates each user considers optimal.

A. Bilateral Equilibrium

We start by considering users' behavior in bilateral schemes, given a vector of exchange ratios $(\gamma_{ij}, i, j \in U)$. User i solves the Bilateral Peer Optimization problem given in Figure 1.[1] In the definition of this optimization problem, in addition to the definition of upload and download rates, each user i faces one constraint for *each potential peer j* with which i might exchange content: the restriction that $\sum_f r_{jif} = \gamma_{ij} \sum_f r_{ijf}$ ensures that the rate at which i downloads from j is exactly the exchange ratio times the rate at which i uploads to j.

We can now define bilateral equilibrium.

Definition 1 *The rate allocation $\boldsymbol{r}^* \in \mathcal{X}$ and the exchange ratios $\boldsymbol{\gamma}^* = (\gamma^*_{ij}, i, j \in U)$ with $\gamma^*_{ij} \cdot \gamma^*_{ji} = 1$ for all i, j, constitute a Bilateral Equilibrium (BE) if, for each user i, $\boldsymbol{r}^*$ solves the Bilateral Peer Optimization problem given exchange ratios $\boldsymbol{\gamma}^*$.*

Definition 1 requires that (i) all users have optimized with respect to the exchange ratios, and (ii) the market clears. Even though the market clearing condition is not explicitly stated, it is implicitly required, since the same vector $\boldsymbol{r}^*$ is an optimal solution of the Bilateral Peer Optimization problems of all users.

We do not expect a BE to exist in general. For example, this is trivially the case if no pair of users has reciprocally desired files; *i.e.*, if for every pair i, j either $S_i \cap T_j = \emptyset$ or $S_j \cap T_i = \emptyset$. Existence can be assured if we assume that every user can find every file he desires through bilateral trade. This is formalized in Assumption 2.

Assumption 2 *For every user i and every file $f \in T_i$ there exists a user j such that $f \in S_j$ and $T_j \cap S_i \neq \emptyset$.*

Proposition 1 *If Assumption 2 holds, then a BE exists.*

B. Multilateral Equilibrium

By contrast, in a *multilateral price-based exchange*, the system maintains one price per peer, and users optimize with respect to these prices.[2] We denote the price of user i by p_i. Figure 1 also gives the Multilateral Peer Optimization problem. Note that the first three constraints (giving download and upload rates and ensuring that the rate allocation is feasible) are identical to the Bilateral Peer Optimization; only the last constraint is different. While the bilateral exchange requires user i to download only from those users to whom he uploads, no such constraint is imposed on multilateral exchanges: user i accrues capital for uploading, and he can spend this capital however he wishes for downloading.

[1] Note that we allow users to bilaterally exchange content over multiple files, even though this more general design is not typically supported by swarming systems like BitTorrent; in BitTorrent, a single file is split into subpieces called chunks, and users exchange chunks belonging to the same file.

[2] It can be shown that this is equivalent to having either one price per file, or one price per peer per file, in our setting [5]. As explained in that paper, the choice of one price per peer affords certain advantage for system design, so we adopt it as our approach in this paper.

We next give the definition of ME, which corresponds to the concept of competitive equilibrium in economics [22].

Definition 2 *The rate allocation $\boldsymbol{r}^*$ and the user prices $\boldsymbol{p}^* = (p_i^*, i \in U)$ with $p_i^* > 0$ for all $i \in U$ constitute a Multilateral Equilibrium (ME) if, for each user i, $\boldsymbol{r}^*$ solves the Multilateral User Optimization problem given prices $\boldsymbol{p}^*$.*

Similar to Definition 1, Definition 2 requires that (i) all users have optimized with respect to prices, and (ii) the market clears. Again, even though the market clearing condition is not explicitly stated, it is implicitly required, since the same vector $\boldsymbol{r}$ is used in the optimization problems of all users.

Our model is closely related to exchange economies [22]. In an *exchange economy*, there is a finite number of agents and a finite number of commodities. Each agent is endowed with a bundle of commodities and has a preference relation on the set of commodity vectors. Given a price vector, each agent finds a vector of commodities to exchange that maximizes his utility. In particular, if $\boldsymbol{p}$ is the vector of prices and agent i has endowment $\boldsymbol{w}_i$, he sells it at the market and obtains wealth $\boldsymbol{p} \cdot \boldsymbol{w}_i$. Then the agent buys goods for his consumption at the same price (he may buy back some of the goods he sold).

A straightforward reformulation reveals that our model shares much in common with a standard exchange economy: it is *as if* agent i has B_i units of his own "good", priced at p_i. He can trade this for goods from other users on the open market at prices $\boldsymbol{p}$. With this interpretation, $B_i - y_i$ is the amount of his own good that he chooses to keep. However, notice that this is not a standard exchange economy, as the upload rate is not a true commodity; rather, the commodities are the rates of specific files that are uploaded. Since B_i imposes a *joint* constraint on the upload rates of these files, our model is a generalization of the standard exchange economy.

Our goal is to show that a ME exists. However, as with BE, we do not expect equilibria to exist without any restrictions on the sets S_i and T_i of files being uploaded and downloaded, respectively, by user i. For example, suppose there is a file that some users want to download, but no user has the file available for upload. Then in general, such a file has positive demand, while supply is always zero. Thus the excess demand for such a file is positive unless its price is sufficiently high. Setting a sufficiently high price is equivalent to considering a system without that file.

To avoid such pathological situations, we introduce a natural diversity assumption. We define the *user graph* as the directed graph $G = (V, E)$ with $V = U$, and $E = \{(i, j) : S_i \cap T_j \neq \emptyset\}$. In other words, G is a graph where nodes correspond to users. There is a directed edge from user i to user j if i has a file that j desires.

Assumption 3 *The user graph consists only of strongly connected components.*

Proposition 2 *If Assumption 3 holds, then a ME exists.*

IV. Efficiency of Equilibria

This section rigorously analyzes the efficiency properties of bilateral and multilateral exchange. We assume users explicitly react to exchange ratios or prices, and we compare the schemes through their resulting equilibria. In order to proceed, we first formally define Pareto efficiency.

Definition 3 *Given rate allocations $\boldsymbol{r}, \boldsymbol{r}' \in \mathcal{X}$, let $\boldsymbol{x}, \boldsymbol{x}'$ be the corresponding download rates, and let $\boldsymbol{y}, \boldsymbol{y}'$ be the corresponding upload rates. Then $\boldsymbol{r}$* Pareto dominates *$\boldsymbol{r}'$ if $v_i(\boldsymbol{x}_i, y_i) \geq v_i(\boldsymbol{x}'_i, y_i)$ for all i, with strict inequality for at least one i.*

A rate allocation $\boldsymbol{r} \in \mathcal{X}$ is Pareto efficient *if it is not Pareto dominated by any other rate allocation $\boldsymbol{r}' \in \mathcal{X}$.*

Thus a rate allocation is Pareto efficient if there is no way to increase the utility of some user without decreasing the utility of some other user. An ME allocation is always Pareto efficient; this is the content of the first fundamental theorem of welfare economics [22]. For completeness, we include the result here.

Proposition 3 *If the rate allocation $\boldsymbol{r}^*$ and the user prices $(p_i^*, i \in U)$ with $p_i^* > 0$ for all $i \in U$ constitute a ME, then the allocation $\boldsymbol{r}^*$ is Pareto efficient.*

In some systems where ME exists, BE may not exist, so in these cases ME is clearly a more desirable outcome. However, even when BE exist they may not be Pareto efficient, as the following example shows.

Example 1 *Consider a system with n users and n files, for $n > 2$. Each user i has file f_i and wants files f_{i+1} and f_{i-1}. The utility of user i is $v_i(x_{i,f_{i-1}}, x_{i,f_{i+1}}, y_i) = x_{i,f_{i-1}} + 4x_{i,f_{i+1}} + \ln(2 - y_i)$,* i.e., *user i wants the files of both user $i+1$ and user $i-1$, but derives a higher utility from the file of user $i+1$.*

We first consider a symmetric BE with exchange ratios $\gamma^*_{i,i+1} = 2$ and $\gamma^*_{i,i-1} = 1/2$. The equilibrium rates are $r^*_{i-1,i} = 1$ and $r^*_{i+1,i} = 1/2$, and the download rates are $x^*_{i,f_{i-1}} = 1$ and $x^*_{i,f_{i+1}} = 1/2$. The utility of each user i is $3 - \ln(2) \approx 2.3$. On the other hand, prices $p_i^* = 1$ for all i, and rates $r^*_{i+1,i} = 1.75$, $r^*_{i-1,i} = 0$ constitute a ME. The utility of each user is $7 - \ln(4) \approx 5.61$, i.e., *significantly larger than the utility of a user at the BE. This demonstrates that the BE allocation is not Pareto efficient.*

The previous example shows that BE may not be Pareto efficient. By changing the utility function of a user in this example, we next provide an example of a BE rate allocation that is Pareto efficient.

Example 2 *Consider a system with n users and n files, for $n > 2$. Each user i has file f_i and wants files f_{i+1} and f_{i-1}. The utility of user i is $v_i(x_{i,f_{i-1}}, x_{i,f_{i+1}}, y_i) = x_{i,f_{i-1}} + x_{i,f_{i+1}} + \ln(2 - y_i)$.*

We consider a symmetric BE with exchange ratios $\gamma^*_{i,i+1} = 1$ and $\gamma^*_{i,i-1} = 1$. The equilibrium rates are $r^*_{i-1,i} = 1/2$ and $r^*_{i+1,i} = 1/2$. The BE rate allocation is Pareto efficient. In particular, it corresponds to a ME: prices $p_i^* = 1$ for all i, and rates $r^*_{i+1,i} = 1/2$, $r^*_{i-1,i} = 1/2$ constitute a ME.

Thus BE may be inefficient, while ME always have Pareto efficient allocations (Proposition 3). In Example 2, the BE rate allocation is Pareto efficient and corresponds to a ME. Our main result is that *a BE allocation is Pareto efficient if and only if it is an ME allocation*. In particular, if a BE allocation is Pareto efficient, then there exist "supporting prices", *i.e.*, prices such that the BE rate allocation is optimal for the Multilateral Peer Optimization problem of each user. Informally, Pareto efficiency represents the "gap" between BE and ME.

Proposition 4 *Assume that for every user i and any fixed $\boldsymbol{x_i}$, $v_i(\boldsymbol{x_i}, y_i) \to -\infty$ as $y_i \to B_i$. Let $(\boldsymbol{r}^*, \boldsymbol{\gamma}^*)$ be a BE. The rate allocation $\boldsymbol{r}^*$ is Pareto efficient if and only if there exists a price vector $\boldsymbol{p}$ such that $\boldsymbol{r}^*$ and $\boldsymbol{p}$ constitute a ME.*

Proposition 4 assumes that $v_i(\boldsymbol{x_i}, y_i) \to -\infty$ as $y_i \to B_i$ for every user i and every fixed $\boldsymbol{x_i}$. This assumption ensures that the total upload rate of a user is strictly smaller than his upload capacity at the BE. This is a reasonable assumption for a peer-to-peer setting, since we do not expect users to use all their upload capacity. We note that if the total upload rate of a user is equal to his upload capacity, then there may exist Pareto efficient BE that do not correspond to ME, simply because users have already "maxed out" their available upload capacity.

We provide an overview of the proof of Proposition 4, which demonstrates an interesting connection between equilibria and Markov chains; the details of the proof are provided in the Appendix. From a BE rate allocation $\boldsymbol{r}^*$, we construct a transition rate matrix $\boldsymbol{Q}$ for a continuous time Markov chain, such that $Q_{ij} = \sum_f r^*_{ijf}$ if $i \neq j$, and $Q_{ii} = -\sum_{j,f} r^*_{ijf}$. We first observe that $\boldsymbol{\pi Q} = 0$ implies that the multilateral budget constraint is satisfied with price vector $\boldsymbol{\pi}$; therefore, for any invariant distribution $\boldsymbol{\pi}$, $\boldsymbol{r}^*$ is feasible for the Multilateral Peer Optimization problem of every user when prices are equal to $\boldsymbol{\pi}$. We then show that if $\boldsymbol{r}^*$ is also Pareto efficient, there exists an invariant distribution of $\boldsymbol{Q}$, say $\boldsymbol{p}$, such that $\boldsymbol{r}^*$ is an optimal solution of the Multilateral Peer Optimization problem of each user when the prices are equal to $\boldsymbol{p}$. We conclude that $\boldsymbol{r}^*$ and $\boldsymbol{p}$ constitute a ME.

A key step of the proof is to show that Pareto efficiency of $\boldsymbol{r}^*$ implies *reversibility* of $\boldsymbol{Q}$. This is proven by contradiction: if $\boldsymbol{Q}$ is not reversible, then we can find a cycle of peers that can change their rates only along successive pairs of peers on the cycle and, in doing so, make all their utilities strictly higher.[3]

Now let $\boldsymbol{\pi}$ be an invariant distribution of $\boldsymbol{Q}$ with all entries positive.[4] If the matrix $\boldsymbol{Q}$ is reversible, then $\gamma^*_{ij} = \pi_i/\pi_j$ for all pairs of users i and j that trade at the BE. We conclude that if $\boldsymbol{r}^*$ is Pareto efficient, then $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem for each user given prices $\boldsymbol{\pi}$ *if the user is restricted to trade with peers he trades with at the BE*. Much of the complexity in the proof is to show that this result holds even if user i is not restricted to trade only with those users it transacts with at the BE.

The matrix corresponding to the BE allocation of Example 1 is not reversible, which implies that the BE allocation is not Pareto efficient. On the other hand, the matrix corresponding to the BE allocation of Example 2 is reversible, and the BE allocation is Pareto efficient and corresponds to a ME allocation.

[3]This proof is closely related to Kolmogorov's cyclic characterization of reversible Markov chains; see, for example, [30] for details.

[4]It is straightforward to check such a distribution exists; see the proof for details.

V. Participation in Trade

Bilateral exchange may be particularly restrictive because a pair of users can exchange only if each has a file that the other wants. On the other hand, while allowing multilateral exchange significantly increases the number of possible exchanges and potentially increases the number of users that can trade, it comes at the cost of increased complexity. In this section, *we compare bilateral and multilateral exchange through the corresponding percentages of users that can trade*.[5] Though distinct from Pareto efficiency, this metric provides *quantitative* insight into the comparison of the two types of exchange. In particular, we expect that systems that perform well will also generally encourage high levels of participation. We characterize regimes where bilateral exchange performs very well with respect to this metric, and for which, as a result, it may not be worth the effort to use multilateral exchange.

In Section V-A, we introduce the framework we use to study the percentage of users that can trade in BE and ME. Our analysis is based on a random model, where we assume file popularity follows a power law. In Section V-B we carry out an asymptotic theoretical analysis as the number of users and files grows large. In Section V-C, we complement our theoretical analysis by studying file popularity from a large BitTorrent dataset; here we find that participation in bilateral exchange improves significantly if each user shares a sufficiently large number of files.

A. Framework

1) Definitions: We start by formally defining the quantities we compare. For simplicity, we consider settings where each user is interested in downloading one file, *i.e.*, $|T_i| = 1$ for all $i \in U$. This assumption significantly simplifies the analysis, since we do not need to consider how a user's utility function depends on different files.[6] In this section, therefore, we need not specify utility functions in determining how much bilateral exchange restricts trade. For a given peer-to-peer system, we define the *system profile* to consist of the specification of which files each user desires and possesses, *i.e.*, $\mathcal{P} = \{T_i, S_i, i \in U\}$.

We say that user i *can trade bilaterally under* $\mathcal{P}$ if there exists some user j such that $S_i \cap T_j \neq \emptyset$ and $S_j \cap T_i \neq \emptyset$, that is, if i and j have reciprocally desired files. Given a system profile $\mathcal{P}$, let $\rho_{BE}(\mathcal{P})$ be the percentage of users that cannot trade bilaterally. We note that $\rho_{BE}(\mathcal{P})$ is equal to the percentage of users that need to be removed from the system so that a BE exists for $\mathcal{P}$. The condition $\rho_{BE}(\mathcal{P}) = 0$ is equivalent to Assumption 2 when each user desires exactly one file, as we assume in this section.

Similarly, we say that user i *can trade multilaterally under* $\mathcal{P}$ if there exist users $k_1, k_2, ..., k_n$ such that $S_{k_j} \cap T_{k_{j+1}} \neq \emptyset$ for $j = 1, ..., n$; $S_i \cap T_{k_1} \neq \emptyset$ and $S_{k_n} \cap T_i \neq \emptyset$. In words,

[5]We note that this metric does not account for differences in bandwidth.

[6]If marginal utilities at zero are infinite, then a setting with $|T_i| > 1$ can be reduced to a problem where each user only desires one file. For each user i, we construct $|T_i|$ users in the new problem each of which desires one file. Then, a user in the original problem can trade bilaterally if all users corresponding to him in the new problem can trade bilaterally.

user i is able to trade multilaterally if and only if there exists a cycle of users starting (and ending) at i such that each user possesses a file that is desired by the next user in the cycle. Clearly, if user i can trade bilaterally under $\mathcal{P}$, then he can also trade multilaterally under $\mathcal{P}$. Let $\rho_{ME}(\mathcal{P})$ be the percentage of users that cannot trade multilaterally.

We note that $\rho_{ME}(\mathcal{P})$ is equal to the percentage of users that need to be removed from the system so that a ME exists for $\mathcal{P}$. The condition $\rho_{ME}(\mathcal{P}) = 0$ is weaker than Assumption 3; however, it is sufficient for ME existence when each user desires exactly one file. For details, see Lemma 1 in the Appendix.

2) Random Model: We assume that the system profile $\mathcal{P}$ is chosen according to some distribution that depends on the popularity of different files, and that the sets S_i and T_i are chosen independently for each user i. We denote by q_i the *popularity* of the i-th file and assume that the *probability* that the i-th file is desired or possessed by a user is proportional to q_i. We assume that each q_i does not depend on the number of files in the system. On the other hand, the probability that the i-th file is chosen clearly depends on the number of files K in the system, since it is $q_i / \sum_{j=1}^{K} q_j$.

We are interested in comparing the expected proportions of peers that cannot trade bilaterally and multilaterally—that is, the expected values of $\rho_{BE}(\mathcal{P})$ and $\rho_{ME}(\mathcal{P})$—for given file popularities.

B. Asymptotic Analysis

This section theoretically compares the two types of exchange through the expected percentages of users that cannot trade. We focus on large systems, and consider the asymptotic regime where the number of files and users in the system becomes large.

We assume the files that users possess and desire are drawn independently from a Zipf file-popularity distribution that is identical for each user. Our motivation to study this distribution comes from the fact that Zipf's law has been observed in many settings, and has been suggested as a good model for file popularity (*e.g.*, [9, 1, 4]).[7] Zipf's law states that the popularity of the r-th largest occurrence is proportional to a power of its inverse rank. We adjust this definition to our setting.

Definition 4 *File popularity has a* Zipf distribution *with exponent s if the r-th most popular file has popularity $q_r = r^{-s}$.*

Note that $s = 0$ corresponds to the uniform distribution. On the other hand, the distribution becomes more concentrated as s increases.

Recall that we are interested in the expected percentage of users that cannot trade. This is a function of the number of users N, the number of files K, and the Zipf exponent s. Let $\overline{\rho}_{BE}(K, N, s)$ and $\overline{\rho}_{ME}(K, N, s)$ be the expected percentages of users that cannot trade bilaterally and multilaterally, respectively. In particular, $\overline{\rho}_{BE}(K, N, s)$ (resp., $\overline{\rho}_{ME}(K, N, s)$) is the expected value of $\rho_{BE}(\mathcal{P})$ (resp., $\rho_{ME}(\mathcal{P})$) over system profiles.

[7]Gummadi *et al.* find that peer-to-peer file popularity follows a flattened Zipf-link distribution [14]; however, the Zipf distribution is still the closest approximation for which analytical work is possible.

We consider a sequence of peer-to-peer systems indexed by N. The Nth system has N users and $K(N)$ files, where $K(N)$ is a non-decreasing function of N. The function $K(N)$ represents how the number of files scales with the number of users. For simplicity, we suppress the dependence of K on N. We study an asymptotic regime where $N \to \infty$.

Since the number of users that cannot trade bilaterally is always greater than or equal to the number of users that cannot trade multilaterally, we have $\overline{\rho}_{BE}(K, N, s) \geq \overline{\rho}_{ME}(K, N, s)$. The following propositions imply that in a large system $\overline{\rho}_{BE}(K, N, s) - \overline{\rho}_{ME}(K, N, s)$ may be significant when $s < 1$, but is always negligible when $s > 1$.

Proposition 5 *If $s > 1$, then $\overline{\rho}_{BE}(K, N, s) \to 0$ as $N \to \infty$ for any non-decreasing K.*

Since $\overline{\rho}_{ME}(K, N, s) \leq \overline{\rho}_{BE}(K, N, s)$, we conclude that if files are chosen according to a Zipf distribution with $s > 1$ then both $\overline{\rho}_{BE}(K, N, s) \to 0$ and $\overline{\rho}_{ME}(K, N, s) \to 0$ as $N \to \infty$. We note that this result holds regardless of the number of files that peers possess. When $s > 1$, bilateral exchange performs very well asymptotically even if each user only possesses one file for any scaling of K and N.

We note that the result of Proposition 5 can be generalized to all popularity distributions for which $\sum_{i=1}^{\infty} q_i < \infty$. The Zipf distribution with $s > 1$ is of course a special case. In particular, it is easy to adapt the proof of Proposition 5 to show that if the file popularities q_i satisfy $\sum_{i=1}^{\infty} q_i < \infty$,[8] then the expected proportion of peers that cannot trade bilaterally approaches zero as $N \to \infty$ for any scaling of the number of peers N and the number of files K.

This is an interesting result: even though bilateral exchange significantly restricts trade compared to multilateral exchange, almost all users can trade in expectation under both types of exchange when the system is large and file popularity follows a Zipf distribution with exponent $s > 1$. The intuition behind this result is that when s is large, the popularity distribution is more concentrated, *i.e.*, the most popular files are chosen with relatively high probability. As a result, for any user i, both T_i and S_i probably consist of one of the most popular files, and it is more likely that there exists a user j such that i and j have reciprocally desired files.

When $s < 1$, the asymptotic behavior can be quite different, as the following proposition shows.

Proposition 6 *Assume $0 \leq s < 1$, and $|S_i| = |T_i| = 1$ for all $i \in U$. As $N \to \infty$:*

(i) *If $K/\sqrt{N} \to \infty$, then $\overline{\rho}_{BE}(K, N, s) \to 1$.*
(ii) *If $K/\sqrt{N} \to 0$, then $\overline{\rho}_{BE}(K, N, s) \to 0$.*
(iii) *If $K \log K / N \to 0$, then $\overline{\rho}_{ME}(K, N, s) \to 0$.*

When $0 \leq s < 1$ and K scales slower than $\sqrt{N}$, both bilateral and multilateral exchange perform well. By contrast, the case where K scales faster than $\sqrt{N}$ but $K \log K$ scales slower than N is of particular interest. In this case, according to Proposition 6, $\overline{\rho}_{BE}(K, N, s) \to 1$ and $\overline{\rho}_{ME}(K, N, s) \to 0$ as $N \to \infty$. That is, when the system is large almost all users *can* trade multilaterally but *cannot* trade bilaterally. Thus, for

[8]We are assuming that the file popularities do not depend on the number of files in the system (K).

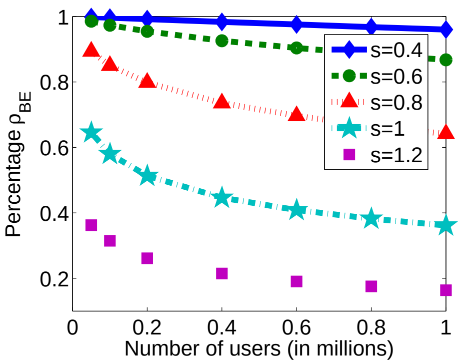

Fig. 2. Percentages of users (from simulations) that cannot trade bilaterally (ρ_{BE}), when file popularities follow a Zipf distribution with exponent $s \in \{0.4, 0.6, 0.8, 1, 1.2\}$. Users desire and possess one file, *i.e.*, $|T_i| = |S_i| = 1$ for all users i. There are 7,323 files in the system and the number of users is shown on the horizontal axis.

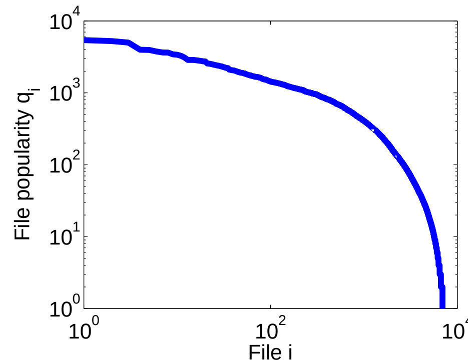

Fig. 3. The popularity of each file (q_i for all files i)—that is, the number of times file i was downloaded—shown in decreasing popularity on a log-log scale.

this case, multilateral exchange performs significantly better than bilateral exchange in terms of the number of users that can trade.

Figure 2 shows the percentages of users (from simulations) that cannot trade bilaterally when popularities follow a Zipf distribution for various values of the exponent s, assuming that users desire and possess one file. (We assume the system consists of 7,323 files, as this is the number of files in the dataset considered in the next section.) We observe that bilateral exchange does not perform well when the exponent s is small, which agrees with our theoretical results. As the exponent increases, the performance of bilateral exchange improves. When the exponent is greater than one, bilateral exchange performs reasonably well.

In the Appendix, we show that if $s = 0$, then the same conclusions hold even if peers possess multiple files (see Proposition 7). In fact, the proof of Proposition 7 shows that when each peer possesses σ files,

$$\overline{\rho}_{BE}(K, N, 0) = \left(1 - \frac{\sigma^2}{K(K-1)}\right)^{N-1}$$

for any K and N. This is an intriguing result, because it implies that the performance of bilateral equilibrium improves *quadratically* in the number of files that individuals possess. This suggests that small increases in the number of files that agents are willing to trade can lead to significant improvements in system performance. Indeed, we observe precisely this phenomenon in the next section's analysis.

C. Data Analysis

This section quantitatively compares bilateral and multilateral exchange using data on BitTorrent peer-to-peer file-sharing collected by Piatek *et al.* [29]. We find that a significant percentage of users cannot trade bilaterally when each user is sharing one file; however, the percentage becomes negligible as peers share more files. We conclude by discussing this finding's implications for the design of peer-to-peer content exchange systems.

The dataset consists of 1,364,734 downloads, 679,523 users and 7,323 files. We use the number of downloads of each file in the dataset to estimate the popularities of different files. We thus abstract from the details of the specific BitTorrent trace, and only use the information on the preferences of the users for different files in order to compare bilateral and multilateral exchange through simulations.

The estimated popularities are shown in Figure 3. As before, we assume that the probability that a given file is selected is proportional to its popularity. We then use these probabilities to generate system profiles and compute the percentages of users that cannot trade bilaterally and multilaterally. We assume that there are 7,323 files with the given distribution, and vary the number of users in the system.

The algorithm we use to compute ρ_{BE} is exact: for every user i we check whether there is some user j such that i and j have reciprocally desired files. Computing the exact value of ρ_{ME} for a large system appears computationally intractable. Therefore, we use an approximation: we recursively remove peers that either possess files not desired by others or desire files not possessed by others, since such peers cannot trade multilaterally. Simulations for small numbers of users suggest that this algorithm provides a very good approximation for ρ_{ME}.[9]

Bilateral and multilateral trade. We first assume that each user possesses and desires exactly one file, *i.e.*, $|T_i| = |S_i| = 1$ for every $i \in U$. Figure 4 shows the percentages of users that cannot trade bilaterally and multilaterally from simulations for various numbers of users in the system. We observe that a significant majority of users cannot trade bilaterally, while nearly all users can trade multilaterally. Finally, as the number of users increases, the percentages of users that can trade increase for both bilateral and multilateral exchange.

Trading trilaterally. Figure 4 also shows the percentage ρ_{TE} of peers that cannot trade in *triangles*, *i.e.*, triples (i, j, k), where i uploads to j, j uploads to k, and k uploads to i.[10] We observe that a very large percentage of peers is able to trade in triangles when there any at least 600,000 peers in the

[9] For instance, suppose there are 1,000 peers and 200 files in the system whose popularities are equal to the popularities of the 200 most popular files of the dataset. In 100 simulations, 972 peers can trade multilaterally on average, while our heuristic finds that 976 peers can trade multilaterally on average (99.6% accuracy).

[10] We estimate ρ_{TE} by sampling at least a few thousand peers and using an exact algorithm to compute the proportion of the sampled peers that cannot trade in triangles.

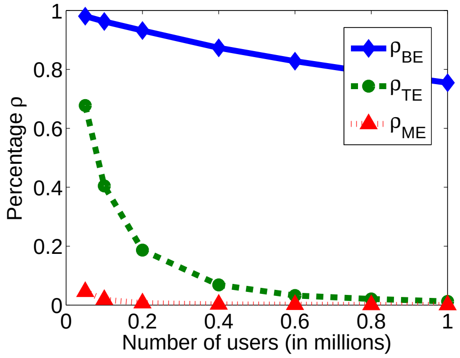

Fig. 4. Percentages of users (from simulations) that cannot trade bilaterally, trilaterally, and multilaterally when users desire and possess one file, *i.e.*, $|T_i| = |S_i| = 1$ for all i. The horizontal axis shows the number of users in the system.

system.[11]

Uploading multiple files. We next assume each user desires one file and possesses multiple files. As the number of files that each user has increases, the number of possible trades increases, and as a result the percentage of users that can trade bilaterally increases. In Figure 5 (top), we show the percentages of users that cannot trade bilaterally when each user desires one file ($|T_i| = 1$) and possesses multiple files ($|S_i| = \{2, 5, 10, 20\}$). In these experiments, all users possess the same number of files, *i.e.*, $|S_i| = |S_j|$ for all $i, j \in U$. (Note that the case of $|S_i| = 1$ is already shown in Figure 4.)

From these simulations, we observe a significant decrease in the percentage of users that cannot trade when $|S_i|$ increases from 1 to 20. We can illustrate this by considering the minimum required number of users in the system so that at most 10% are not able to trade: 1,000,000 users are required when each user has 5 files, but only 50,000 users are needed when each user possesses 20 files.

Distribution of $|S_i|$ across users. Our simulations up to now have assumed that all users in the system possess the same number of files, *i.e.*, $|S_i| = |S_j|$ for all i, j. We next assume that the number of files that users possess vary across different users, inferring this distribution for $|S_i|$ from the dataset.[12] We are interested in whether the percentage of users that can trade bilaterally increases as the variance of the distribution of $|S_i|$ increases (assuming that the mean remains the same). At first it may seem plausible that users with very large $|S_i|$ would be able to accommodate a lot of trades and as a result ρ_{BE} should increase as the $|S_i|$'s become more dispersed. However, this is not the case, as we discuss next.

The dataset shows that most users, in fact, possess only a few files. Only 32% of users possess more than a single file, and only 2% of users possess more than 10 files. There are, however, a few users that have more than 400 files. Since the mean value of $|S_i|$ in the dataset is 2.0084, we are interested in whether ρ_{BE} increases compared to the case that $|S_i| = 2$ for all i. This comparison is shown in Figure 5 (bottom).

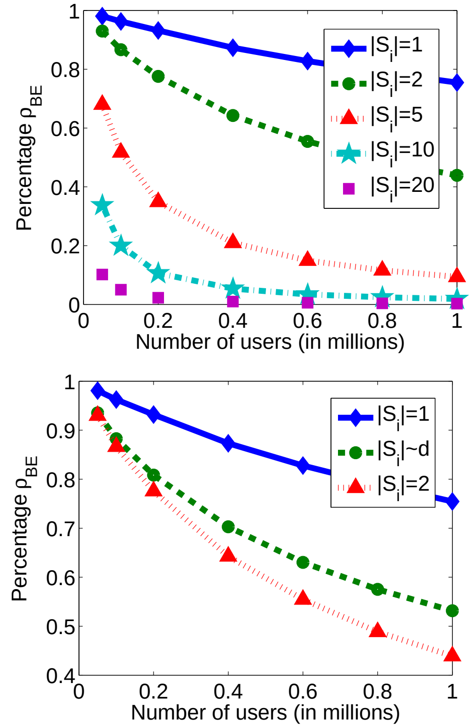

Fig. 5. Percentages of users (from simulations) that cannot trade bilaterally when each user desires one file ($|T_i| = 1$) and possesses multiple files. The legend shows $|S_i|$ for each line. The horizontal axis shows the number of users in the system. In the top graph, all users possess the same number of files ($|S_i| \in \{1, 2, 5, 10, 20\}$). In the bottom graph, we consider the case where different users possess different numbers of files, where this number is drawn from the dataset distribution d (denoted by $|S_i| \sim d$).

We observe that when $|S_i|$ is drawn from the dataset distribution d, the percentages of users that cannot trade bilaterally are between the cases of $|S_i| = 1$ and $|S_i| = 2$, even though the expected value of $|S_i|$ d is slightly greater than 2. This occurs because, even though some peers have a large number of files and thus are more likely to be able to trade bilaterally when the distribution of $|S_i|$ is more dispersed, the percentage of peers that only have one file also increases. Moreover, since each peer desires one file, the probability that a peer that has one file is matched with a peer with multiple files does not significantly increase.

VI. Related Work

In this paper, we have provided a formal comparison of peer-to-peer system designs, and have studied the advantages and disadvantages of bilateral and multilateral exchange. Menasché *et al.* investigate direct and indirect reciprocity in peer-to-peer systems [23], which correspond to bilateral and multilateral exchange in our model. They upper bound the efficiency loss of direct reciprocity (in the absence of relays) assuming that users are willing to download files they do not desire for bartering purposes. On the other hand, we compare bilateral and multilateral exchange through equilibrium outcomes and through the expected percentage of peers that can trade, and we do not assume that users download files they do not desire.

[11] Analytically, it can be shown that if we allow trade in triangles, then performance under uniform file popularity (*i.e.*, $s = 0$) is good as long as $N^2/(K^3 \log K) \to \infty$ as $N \to \infty$ (see Prop. 8 in the Appendix). This is a significant improvement on the corresponding result for bilateral equilibrium.

[12] We assume that the number of files that a user possesses is equal to the number of files he downloads in the dataset; note that this may be a bit optimistic, as it ignores the possibility of deletions over time from a user's set of shared files.

The "gap" between bilateral and multilateral exchange in terms of both efficiency and complexity has motivated the study of incentive mechanisms that lie between the two types of exchange in terms of both metrics. Through trace-driven analysis and measurements of a deployment on PlanetLab, Piatek *et al.* find that allowing trades to pass through one intermediary improves performance and incentives relative to BitTorrent [29]. Liu et al. study a similar mechanism assuming that peers belong to an underlying social network [21]. Finally, the performance implications of bundling have been considered [24].

Our work is also related to the study of equilibria in economies where not all trades are allowed. Kakade et al. introduce a graph-theoretic generalization of classical Arrow-Debreu economics, in which an undirected graph specifies which consumers or economies are permitted to engage in direct trade [20]; however, the inefficiencies of bilateral exchange do not arise in their model. The monetary economics literature has long studied how money reduces the double coincidence problem. The implementation of a competitive equilibrium is a central theme in this literature. The superiority of monetary exchange has been studied [32], and dynamics of bilateral trading processes have been considered [25, 11]. The transactions role of money is surveyed in [26].

Finally, as discussed in the introduction, we note that a number of studies consider incentives in peer-to-peer systems (*e.g.*, [13, 10, 33, 8, 3, 12, 29]). Our work contributes to this broad line of literature.

VII. Conclusion

This paper provides a formal comparison of two peer-to-peer system designs: bilateral barter systems such as BitTorrent, and a market-based exchange of content enabled by a price mechanism to match supply and demand. Our results demonstrate that even though bilateral equilibria are not Pareto efficient in general, bilateral exchange may perform very well in terms of the expected percentage of users that can trade for certain file probability distributions. Moreover, our data analysis shows a significant increase in the percentage of users that can trade bilaterally when each user shares multiple files. On one hand, this insight validates the performance gains of splitting files into chunks, as is currently done in BitTorrent. The potential improvement in the expected percentage of users that can trade should narrow the efficiency gap between BitTorrent and currency-based approaches. More generally, our insight suggests that bilateral incentives in BitTorrent could be made even stronger if the protocol considered exchanges across *different* files, rather than restricting exchange in a single swarm to chunks of the same file.

We conclude by noting that our work has considered a static "snapshot" view of a file-sharing system. Thus, a significant open issue concerns a *dynamic* comparison of the relative efficiency benefits of bilateral barter and multilateral, money-based exchange. In a dynamic system, there may be other considerations that limit the effectiveness of increasing the number of files that users choose to upload. For example, consider a natural setting in which the number of pieces of content in the system is quite large, but each user is interested only in a small fraction of these at any given time. Then, the number of *simultaneous* matches possible may be quite small, even if all content possessed by the users is available for upload. In such systems, money plays another important role: it can act as a *store of value* (*e.g.*, see [19]) over time, allowing a user to upload *now* and earn the ability to download *later*. In this sense, multilateral, money-based exchange may possess an important efficiency advantage in a dynamic view of the system. Quantifying this advantage (in the sense of Section V) remains an important direction for future work.

Acknowledgments

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APPENDIX

Note: Throughout the Appendix we write $\boldsymbol{x} \gg 0$ if all components of $\boldsymbol{x}$ are positive.

Proof of Proposition 1: We first define the concept of restricted BE and show that such an equilibrium always exists. We then use the exchange ratios of the restricted BE to construct a BE according to Definition 1.

The rate allocation $\boldsymbol{r}^*$ and the exchange ratios $\boldsymbol{\gamma}^*$ constitute a *restricted BE* if

1) $\gamma^*_{ij} = 0$ if $S_i \cap T_j = \emptyset$ or $S_j \cap T_i = \emptyset$; and $\gamma^*_{ij} \cdot \gamma^*_{ji} = 1$ otherwise.
2) For each user i, $\boldsymbol{r}^*$ solves the Bilateral Peer Optimization problem given exchange ratios $\boldsymbol{\gamma}^*$.

Thus at a restricted BE all exchange ratios between peers that cannot trade bilaterally are set to zero.

We show that a restricted BE exists under Assumption 1. Let $E = \{(i,j) : T_i \cap S_j \neq \emptyset, T_j \cap S_i \neq \emptyset\}$ be the set of tuples of users with reciprocally desired files. In a restricted BE, $\gamma_{ij} > 0$ if and only if $(i,j) \in E$. We consider an equivalent formulation with a price p_{ij} for every tuple $(i,j) \in E$, representing the price that user j pays to download from user i; see Section II for details. The exchange ratio between i and j is $\gamma_{ij} = p_{ij}/p_{ji}$. In particular, without loss of generality we assume that the budget constraint in the Bilateral Peer Optimization of user i is replaced by

$$p_{ji} \sum_f r_{jif} = p_{ij} \sum_f r_{ijf}.$$

For this proof, let $\boldsymbol{p} = (p_{ij}, (i,j) \in E)$. We ignore pairs of users that are not in E (since by definition such users cannot trade bilaterally), and show that it is possible to have some $\boldsymbol{p} \gg 0$ such that the market clears.

For the purposes of this proof, let $\boldsymbol{r}^i(\boldsymbol{p})$ be the optimal solution for the Bilateral Peer Optimization problem of user i when the exchange ratios are equal to $\gamma_{ij} = p_{ij}/p_{ji}$. If $\boldsymbol{r}$ and $\boldsymbol{p}$ constitute a BE, then $\boldsymbol{r} \in \boldsymbol{r}^i(\boldsymbol{p})$ for all $i \in U$. We note that each r^i_{ijf} is in general a correspondence. We define *excess demand* for each $(i,j) \in E$ as

$$z_{ij}(\boldsymbol{p}) = \sum_f r^j_{ijf}(\boldsymbol{p}) - \sum_f r^i_{ijf}(\boldsymbol{p}).$$

Our proof follows a similar approach to the standard proof of existence of competitive equilibrium [22]. We first show that the excess demand $\boldsymbol{z}$ has the following properties:

1) For every $\boldsymbol{p}$ and $\boldsymbol{z} \in \boldsymbol{z}(\boldsymbol{p})$, $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$.

2) $\boldsymbol{z}(\cdot)$ is convex-valued.
3) $\boldsymbol{z}(\cdot)$ is homogeneous of degree 0.
4) $\boldsymbol{z}(\cdot)$ is upper-hemicontinuous.
5) There is $s > 0$ such that $z_{ij} > -s$ for any $\boldsymbol{z} \in \boldsymbol{z}(\boldsymbol{p})$ and $\boldsymbol{p}$.
6) If $\boldsymbol{p}^n \to \boldsymbol{p} \neq 0$, $\boldsymbol{z}^n \in \boldsymbol{z}(\boldsymbol{p}^n)$, and for some $(i,j) \in E$, $p_{ij} > 0$ while $p_{ji} = 0$, then
$$\max\{z_{ij}^n : (i,j) \in E\} \to \infty.$$

By Assumption 1, the budget constraint of each user binds. The budget constraint of user i is
$$p_{ji} \sum_f r_{jif}^i(\boldsymbol{p}) = p_{ij} \sum_f r_{ijf}^i(\boldsymbol{p}).$$
By summing over all users, we obtain Property 1.

Fix a price vector $\boldsymbol{p} \gg 0$. By Assumption 1, $v(\cdot)$ is strictly concave; therefore $r_{ijf}^i(\boldsymbol{p})$ and $r_{ijf}^j(\boldsymbol{p})$ are convex-valued. Thus the aggregate excess demand $\boldsymbol{z}(\cdot)$ is a convex-valued correspondence (Property 2).

Consider a price vector $\boldsymbol{p} \gg 0$, and fix a constant $t > 0$. It is clear that the feasible region of the Bilateral Peer Optimization problem remains unchanged if we replace the price vector $\boldsymbol{p}$ by $t\boldsymbol{p}$. Thus the aggregate excess demand is homogeneous of degree zero (Property 3).

By Assumption 1, $v(\cdot)$ is a continuous function. From the Theorem of the Maximum [7] it follows that $r_{ijf}^i(\boldsymbol{p})$ and $r_{jif}^i(\boldsymbol{p})$ are upper hemicontinuous correspondences. The aggregate excess demand for $(i,j) \in E$ is a linear combination of the rates $r_{ijf}^j(\boldsymbol{p})$ and $r_{ijf}^i(\boldsymbol{p})$, and therefore is also upper hemicontinuous (Property 4).

The upload rate of any user i is upper bounded by his upload rate constraint B_i, so the total supply is upper bounded and the excess demand is bounded from below (Property 5).

Suppose that $\boldsymbol{p}^n \to \boldsymbol{p} \neq 0$, and $p_{ij} > 0$ while $p_{ji} = 0$ for some $(i,j) \in E$. Let $f \in T_i \cap S_j$. As $\boldsymbol{p}^n \to \boldsymbol{p}$ the amount of f that user i can afford approaches infinity. On the other hand, the total possible supply is bounded above by the upload rate constraint of user j. Thus $\max\{z_{ij}^n : (i,j) \in E\} \to \infty$, establishing Property 6.

Using Properties 1-6 we show that there exists a restricted BE. Let
$$\Delta = \{\boldsymbol{p} \in R_+^{|E|} : p_{ij} + p_{ji} = 1, (i,j) \in E\}$$
$$\Delta^n = \{\boldsymbol{p} \in \Delta : p_{ij} \geq 1/n, (i,j) \in E\}$$
We observe that Δ^n is compact. Then (from Property 4) for each n, there exists $\boldsymbol{r}^n > 0$ such that $\boldsymbol{z}(\boldsymbol{p}) \subset [-\boldsymbol{r}^n, \boldsymbol{r}^n]^{|E|}$ if $\boldsymbol{p} \in \Delta^n$. For each n, define $\boldsymbol{f}^n : \Delta^n \times [-\boldsymbol{r}^n, \boldsymbol{r}^n]^{|E|} \to \Delta^n \times [-\boldsymbol{r}^n, \boldsymbol{r}^n]^{|E|}$ by
$$\boldsymbol{f}^n(\boldsymbol{p}, \boldsymbol{z}) = \{\boldsymbol{q} \in \Delta^n : \boldsymbol{z} \cdot \boldsymbol{q} \geq \boldsymbol{z} \cdot \boldsymbol{q}', \forall \boldsymbol{q}' \in \Delta^n\} \times \boldsymbol{z}(\boldsymbol{p}).$$
For each n, the correspondence $\boldsymbol{f}^n$ is convex-valued and upper-hemicontinuous. We can now apply Kakutani's fixed point theorem to conclude that for each n, $\boldsymbol{f}^n(\cdot)$ has a fixed point, which we denote by $(\boldsymbol{p}^n, \boldsymbol{z}^n)$.

The sequence $\boldsymbol{p}^n$ in Δ has a subsequence that converges, because Δ is compact. By Property 5 and the fact that $\boldsymbol{z}^n$ is bounded, the limit must be in the interior of Δ. Therefore, by taking a subsequence if necessary, we can assume that $\boldsymbol{p}^n \to \boldsymbol{p}^*$ and $\boldsymbol{z}^n \to \boldsymbol{z}^*$, where $\boldsymbol{p}^*$ is in the interior of Δ. Now observe that $\boldsymbol{z}^* = \boldsymbol{z}(\boldsymbol{p}^*)$, and (by Property 1) $0 = \boldsymbol{z}^* \cdot \boldsymbol{p}^* \geq \boldsymbol{z}^* \cdot \boldsymbol{p}$ for any $\boldsymbol{p} \in \Delta$. Now observe that $p_{ji}^* z_{ji}^* = -p_{ij}^* z_{ij}^*$, so we must in fact have $z_{ij}^* = 0$ for all i,j. We conclude that the limit $\boldsymbol{p}^*$ is a restricted BE price vector, with corresponding allocation $\boldsymbol{r}^* \in \boldsymbol{r}^i(\boldsymbol{p}^*)$ for all i.

We have shown that a restricted BE exists under Assumption 1. We now show how to construct a BE (according to Definition 1) when Assumption 2 holds. Suppose $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{\gamma}}$ constitute a restricted BE. Let $\boldsymbol{r}^* = \tilde{\boldsymbol{r}}$. For pairs of users i,j such that $S_i \cap T_j \neq \emptyset$ and $S_j \cap T_i \neq \emptyset$, set $\gamma_{ij}^* = \tilde{\gamma}_{ij}$. Having set exchange ratios for all pairs of users that can trade bilaterally, we now consider users that cannot trade bilaterally. If $S_i \cap T_j = \emptyset$ and $S_j \cap T_i = \emptyset$, set $\gamma_{ij}^* = \gamma_{ji}^* = 1$. If $S_i \cap T_j \neq \emptyset$ and $S_j \cap T_i = \emptyset$, set
$$\gamma_{ij}^* = \varepsilon + \max_{k : S_k \cap T_i \neq \emptyset, S_i \cap T_k \neq \emptyset} \{\gamma_{kj}^*\},$$
and $\gamma_{ji}^* = 1/\gamma_{ij}$. If Assumption 2 holds, then $\boldsymbol{r}^*$ solves the Bilateral Peer Optimization problem of every user with respect to exchange ratios $\boldsymbol{\gamma}^*$. In particular, user i can find every file in T_i through bilateral trade at the same exchange ratios as in the restricted BE. Exchange ratios with users that i cannot trade with bilaterally are set so that they do not affect i's optimization problem. ∎

Proof of Proposition 2: If Assumption 3 holds, then either the user graph is strongly connected, or the system can be decomposed to subsystems for which the user graphs are strongly connected. Therefore, without loss of generality, in this proof we assume that the user graph is strongly connected.

For the purposes of this proof, let $\boldsymbol{r}^i(\boldsymbol{p})$ be the optimal solution of the Multilateral Peer Optimization problem of user i when the price vector is $\boldsymbol{p} = (p_i, i \in U)$. If $\boldsymbol{r}$ and $\boldsymbol{p}$ constitute a ME, then $\boldsymbol{r} \in \boldsymbol{r}^i(\boldsymbol{p})$ for all $i \in U$.

We define excess demand for the upload rate of each user $i \in U$ as
$$z_i(\boldsymbol{p}) = \sum_{f,j} r_{ijf}^j(\boldsymbol{p}) - \sum_{f,j} r_{ijf}^i(\boldsymbol{p}).$$
We show that the aggregate excess demand correspondence $\boldsymbol{z}(\cdot)$ defined on $(0,\infty)^{|U|}$ satisfies the following properties:

1) For every $\boldsymbol{p} \gg 0$ and $\boldsymbol{z} \in \boldsymbol{z}(\boldsymbol{p})$, $\boldsymbol{p} \cdot \boldsymbol{z} = 0$.
2) $\boldsymbol{z}(\cdot)$ is convex-valued.
3) $\boldsymbol{z}(\cdot)$ is homogeneous of degree 0.
4) $\boldsymbol{z}(\cdot)$ is upper hemicontinuous.
5) There is an $s > 0$ such that $z_j > -s$ for any $\boldsymbol{z} \in \boldsymbol{z}(\boldsymbol{p})$, for every file $j \in F$ and every price vector $\boldsymbol{p} \gg 0$.
6) If $\boldsymbol{p}^m \to \boldsymbol{p} \neq \boldsymbol{0}$, $\boldsymbol{z}^m \in \boldsymbol{z}(\boldsymbol{p}^m)$ and $p_j = 0$ for some j, then $\max\{z_j^m : j \in F\} \to \infty$.

Then the existence of a ME follows from standard results in microeconomics; see, *e.g.*, [22], Exercise 17.C.2.

By Assumption 1, the budget constraint of each user binds. The budget constraint of user i is
$$\sum_{j,f} p_j r_{jif}^i(\boldsymbol{p}) = p_i \sum_{j,f} r_{ijf}^i(\boldsymbol{p}).$$
By summing over all users, we obtain Property 1.

Fix a price vector $\boldsymbol{p} \gg 0$. By Assumption 1, $v(\cdot)$ is strictly concave; therefore $r_{ijf}^i(\boldsymbol{p})$ and $r_{ijf}^j(\boldsymbol{p})$ are convex-valued. Thus the aggregate excess demand $\boldsymbol{z}(\cdot)$ is a convex-valued correspondence (Property 2).

Consider a price vector $\boldsymbol{p} \gg 0$, and fix a constant $t > 0$. It is clear that the feasible region of the multilateral user optimization problem remains unchanged if we replace the price vector $\boldsymbol{p}$ by $t\boldsymbol{p}$. Thus the aggregate excess demand is also homogeneous of degree zero (Property 3).

We now show that the aggregate excess demand correspondence is upper hemicontinuous. By Assumption 1, $v(\cdot)$ is a continuous function. From the Theorem of the Maximum [7] it follows that $r^i_{ijf}(\boldsymbol{p})$ and $r^i_{jif}(\boldsymbol{p})$ are upper hemicontinuous correspondences. The aggregate excess demand for user i is a linear combination of the rates $r^j_{ijf}(\boldsymbol{p})$ and $r^i_{ijf}(\boldsymbol{p})$, and therefore is also upper hemicontinuous (Property 4).

The upload rate of any user i is upper bounded by his upload rate constraint B_i, so the total supply is upper bounded and the excess demand is bounded from below (Property 5).

If $\boldsymbol{p}^m \to \boldsymbol{p} \neq 0$ and $p_j = 0$, then $p_k > 0$ for some k. Because of Assumption 3, there is a sequence of users $1, 2, ..., n \in U$ such that $T_i \cap S_{i+1} \neq \emptyset$. Thus, there is a user i such that p_i approaches a strictly positive limit and p_{i+1} approaches zero. Let $f \in T_i \cap S_{i+1}$. The budget of user i approaches a strictly positive limit as $\boldsymbol{p}^m \to \boldsymbol{p}$ and the amount of f he can afford goes to infinity. On the other hand, the total possible supply is bounded above by the upload rate constraints of user $i+1$. Thus $\max\{z^m_j : j \in F\} \to \infty$, establishing Property 6. ■

Proof of Proposition 3: Suppose that $\boldsymbol{r} \in \mathcal{X}$ is a Pareto improvement. Then some user i strictly prefers $\boldsymbol{r}$ to $\boldsymbol{r}^*$. Since $\boldsymbol{r}$ is not an optimal solution for user i under $\boldsymbol{p}$, it must be that

$$\sum_{j,f} p_j r_{jif} > p_i \sum_{j,f} r_{ijf}.$$

All users $k \neq i$ are at least as well off under $\boldsymbol{r}$ as under $\boldsymbol{r}^*$. This implies that

$$\sum_{j,f} p_j r_{jkf} \geq p_k \sum_{j,f} r_{kjf},$$

because the utilities are increasing in the total rates of files that users are interested in. In particular, consider a user k who gets exactly the same utility under $\boldsymbol{r}$ and $\boldsymbol{r}^*$: if $\sum_{j,f} p_k r_{jkf} < p_k \sum_{j,f} r_{kjf}$, then there is a rate allocation that satisfies k's budget constraint and k strictly prefers to $\boldsymbol{r}$, which implies that $\boldsymbol{r}^*$ is not optimal.

Summing over all users,

$$\sum_k \sum_{j,f} p_j r_{jkf} > \sum_k p_k \sum_{j,f} r_{kjf},$$

which is a contradiction. We conclude that a ME allocation is Pareto efficient. ■

Proof of Proposition 4: Define $r^*_{ij} \equiv \sum_f r^*_{ijf}$, the total rate that user i sends to user j. We define the matrix $\boldsymbol{Q}$ such that $Q_{ij} = r^*_{ij}$ if $i \neq j$; and $Q_{ii} = -\sum_j r^*_{ij}$. By construction, $\boldsymbol{Q}$ is a transition rate matrix of a continuous time Markov chain with no transient subclasses, since $r^*_{ij} > 0$ implies that $r^*_{ji} > 0$ (by the definition of BE). In what follows we consider the communicating classes of $\boldsymbol{Q}$: if $r^*_{ij} > 0$, then users i and j are in the same communicating class. For the purposes of this proof, let $\mathcal{N}_i(\boldsymbol{r}^*)$ be the set of peers with which i trades under $\boldsymbol{r}^*$, *i.e.*, $\mathcal{N}_i(\boldsymbol{r}^*) = \{j \in U : r^*_{ji} > 0\}$. Note that $\mathcal{N}_i(\boldsymbol{r}^*)$ is a subset of the communicating class containing i.

Let $\boldsymbol{\pi}$ be an invariant distribution of $\boldsymbol{Q}$, *i.e.*, $\boldsymbol{\pi Q} = 0$. We observe that $\boldsymbol{\pi Q} = \boldsymbol{0}$ implies that the budget constraint in the Multilateral Peer Optimization problem is satisfied with prices $\boldsymbol{\pi}$; therefore for any invariant distribution $\boldsymbol{\pi}$, $\boldsymbol{r}^*$ is feasible for the Multilateral Peer Optimization problem of every user when prices are equal to $\boldsymbol{\pi}$. We show that for some invariant distribution $\boldsymbol{p}$ of $\boldsymbol{Q}$, $\boldsymbol{r}^*$ and $\boldsymbol{p}$ constitute a ME. In particular, we show that for each user i, $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem under $\boldsymbol{p}$.

This is done in three steps. First, we show that if $\boldsymbol{r}^*$ is Pareto efficient, then $\boldsymbol{Q}$ corresponds to a reversible Markov chain. This implies that if $\boldsymbol{\pi}$ is an invariant distribution of $\boldsymbol{Q}$ with all components strictly positive, then $\gamma^*_{ij} = \pi_i/\pi_j$ whenever $r^*_{ij} > 0$, and as a result $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem of user i given prices $\boldsymbol{\pi}$ *if user i is restricted to trade with users in $\mathcal{N}_i(\boldsymbol{r}^*)$* (Step 1). We then show that if user i is restricted to trade with users in the same communicating class under prices $\boldsymbol{\pi}$, then $\boldsymbol{r}^*$ is an optimal solution of the Multilateral Peer Optimization problem (Step 2). Step 2 completes the proof if $\boldsymbol{Q}$ consists of one communicating class. Finally, we show that if there are multiple communicating classes, there exists an invariant distribution $\boldsymbol{p}$ (derived as a convex combination of the invariant distributions corresponding to the communicating classes) such that $\boldsymbol{r}^*$ is an optimal solution of the Multilateral Peer Optimization problem of each user (Step 3). We show each of these steps by demonstrating that if the desired conclusion of the step does not hold, then there exists a rate vector $\boldsymbol{r}$ that Pareto improves $\boldsymbol{r}^*$—violating the assumption that $\boldsymbol{r}^*$ is Pareto efficient.

Before beginning the proof, we derive a simple condition that allows us to test whether a new rate vector improves user i's utility. Suppose $\boldsymbol{r}^*$ solves the Bilateral Peer Optimization problem of user i under $\boldsymbol{\gamma}^*$. Let $(x^*_{if}, f \in T_i)$ and y^*_i be the corresponding download and upload rates for user i. Consider a rate allocation $\boldsymbol{r}$ where $(x_{if}, f \in T_i)$ and y_i are the corresponding download and upload rates for user i. Fix a file $f \in T_i$, and assume that $x_{ig} = x^*_{ig}$ for all files $g \neq f$. If $x_{if} - x^*_{if}$ and $y_i - y^*_i$ are sufficiently small, we can use Taylor's approximation to conclude that user i is strictly better off under $\boldsymbol{r}$ if

$$(x_{if} - x^*_{if}) \frac{\partial v_i(\boldsymbol{x}^*_i, y^*_i)}{\partial x_{if}} > (y_i - y^*_i) \frac{\partial v_i(\boldsymbol{x}^*_i, y^*_i)}{\partial y_i}. \quad (1)$$

Suppose that $r^*_{jif} > 0$ for some j, which implies that $x^*_{if} > 0$ and $y^*_i > 0$. The optimality conditions for the Bilateral Peer Optimization problem of user i then yield

$$\frac{\partial v_i(x^*_{ig}, g \in T_i)}{\partial x_{if}} = \frac{1}{\gamma^*_{ij}} \frac{\partial v_i(\boldsymbol{x}^*_i, y^*_i)}{\partial y_i}.$$

(Here we use the fact that $v_i(\boldsymbol{x}_i, y_i) \to \infty$ as $y_i \to B_i$ to ignore the constraint $y_i \leq B_i$.) Combining this with (1), we see that user i strictly prefers $\boldsymbol{r}$ to $\boldsymbol{r}^*$ if

$$\frac{x_{if} - x^*_{if}}{y_i - y^*_i} > \gamma^*_{ij}, \quad (2)$$

assuming that $x_{if} - x^*_{if}$ and $y_i - y^*_i$ are sufficiently small. Thus if $r^*_{jif} > 0$, and we increase the download rate of file f to user i as well as the total upload rate of user i such that the previous condition holds, then user i is strictly better off.

Step 1. *If $\boldsymbol{r}^*$ is Pareto efficient, then $\boldsymbol{Q}$ is reversible. Further,* *if $\boldsymbol{\pi} \gg 0$ is an invariant distribution of $\boldsymbol{Q}$, then $\boldsymbol{r}^*$ solves the Multilateral Optimization Problem of user i given prices $\boldsymbol{\pi}$ if user i is restricted to trade only with the users in $\mathcal{N}_i(\boldsymbol{r}^*)$.* Let $\boldsymbol{\pi}$ be a strictly positive invariant distribution of $\boldsymbol{Q}$, *i.e.*, $\boldsymbol{\pi} \gg 0$ and $\boldsymbol{\pi} \cdot \boldsymbol{Q} = \mathbf{0}$. If $\boldsymbol{Q}$ is reversible, then the detailed balance equations hold for every $i, j \in U$, *i.e.*, $\pi_i r^*_{ij} = \pi_j r^*_{ji}$. We note that the detailed balance equations trivially hold if $r^*_{ij} = 0$, because then also $r^*_{ji} = 0$. On the other hand, the budget constraint of the Bilateral Peer Optimization problem of user i implies that $\gamma^*_{ki} r^*_{ki} = r^*_{ik}$. We conclude that $\boldsymbol{Q}$ is reversible if and only if $\gamma^*_{ij} = \pi_i/\pi_j$ whenever $r^*_{ij} > 0$.

We show that Pareto efficiency of $\boldsymbol{r}^*$ implies reversibility of $\boldsymbol{Q}$. Assume that $\boldsymbol{Q}$ is not reversible. Then $\pi_j/\pi_i > \gamma^*_{ji}$ for some i, j with $r^*_{ij} > 0$. Since π is an invariant distribution, $\boldsymbol{\pi Q} = \mathbf{0}$, and thus $\sum_k (\pi_k/\pi_i) r^*_{ki} = \sum_k r^*_{ik}$. On the other hand, the budget constraint of the Bilateral Peer Optimization problem of user i implies that $\gamma^*_{ki} r^*_{ki} = r^*_{ik}$. Summing over k and substituting, we conclude that

$$\sum_k \gamma^*_{ki} r^*_{ki} = \sum_k \frac{p_k}{p_i} r^*_{ki}.$$

If $\pi_j/\pi_i > \gamma^*_{ji}$ for some i, j with $r^*_{ij} > 0$, the previous equation implies that there exists some user k such that $\pi_{j+1}/\pi_k > \gamma^*_{j+1,k}$ and $r^*_{j+1,k} > 0$. Without loss of generality, we relabel i to be $j+1$, and k to be $j+2$. Then $\pi_{j+1}/\pi_{j+2} > \gamma^*_{j+1,j+2}$. Applying this reasoning recursively, we can find a sequence of users $1, 2, ..., K, K+1$ such that $1 \equiv K+1$ and $\pi_k/\pi_{k+1} > \gamma^*_{k,k+1}$ for all k.

We show how the utility of each user in $D = \{1, 2, ..., K\}$ can increase while the rate allocation to users outside D remains the same. In particular, we increase $r^*_{k,k-1}$ and y^*_k by a_k for all $k \in D$, as illustrated in the first part of Figure 6 (for $K = 3$). We note that users' upload capacity constraints do not bind at the BE, a consequence of the assumption that $v_i(\boldsymbol{x}_i, y_i) \to \infty$ as $y_i \to B_i$. Therefore, it is feasible to slightly increase the upload rates of all users. Applying (2), user k is better off if

$$\frac{a_{k+1}}{a_k} > \gamma^*_{k,k+1}.$$

Since $\pi_k/\pi_{k+1} > \gamma^*_{k,k+1}$, it follows that $\prod_k \gamma^*_{k,k+1} < 1$. Then, it is possible to make all users in the set D better off by, *e.g.*, choosing δ and ε small enough, and setting $a_1 = \delta$; $a_{k+1} = \gamma_{k,k+1} \cdot a_k + \varepsilon$, for all $k \in S$.

We conclude that if $\boldsymbol{r}^*$ is the rate allocation of a BE and is Pareto efficient, then $\boldsymbol{Q}$ is reversible. Further, if $\boldsymbol{\pi} \gg 0$ is an invariant distribution of $\boldsymbol{Q}$, then $\gamma^*_{ij} = \pi_i/\pi_j$ whenever $r^*_{ij} > 0$. This means that $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem of user i given prices $\boldsymbol{\pi}$ if he is restricted to trade with peers in $\mathcal{N}_i(\boldsymbol{r}^*)$. The remainder of the proof shows that there exists an invariant distribution $\boldsymbol{p}$ such that $\boldsymbol{r}^*$ is optimal for the Multilateral Peer Optimization problem under $\boldsymbol{p}$.

Step 2. *Assume $\boldsymbol{r}^*$ is Pareto efficient, and let $\boldsymbol{\pi} \gg 0$ be an invariant distribution of $\boldsymbol{Q}$. Then $\boldsymbol{r}^*$ solves the Multilateral Optimization Problem of user i given prices $\boldsymbol{\pi}$ if user i is restricted to trade only with the users in the same communicating class.* Let $\boldsymbol{\pi} \gg 0$ be an invariant distribution of $\boldsymbol{Q}$, and consider the Multilateral Peer Optimization problems when prices are given by $\boldsymbol{\pi}$. Suppose that $\boldsymbol{r}^*$ is not optimal for the Multilateral Peer Optimization problem of some user i. Then by Step 1 there must exist a user j such that $r^*_{ji} = 0$ with which i wants to exchange under $\boldsymbol{\pi}$.

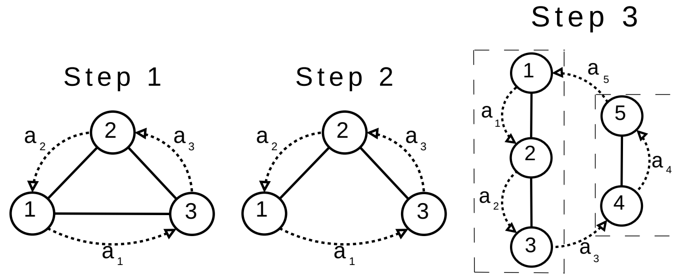

Fig. 6. Pareto improvements when the BE allocation does not correspond to a ME allocation for Steps 1, 2, and 3 of the proof of Proposition 4 respectively. A pair of users that trade at the BE is connected with a solid line. Dotted arrows show the rates that increase for the Pareto improvement: user i increases his upload rate and the rate he sends to user $i-1$ by a_i. In the third figure (Step 3) there are two communicating classes — each class is included in a dashed box.

In this step we consider the case that i and j are in the same communicating class. Then there exists a sequence of users between i and j such that each two consecutive users trade at the BE. Without loss of generality we relabel user i by K, user j by 1, and the users in the sequence by $2, 3, ..., K-1$. Then, $r^*_{j,j-1} > 0$ for $j = 2, 3, ..., K$. We show that there is a Pareto improvement, where the utilities of all users in the set $S = \{1, 2, ..., K\}$ strictly increase, while utilities of users outside S remain the same.

Let a_j be the amount by which we increase rate $r^*_{j,j-1}$ for $j = 2, ..., K$. We assume that all users in the set increase this rate by increasing their upload rates. In particular, user j increases his upload rate by a_j, and the download rate he receives from user $j+1$ increases by a_{j+1}. This is illustrated for $K = 3$ in the second part of Figure 6. Applying (2), user $j \neq K$ is better off if $a_{j+1}/a_j > \gamma^*_{j,j+1} \equiv \pi_j/\pi_{j+1}$ (the last part follows from the reversibility of $\boldsymbol{Q}$). To conclude this step, we show that user K is better off if $a_1/a_K > \pi_K/\pi_1$. Then, as in Step 1, it is possible to find a_i for $i \in S$, such that all users in S are better off.

Now consider user K. Let $f \in T_K$ be a file that user K wants to get from user 1 under prices $\boldsymbol{\pi}$; then clearly $f \in S_1$. There are two cases to consider, depending on whether user K downloads file f at the BE.

1) $r^*_{jKf} > 0$ for some j. Then, by (2), we conclude that user K is better off if $a_1/a_K > \gamma^*_{Kj}$. Moreover, since K prefers to get f from 1 under $\boldsymbol{\pi}$ it must be that $\pi_K/\pi_1 > \pi_K/\pi_j = \gamma^*_{Kj}$ (where the last equality follows by the reversibility of $\boldsymbol{Q}$). Thus, user K is strictly better off if $a_1/a_K > \pi_K/\pi_1$.
2) $\sum_j r^*_{jKf} = 0$, *i.e.*, K does not download file f at rate allocation $\boldsymbol{r}^*$. Under $\boldsymbol{\pi}$, user K is strictly better off downloading a positive amount of f from user 1. It is straightforward to check from the Multilateral Peer Optimization problem of user K that:

$$\frac{\partial v_K(x^*_{Kg}, g \in T_K)}{\partial x_{Kf}} > \frac{\pi_1}{\pi_K} \frac{\partial v_K(\boldsymbol{x}^*_K, y^*_K)}{\partial y_K}.$$

Combining this with (1) we conclude that user K is better off if $a_1/a_K > \pi_K/\pi_1$.

In either case, user K is better off if $a_1/a_K > \pi_K/\pi_1$. This concludes the proof of this step.

Step 3. *If $\boldsymbol{r}^*$ is Pareto efficient, there exists an invariant distribution $\boldsymbol{\pi} \gg 0$ such that $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem of every user i given prices $\boldsymbol{\pi}$.*

To complete the proof, we extend the result of Step 2 across communicating classes. Let $\boldsymbol{\pi}_c$ be the invariant distribution of the rate matrix $\boldsymbol{Q}$ restricted to communicating class c. We show that there exist coefficients $\rho_c > 0$ such that $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem of each user under $\boldsymbol{p} \equiv \sum_c \rho_c \boldsymbol{\pi}_c$.

We start by deriving conditions that the coefficients ρ_c must satisfy to ensure optimality. Consider two communicating classes c and c'. If $(\cup_{i\in c} T_i) \cap (\cup_{j \in c'} S_j) \neq \emptyset$, then some users from class c are interested in files that are possessed by users in class c'. To ensure that $\boldsymbol{r}^*$ is optimal for the Multilateral Peer Optimization problems of these users, the ratio $\rho_{c'}/\rho_c$ should be sufficiently large. Let $\xi_{c',c}$ be the smallest possible ratio $\rho_{c'}/\rho_c$ such that no users from class c would download files from class c' if the prices were $\rho_c \boldsymbol{\pi}_c + \rho'_c \boldsymbol{\pi}'_c$.

Suppose that there do not exist coefficients ρ_c such that $\boldsymbol{r}^*$ is an optimal solution of the Multilateral Peer Optimization problem of each peer. Then, there exists a directed cycle of classes such that (1) $(\cup_{i\in c} T_i) \cap (\cup_{j \in c'} S_j) \neq \emptyset$ for each two consecutive classes in the cycle, and (2) the product of $\xi_{c',c}$ along the cycle is strictly greater than 1. This implies the existence of a vector $\boldsymbol{\rho}$ such that $\rho_{c'}/\rho_c < \xi_{c',c}$ for every pair of consecutive classes along the directed cycle. In particular, when prices are $\boldsymbol{p} \equiv \sum_c \rho_c \boldsymbol{\pi}_c$, for each pair of consecutive classes along the cycle c and c', there is a user n_c in class c that wants to trade with user $m_{c'}$ from class c'. We construct a set S that includes users n_c, m_c as well as the users between them, *i.e.*, users $i_{c1}, ..., i_{cl}$ such that $n_c \equiv i_{c1}$, $m_c \equiv i_{cl}$ and $r^*_{i_{cj}, i_{c,j+1}} > 0$. We relabel users in S by $\{1, 2, ..., K\}$ such that if i and $i+1$ are in different communicating classes (say c and c') then $i = n_c$ and $i + 1 = m_{c'}$, *i.e.*, user i wants to trade with user $i + 1$.

We demonstrate a Pareto improvement where user $i \in S$ increases his upload rate and the rate he sends to user $i - 1$ by a_i. In Figure 6 we illustrate an example with two communicating classes. We demonstrate that it is possible to reallocate rates in a way that strictly increases the utilities of all users in S and does not change the utilities of users outside S. From (2) we see that a user $j \neq n_c$ can be made better off if $a_{j+1}/a_j > p_j/p_{j+1}$. A user $j \equiv n_c$ for some j can be made better off if $a_{j+1}/a_j > p_j/p_{j+1}$ (this can be shown by applying the same argument we used for user K in Step 2). As in Steps 1 and 2, since the product of all left hand sides is equal to 1 while the product of all right hand sides is strictly less than 1, it is possible to find a vector $\boldsymbol{a}$ that satisfies all these inequalities. This concludes the proof. ■

Lemma 1 *If $|T_i| = 1$ for all i, $\rho_{ME}(\mathcal{P}) = 0$, and Assumption 1 holds, then there exists a ME.*

Proof of Lemma 1: We consider the user graph that was defined in Section III-B, *i.e.*, the directed graph $G = (V, E)$ where V is the set of peers, and $E = \{(i, j) : S_i \cap T_j \neq \emptyset\}$. If G consists only of strongly connected components, then Assumption 3 holds and therefore a ME exists.

Now suppose that Assumption 3 does not hold, *i.e.*, there exist connected components of G that are not strongly connected. For the purpose of ME existence, we can consider each connected component separately, since peers from one connected component are not interested in files possessed by peers in another connected component.

Consider a connected component of G that is not strongly connected. Let C be the set of peers in this component. C can be partitioned into strongly connected subsets $C_1, C_2, ..., C_K$. We consider the partition where K is minimized, *i.e.*, each strongly connected component has the maximum possible size. This partition is uniquely defined. We consider the directed graph $G' = (V', E')$ with $V' = \{1, 2, ..., K\}$, and $E' = \{(k, k') : i \in C_k, j \in C_{k'}, S_i \cap T_j \neq \emptyset\}$. We observe that G' is a directed acyclic graph (cycles would imply that we have not chosen the minimal partition).

We have that $|C_k| \geq 2$ for $k \in \{1, ..., K\}$, because $\rho_{ME}(\mathcal{P}) = 0$ and the minimal partition is considered. Since $|T_i| = 1$ for all i, each user in C_k can find the file he desires in C_k. Then Assumption 1 implies that a ME exists for each C_k.

We next show that it is possible to scale prices of the ME in each C_k so that a ME exists for C. This is done in the following steps.

(1) Let $\mathcal{A}$ be the set of sinks of G', and let $\overline{p} = 1$.
(2) Find a ME for each C_k with $k \in \mathcal{A}$, and scale the prices of this ME so that the minimum price is greater than $\overline{p}$.
(3) Let $\overline{p}$ be the maximum price over all peers in $\cup_{k\in\mathcal{A}} C_k$. Remove all peers in $\mathcal{A}$ from G' and let $\mathcal{A}$ be the set of the new sinks.
(4) Repeat steps (2)-(3) until G' becomes empty.

■

Proof of Proposition 5: Let

$$\theta(K, s) \equiv \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} (ij)^{-s}. \tag{3}$$

The probability that $S_k = \{i\}$ and $T_k = \{j\}$ is equal to

$$\frac{(ij)^{-s}}{\theta(K, s)} \equiv p_{ij}.$$

Peer k cannot trade bilaterally if there exists no peer k' such that $S_{k'} = T_k$ and $T_{k'} = S_k$. The event $S_{k'} = T_k$ and $T_{k'} = S_k$ occurs with probability $(1 - p_{ji})^{N-1}$ (since there are $N - 1$ peers to choose from). Since (S_k, T_k) is chosen independently for each peer k, the expected percentage of users that cannot trade bilaterally when there are K files and N users is:

$$\overline{\rho}_{BE}(K, N, s) = \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} \frac{(ij)^{-s}}{\theta(K, s)} \left(1 - \frac{(ij)^{-s}}{\theta(K, s)}\right)^{N-1}. \tag{4}$$

We first assume that $K \not\to \infty$ as $N \to \infty$. Then $\overline{\rho}_{BE}(K, N, s)$ is the sum of a finite number of terms, each of which approaches 0 as $N \to \infty$. Thus, $\overline{\rho}_{BE}(K, N, s) \to 0$ as $N \to \infty$.

Now assume that $K \to \infty$ as $N \to \infty$ and let

$$\theta(s) \equiv \sum_{i\neq j: i,j \in \{1,2,...\}} (ij)^{-s}.$$

We observe that $\theta(K,s) \uparrow \theta(s)$ as $K \to \infty$. Since $s > 1$, $\theta(s)$ is finite. We have:

$$\overline{\rho}_{BE}(K,N,s) \le \frac{1}{\theta(K,s)} \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} (ij)^{-s} \left(1 - \frac{(ij)^{-s}}{\theta(s)}\right)^{N-1}.$$

Let

$$B_N \equiv \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} (ij)^{-s} \left(1 - \frac{(ij)^{-s}}{\theta(s)}\right)^{N-1}.$$

Since $\theta(s)$ is finite and $\theta(K,s) < \theta(s)$, it suffices to show that $B_N \to 0$ as $N \to \infty$. We observe that for any fixed $\overline{K}$,

$$B_N < \sum_{i\neq j: i\cdot j \le \overline{K}} \left(1 - \frac{(ij)^{-s}}{\theta(s)}\right)^{N-1} + \sum_{i\neq j: i\cdot j > \overline{K}} (ij)^{-s}.$$

The first term approaches zero as $N \to \infty$; the second term does not depend on N, and approaches zero as $\overline{K} \to \infty$. Thus, first taking the limit as $N \to \infty$, then taking the limit as $\overline{K} \to \infty$, the result follows. ■

The following result is used in the proof of Proposition 6.

Lemma 2 *If $y \in [0,1)$ and $N > 0$, then*

$$(1-y)^N \le \frac{1}{1 + N \cdot y}.$$

Proof of Lemma 2: Let

$$f(y) \equiv (1-y)^{-N} - (1 + Ny).$$

It suffices to show that $f(y) \ge 0$ for $y \in [0,1]$. We observe that $f(0) = 0$ and

$$f'(y) = N\left((1-y)^{-N-1} - 1\right) \ge 0,$$

for $y \in [0,1)$. This completes the proof. ■

Proof of Proposition 6: We follow the same notation as in the proof of Proposition 5; in particular, we define $\theta(K,s)$ as in (3) and conclude that $\overline{\rho}_{BE}$ is given by (4).

We observe that

$$\theta(K,s) \le \left(\sum_{i=1}^{K} i^{-s}\right)^2 = \left(1 + \sum_{i=2}^{K} i^{-s}\right)^2 \le \left(1 + \int_{i=1}^{K} i^{-s}\right)^2 \le \frac{K^{1-s}}{(1-s)^2};$$

$$\theta(K,s) \ge \left(\sum_{i=1}^{K} i^{-s}\right) \cdot \left(\sum_{i=2}^{K} i^{-s}\right) \ge \left(\int_{1}^{K} x^{-s} dx\right) \cdot \left(\int_{2}^{K} x^{-s} dx\right) = \frac{K^{2(1-s)}}{(1-s)^2}\left(1 - \frac{1 + 2^{1-s}}{K^{2(1-s)}}\right).$$

Thus,

$$\frac{K^{2(1-s)}}{(1-s)^2}\left(1 - \frac{1 + 2^{1-s}}{K^{1-s}}\right) \le \theta(K,s) \le \frac{K^{2(1-s)}}{(1-s)^2}. \quad (5)$$

We first show (i). Let $A_K(\delta) \equiv \left\{\frac{\lceil \delta K \rceil}{K}, \frac{\lceil \delta K \rceil + 1}{K}, ..., 1\right\}$. We have:

$$\overline{\rho}_{BE}(K,N,s) \ge \sum_{i=1}^{K} \sum_{j=2}^{K} \frac{(ij)^{-s}}{\theta(K,s)} \left(1 - \frac{(ij)^{-s}}{\theta(K,s)}\right)^{N-1} \ge \frac{(1-s)^2}{K^2} \sum_{u,v \in A_K(\delta)} (uv)^{-s} \left(1 - \frac{(uv)^{-s} K^{-2s}}{\theta(K,s)}\right)^{N-1} \ge \frac{(1-s)^2}{K^2}\left(1 - \frac{(\delta K)^{-2s}}{\theta(K,s)}\right)^{N-1} \sum_{u,v\in A_K(\delta)} (uv)^{-s}. \quad (6)$$

In the previous inequalities we set $u = i/K$, $v = j/K$ and use the upper bound in (5).

Define:

$$\gamma(\delta, K, s) \equiv 1 - \frac{(\delta K)^{-2s}}{\theta(K,s)}.$$

Using the lower bound in (5) and the fact that $K/\sqrt{N} \to \infty$ as $N \to \infty$, we have $\gamma(\delta,K,s)^{N-1} \to 1$ as $N \to \infty$. Also observe that since the cardinality of $A_K(\delta)$ is at least $(1-\delta)^2 K^2$, we have:

$$\frac{1}{K^2} \sum_{u,v\in A_K(\delta)} (uv)^{-s} \ge (1-\delta)^2.$$

It follows from (6) that:

$$\liminf_{N\to\infty} \overline{\rho}_{BE}(K,N,s) \ge (1-s)^2 (1-\delta)^2.$$

Since $\delta > 0$ was arbitrary, the result follows.

We next show (ii). By Lemma 2,

$$\overline{\rho}_{BE}(K,N,s) \le \sum_{i=1}^{K}\sum_{j=1}^{K} \frac{(ij)^{-s}}{\theta(K,s)} \left(1 - \frac{(ij)^{-s}}{\theta(K,s)}\right)^{N-1} \le \sum_{i=1}^{K}\sum_{j=1}^{K} \frac{(ij)^{-s}}{\theta(K,s)} \frac{1}{1 + (ij)^{-s}(N-1)/\theta(K,s)} = \sum_{i=1}^{K}\sum_{j=1}^{K} \frac{1}{(ij)^{s}\theta(K,s) + (N-1)} \le \le \frac{K^2}{N} \to 0 \text{ as } N \to \infty.$$

Finally, we show (iii). Our proof exploits a connection to classical Erdös-Rényi random graphs. Throughout we assume without loss of generality that $N \le K(K-1)$. Let $G(K,N)$ denote a graph drawn uniformly at random from the $\binom{K(K-1)}{N}$ possible directed graphs on K nodes with N edges. We let $H(K,N,\boldsymbol{p})$ denote a random multigraph on K nodes with N edges, where each edge is independently placed from f to g with probability p_{fg}. Typically, $\boldsymbol{p}$ will be a probability distribution. However, in the subsequent analysis it is convenient if we allow the possibility $q(\boldsymbol{p}) = \sum_{f,g} p_{fg} < 1$. If $q(\boldsymbol{p}) < 1$, then we assume that each edge is not placed at all with probability $1 - q(\boldsymbol{p})$.

Now consider a random system with N peers and K files, where each peer desires and has one file. For each peer i, draw an edge from f to g if peer i wants file f and has file g. This is exactly the random multigraph $H(K,N,\boldsymbol{p})$ described in the preceding paragraph, with:

$$p_{fg} = \frac{(fg)^{-s}}{\theta(K,s)},$$

where $\theta(K,s)$ is defined as in (3).

Further, observe that if $H(K,N,\boldsymbol{p})$ is strongly connected, then all peers can trade multilaterally. This follows because if a peer i has f and wants g, then there is an edge from f to g in $H(K,N,\boldsymbol{p})$. If $H(K,N,\boldsymbol{p})$ is strongly connected, then there must exist a path from g to f as well. This path identifies a collection of peers that, together with peer i, form a cycle; thus peer i can trade multilaterally. It suffices to show that with probability approaching 1 as $N \to \infty$, $H(K,N,\boldsymbol{p})$ is strongly connected. (Convergence in probability implies convergence in expectation, as ρ_{ME} is bounded.)

It is known that if $N/(K \log K) \to \infty$, then $\mathbb{P}(G(K,N)$ is connected$) \to 1$ as $N \to \infty$ [27], where we use "connected" to mean strongly connected.[13] In Lemma 3, we use this threshold to establish the same threshold for a special class of $H(K,N,\boldsymbol{p})$ multigraphs, where $p_{fg} = \alpha/(K(K-1))$ for all f,g, with $0 < \alpha \le 1$.

To complete the proof, fix s such that $0 \le s < 1$, and observe that from (5) we have for fixed f and g:

$$\begin{aligned}\frac{(fg)^{-s}}{\theta(K,s)} &\ge \frac{(1-s)^2 (f/K)^{-s}(g/K)^{-s}}{K^2} \\ &\ge \frac{(1-s)^2}{K(K-1)}.\end{aligned}$$

Let $p_{fg} = (fg)^{-s}/\theta(K,s)$, and let $r_{fg} = (1-s)^2/(K(K-1))$. It follows that:

$$\mathbb{P}(H(K,N,\boldsymbol{p}) \text{ is connected}) \ge \mathbb{P}(H(K,N,\boldsymbol{r}) \text{ is connected}).$$

Since the right hand side approaches 1 as $N \to \infty$ by Lemma 3, we conclude that the left hand side approaches 1 as well. Thus the probability that all peers can trade multilaterally approaches 1. ■

The following lemma uses the same definitions as the preceding proof.

Lemma 3 *Suppose* $N/(K \log K) \to \infty$ *as* $N \to \infty$*, and* $p_{fg} = \alpha/(K(K-1))$ *for all* f, g*, where* $0 < \alpha \le 1$*. Then* $\mathbb{P}(H(K,N,\boldsymbol{p})$ *is connected*$) \to 1$ *as* $N \to \infty$*.*

Proof: The random multigraph $H(K,N,\boldsymbol{p})$ differs in two ways from the random graph $G(K,N)$. First, we may sample the same edge twice (this is why $H(K,N,\boldsymbol{p})$ is a multigraph). Second, with probability $1-\alpha$, a given edge may not be placed at all. Informally, neither of these effects change the order scaling of the number of edges needed to ensure connectivity. We now formally justify this intuition.

Where clear from context, to compress notation we let H and G denote $H(K,N,\boldsymbol{p})$ and $G(K,N)$ respectively. Let $\Gamma(H)$ denote the simple graph obtained from H by replacing any multiedges by a single edge. Observe that conditional on $\Gamma(H)$ having N' edges, $\Gamma(H)$ has the same distribution as $G(K,N')$. Thus it suffices to show that almost surely, the number of edges N' in $\Gamma(H)$ satisfies $N'/(K \log K) \to \infty$.

[13] This result is analogous to the same result for undirected Erdös-Rényi random graphs, and can be proven using similar counting arguments for threshold behavior of those graphs; see, *e.g.*, [18, 17].)

Since $N/(K \log K) \to \infty$, we can choose a sequence $M(N)$ such that $N/M \to \infty$ and $M/(K \log K) \to \infty$. (For example, for each k choose N_k such that $N/(K(N) \log K(N)) \ge k^2$ for all $N \ge N_k$. For N such that $N_k \le N < N_{k+1}$, let $M(N) = N/k$.) Since $M/N \to 0$ and $N \le K(K-1)$, it follows that $M/(K(K-1)) \to 0$ as $K \to \infty$. It suffices to show that $\mathbb{P}(\Gamma(H)$ contains at least M edges$) \to 1$ as $N \to \infty$.

Consider the following procedure for sampling with replacement from $K(K-1)$ items. Each time we draw an item, we record its number with probability α; with probability $1-\alpha$ we discard the observation. If we have recorded n distinct items, the time until we record the next distinct item is geometric with parameter $\alpha - \alpha n/(K(K-1))$. Let $T(K)$ denote the time until we observe M distinct items; then:

$$\mathbb{E}[T(K)] = 1 + \frac{1}{\alpha - \alpha/(K(K-1))} + \cdots + \frac{1}{\alpha - \alpha(M-1)/(K(K-1))} \le \frac{M}{\alpha - \alpha M/(K(K-1))}.$$

Since $M/(K(K-1)) \to 0$ as $K \to \infty$, we conclude:

$$\lim_{N\to\infty} \frac{\mathbb{E}[T(K)]}{M} \le \frac{1}{\alpha}.$$

Using Markov's inequality:

$$\mathbb{P}(T(K) > N) \le \frac{\mathbb{E}[T(K)]}{N} = \frac{E[T(K)]}{M} \cdot \frac{M}{N} \to 0$$

as $N \to \infty$. In other words, if we sample N items with replacement from a bin of $K(K-1)$ items as described above, then we obtain M distinct items with probability approaching 1 as $N \to \infty$. It follows that $\mathbb{P}(\Gamma(H)$ contains at least M edges$) \to 1$ as $N \to \infty$, as required. ■

Proposition 7 *Assume* $s = 0$*,* i.e.*, files are chosen uniformly. Moreover, assume* $|S_i| = \sigma$ *and* $|T_i| = 1$ *for all* $i \in U$*. As* $N \to \infty$*:*

(i) *If* $K/\sqrt{N} \to \infty$*, then* $\overline{\rho}_{BE}(K,N,s) \to 1$*.*
(ii) *If* $K/\sqrt{N} \to 0$*, then* $\overline{\rho}_{BE}(K,N,s) \to 0$*.*
(iii) *If* $K \log K/N \to 0$*, then* $\overline{\rho}_{ME}(K,N,s) \to 0$*.*

Proof: If $s = 0$, files are chosen uniformly. If there are K files in the system, the probability that a given user has files $\{f_1, ..., f_\sigma\}$ and wants file g is equal to $1/(\binom{K}{\sigma}(K-\sigma))$ for any set of distinct files $\{f_1, ..., f_\sigma, g\}$. A given user i can trade bilaterally with user j with probability

$$\frac{\sigma\binom{K-2}{\sigma-1}}{\binom{K}{\sigma}(K-\sigma)} = \frac{\sigma^2}{K(K-1)}$$

Thus a user cannot trade bilaterally with probability

$$\overline{\rho}_{BE}(K,N,0) = \left(1 - \frac{\sigma^2}{K(K-1)}\right)^{N-1}.$$

Both (i) and (ii) follow immediately from this expression. Part (iii) follows from Part (iii) of Proposition 6 using a straightforward coupling argument: if agents possess more than one file, ρ_{ME} can only be lowered. ■

Proposition 8 *If $s = 0$, and $N^2/(K^3 \log K) \to \infty$ as $K \to \infty$, then the probability that every peer can participate in trade along a triangle approaches 1 as $N \to \infty$.*

Proof: When $s = 0$, file popularity is uniform. We define $G(K,N)$ and $H(K,N,\boldsymbol{p})$ as in the proof of Part (iii) of Proposition 6, with $p_{fg} = 1/(K(K-1))$. As observed there, we note that if we draw an edge from file f to file g if peer i wants f and has g, then the resulting random multigraph has exactly the same distribution as $H(K,N,\boldsymbol{p})$. In this proof, we argue that every edge of $H(K,N,\boldsymbol{p})$ is part of a directed triangle with probability approaching 1 as $N \to \infty$. Using an argument similar to Lemma 3, it suffices to show that this result holds for $G(K,N)$.

Rather than reasoning about $G(K,N)$, we consider an alternative random graph model where edges are present with a fixed probability r. Formally, let $F(K,r)$ be a random graph on K nodes where each edge is present with probability $r = r(K)$, and edges are i.i.d. (Thus the probability a given graph with N edges is realized is $r^N(1-r)^{K(K-1)-N}$.) We say that $F(K,r)$ *has Property T* if there exists a two hop directed path between every pair of nodes f and g. We first show that if $r^2K/\log K \to \infty$ as $K \to \infty$, then $P(F(K,r)$ has Property T$) \to 1$ as $K \to \infty$. We then use an asymptotic equivalence property to show that the same result holds for $G(K,N)$.

Given a pair of nodes f and g, there are $K-2$ possible two hop paths from f to g, and each exists with probability r^2. Thus the probability there is no two hop path from f to g is $(1-r^2)^{K-2}$, and using a union bound:

$$\begin{aligned}&\mathbb{P}(F(K,r) \text{ does not have property T}) \leq \\ &K(K-1)(1-r^2)^{K-2} \leq \\ &\exp\left(-r^2(K-2)\left(1 - \frac{2\log K}{r^2(K-2)}\right)\right) \to 0 \text{ as } K \to \infty,\end{aligned}$$

Thus if $r^2K/\log K \to \infty$, then $\mathbb{P}(F(K,r)$ has property T$)\to 1$ as $K \to \infty$.

We now apply a standard asymptotic equivalence result for the graphs $F(K,r)$ and $G(K,N)$; informally, such results establish common asymptotic behavior of these models when $r \approx N/(K(K-1))$. Formally, we first observe that Property T is a *monotone property*: if a graph g on K nodes has property T, then any graph on K nodes containing g has Property T as well. Asymptotic equivalence only holds for monotone properties. Consider any sequence $r(N)$ satisfying $r(N) \geq N/K(K-1)$. The assumption of the proposition ensures that $r^2K/\log K \to \infty$ as $N \to \infty$, so for any such sequence, $\mathbb{P}(F(K,r)$ has Property T$) \to 1$ as $N \to \infty$. By Proposition 1.13 in [18], it follows that $\mathbb{P}(G(K,N)$ has Property T$) \to 1$ as $N \to \infty$ as well. This immediately implies that every edge in $G(K,N)$ is part of a directed triangle with probability 1 as $N \to \infty$, as required. ■