Monotonicity and Implementability*

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Abstract

Consider an environment with a finite number of alternatives, and agents with private values and quasi-linear utility functions. A domain of valuation functions for an agent is a monotonicity domain if every finite-valued monotone randomized allocation rule defined on it is implementable in dominant strategies. We fully characterize the set of all monotonicity domains.

1 Introduction

We consider an environment with a finite set of alternatives A, and agents with private values and quasi-linear preferences. We focus on direct revelation mechanisms, which consist of an allocation rule and a payment function. The allocation rule maps each profile of valuations to a probability vector over the set of alternatives.[1] Our interest is in allocation rules that are implementable in dominant strategies. For brevity such rules will be called just implementable.

Monotonicity is a necessary but not sufficient condition for an allocation rule to be implementable.[2] Rochet (1987) (see also (Rockafellar, 1970)) showed that a condition called cyclic monotonicity is both necessary and sufficient for any allocation rule to be implementable.

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[1]What is called here an allocation rule is often called a randomized allocation rule.

[2]An example of a monotone allocation rule which is not implementable is given by Saks and Yu (2005).

Cyclic monotonicity however is a considerably more difficult condition to work with than monotonicity; roughly, monotonicity is a condition on every pair of values, whereas cyclic monotonicity is a condition on every finite sequence of values.[3][4] Therefore, studying in which domains monotonicity is also sufficient for implementing an arbitrary allocation rule is a desired task. Myerson (1981) showed that in a single dimension domain, monotonicity is sufficient for any allocation rule to be implementable.[5] Bikhchandani et al. (2006) proved that in many convex domains, most notably R_+^A, every monotone deterministic allocation rule[6] is implementable. Gui et al. (2004) noticed that by a theorem by Roberts (1979), this result holds for the unrestricted domain $D = R^A$, and they proved in addition that it holds for every cube. Finally, Saks and Yu (2005) extended this result for any convex domain.[7]

In this paper we characterize domains for which every finite-valued (finite range) monotone allocation rule is implementable. Such domains are called *monotonicity domains*. We begin with characterizing *proper monotonicity domains*, which are defined similarly to monotonicity domains, only allocation rules can output also sub-probability vectors (rather than just probability vectors).[8] Using the characterization of proper monotonicity domains we are able to characterize monotonicity domains. Our results do not rule out the possibility that a particular monotone allocation rule can be implementable in a non monotonicity domain.

It can be shown that every domain with a convex closure is a proper monotonicity domain. One way to see this is to extend the result of Saks and Yu (2005) to finite-valued allocation rules which also output sub-probability vectors, and to domains with a convex closure (instead of just convex domains). Deriving these extensions requires some effort, and the supplementary material gives an alternative simpler proof. Our main result is the other direction: if the closure of a domain with dimension at least 2 is not convex, there exists a finite-valued monotone allocation rule, which possibly outputs sub-probability vectors, that is not implementable. The usefulness of this part is that it helps identifies some domains as not being (proper) monotonicity domains. One such domain is the class of gross substitutes preferences.

Both the finite-valued and randomized properties of the allocation rules are necessary;

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[3]For a good background on monotonicity and cyclic monotonicity see (Bikhchandani et al., 2006) and (Vohra, 2007). See also (Jehiel and Moldovanu, 2001), (Jehiel et al., 1996), (Krishna and Maenner, 2001) for characterizations of Bayesian incentive compatible mechanisms.

[4]See (Lavi and Swamy, 2009) who presented a mechanism in a scheduling setting and showed that the cyclic monotonicity holds in order to prove that their mechanism is implementable.

[5]Following Myerson (1981) other authors used the monotonicity condition to prove that their suggested allocation rules are implementable (in single dimensional domains). See e.g. (Goldberg et al., 2006) and (Lehmann et al., 2002).

[6]A deterministic allocation rule always assign probability 1 to some alternative.

[7]Berger et al. (2009) extended Saks and Yu's result to convex valuation functions.

[8]Note that every proper monotonicity domain is also a monotonicity domain.

Indeed, Archer and Kleinberg (2008) and Bikhchandani et al. (2006) each give an example for a monotone allocation rule on a convex domain, with infinitely many outcomes which is not cyclically monotone. Moreover, both Vohra (2010) and Mualem and Schapira (2008) exhibit examples of multi-dimensional non-convex domains in which every deterministic monotone allocation rule is implementable.

Knowing that a domain D is a monotonicity domain can serve as a useful tool in other mechanism design problems. For example, one can use this to find a revenue-optimal dominant strategy incentive compatible mechanism on D. Indeed, proving in the Bayesian setup that every monotone allocation rule is implementable was a key result in finding an optimal single-item auction in (Myerson, 1981). In monotonicity domains it is also easier to deal with the important task of finding concrete characterizations of implementable allocation rules, as the one appearing in Roberts (1979), where it was proved that every implementable deterministic allocation rule is an affine maximizer. Roberts proved the theorem using a condition (positive association of differences), which is very similar in spirit to monotonicity (see Bikhchandani et al. (2006); Lavi et al. (2003) for details).

Recently, researchers in the area of algorithmic mechanism design have been discussing efficiency bounds: Let g be a desired social choice function and f an allocation rule. They are interested in how bad (upper bounds) and how good (lower bounds) can the difference between g and f be (can be measured in various ways), when one insists that f is implementable. Such problems have been extensively analyzed in the computer science literature (see e.g. (Nisan and Ronen, 2001; Archer and Tardos, 2007; Lavi and Swamy, 2007)). Again, knowing that the domain is a monotonicity domain can be useful in analyzing these bounds.

In the next section we present the model and our results. In Section 3 we show that if a domain does not have a convex closure than there exists a finite-valued monotone allocation rule which is not implementable. In other words we complete the characterization of proper monotonicity domains. In Section 4 we characterize monotonicity domains using the characterization of proper monotonicity domains.

2 Model And Results

We restrict our attention to a model with a single agent. This is without loss of generality as all relevant definitions can be interpreted by holding all other agents' types fixed. Let A be a finite set of alternatives. Let R^A be the set of all possible valuation functions on A, that is the set of all real valued functions defined on A. The value of a for an agent with valuation v is thus v_a. It is convenient to represent each alternative a by its associated unit vector $e^a \in R^A$, where $e^a_a = 1$ and $e^a_b = 0$ for every $b \neq a$. Let $Z(A)$ be the set of all probability vectors $z \in R^A$:

$$Z(A) = \{z \in R^A | \ z_a \geq 0 \ \forall a, \ \sum_{a \in A} z_a = 1\}.$$

Let $D \subseteq R^A$, and let $f : D \to Z(A)$. We think of D as the set of all possible valuations of a given agent with a quasi-linear utility function, and f is interpreted as an *allocation rule*. If $f(v) \in \{e^a | a \in A\}$ for every $v \in V$ then f is called a deterministic allocation rule. If an agent with valuation v declares w, alternative a is chosen with probability $f_a(w)$, and therefore she evaluates $f(w)$ by the inner product $\langle v, f(w)\rangle = \sum_{a \in A} v_a f_a(w)$. An allocation rule f is *finite-valued* if its range $\{f(v)|v \in D\}$ is a finite set.

We say that an allocation rule f is *implementable in dominant strategies* (or just implementable) if there exists a payment function $c : D \to R$ such that

$$\langle v, f(v)\rangle - c(v) \geq \langle v, f(w)\rangle - c(w) \quad \forall v, w \in D. \tag{1}$$

Inequality (1) implies that given the payment function c, the agent is better off reporting v over w when her value is v. Writing the same inequality while reversing the order of v and w, and summing with (1), one obtains that:

$$\langle f(v) - f(w), v - w\rangle \geq 0 \quad \text{ for every } v, w \in D. \tag{2}$$

An allocation rule satisfying (2) is called *monotone*.

It was observed by Rochet (1987) that every implementable allocation rule f satisfies a stronger monotonicity property. f is called *cyclically monotone* if for every $k \geq 2$ and for every k vectors in D (not necessarily distinct), $v_1, v_2, \ldots, v_k$ the following holds:

$$\sum_{i=1}^{k} \langle v_i - v_{i+1}, f(v_i)\rangle \geq 0, \tag{3}$$

where v_{k+1} is defined to be v_1. By taking $k = 2$ in (3) it can be seen that every cyclically monotone allocation rule is monotone. The following characterization of implementability was proved by Rochet (1987):

Theorem 1 (Rochet) *An allocation rule is implementable if and only if it is cyclically monotone.*

We say that a domain of valuation functions is a *monotonicity domain* if every finite-valued monotone allocation rule defined on it is implementable. It is well known that every domain of dimension at most one is a monotonicity domain (Myerson, 1981).

To characterize monotonicity domains we need to consider an equivalent definition which relaxes the allocation rule to output also sub-probability vectors. Formally, Let $\bar{Z}(A)$ be the set of all sub-probability vectors $z \in R^A$:

$$\bar{Z}(A) = \{z \in R^A | \ z_a \geq 0 \ \forall a, \ \sum_{a \in A} z_a \leq 1\}.$$

We say that a domain of valuation functions is a *proper monotonicity domain* if every finite-valued monotone function $f : D \to \bar{Z}(A)$ is implementable.[9] To avoid confusion, only functions that always output probability vectors will be called allocation rules.

An important step in characterizing monotonicity domains is due to Saks and Yu (2005):

Theorem 2 (Saks and Yu) *Every deterministic allocation rule on a convex domain is implementable.*

In the supplementary material we give an alternative simpler proof. Our proof holds under weaker assumptions, only requiring the domain to have a convex closure, and the allocation rule to be finite-valued. Furthermore, we allow the allocation rule to output sub-probability vectors. In other words it is shown that *every domain with a convex closure is a proper monotonicity domain*. Other proofs and extensions can be found in (Archer and Kleinberg, 2008) and (Vohra, 2007).

In our main result we complete the characterization of proper monotonicity domains:

Theorem 3 *If a domain with dimension at least 2 does not have a convex closure then there exists a monotone finite-valued function $f : D \to \bar{Z}(A)$ which is not implementable.*

Finally, we use the characterization for proper monotonicity domains to characterize monotonicity domains:

Theorem 11 *A domain D is a monotonicity domain if and only if its projection to the hyperplane $H^A = \{v \in R^A : \sum_{a \in A} v_a = 0\}$ is a proper monotonicity domain.*[10]

3 Domains with a Non-Convex Closure

For every domain D, let M_D be the linear space generated by all differences $v - w$, where $v, w \in D$. The dimension $d(D)$ of D is defined to be the linear dimension of M_D. It is well-known that D is 0-dimensional if and only if D is a singleton. Let $k \geq 1$. It is also well-known that $d(D) \geq k$ if and only if there exist $k+1$ distinct valuations in D, $v_0, v_1, \ldots, v_k$, such that $v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0$ are linearly independent. In this section we prove:

[9] Defining implementability, monotonicity and cyclic monotonicity is similar for functions of the form $f : D \to \bar{Z}$. Moreover, Rochet's theorem holds for such functions.

[10] Intuitively, valuations that project onto the same point in H^A reflect identical preferences under probability distributions. Thus there is a natural connection between allocation rules on D and arbitrary functions on the projection of D to H^A.

Theorem 3 *If the closure of a domain of dimension at least 2 is not convex there exists a monotone finite-valued function $f : D \to \bar{Z}(A)$ which is not implementable. Alternatively, the closure of every proper monotonicity domain of dimension at least* 2 *is convex.*

The proof of Theorem 3 is by construction. We distinguish between domains of dimension 2 and domains of higher dimensions. The proof for $k = 2$ is given in Section 3.1 and the proof for $k \geq 3$ is given in Section 3.2. The reason for distinguishing between the dimensions is subtle and will be explained below. We begin with a sketch the proof of Theorem 3 for domains of dimension $k = 2$:

1. First we construct a monotone function f which is not implementable on a domain which is obtained by removing from R^2 the relative interior of a triangle (see Figure 1a). The range of f contains exactly three outcomes, each obtained in a different region U_0, U_1 and U_2. Furthermore f violates the cyclic monotonicity condition with every three valuations that each belongs to a different intersection of two of the regions $U_0, U_1,$ and U_2. For example, v, w, z in Figure 1a as well as the three vertices of the triangle violate cyclic monotonicity.

2. Next it is shown that if the domain D has a non-convex closure then there exist 3 valuations such that the relative interior of the convex hull generated by them contains a ball which does not intersect D. Subsequently it is shown that the structure identified in the first part can be embedded in D such that the triangle will be located in the ball (see e.g. Figures 1b and 1c).

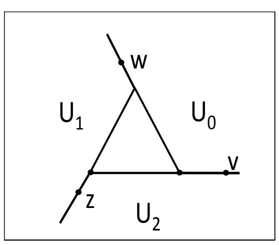

Figure 1a.

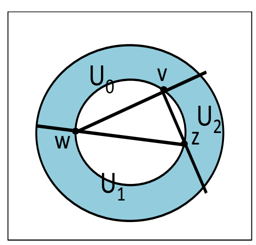

Figure 1b.

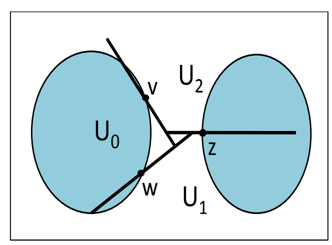

Figure 1c.

Figure 1a depicts the "ideal" case, in which the domain is the entire space except a triangle. In Figures 1b and 1c the triangle is embedded such that its interior does not intersect the domain (the blue region), but the extensions (including the vertices) of the triangle's edges do.

For further intuition regarding our construction and why we distinguish between domains of dimension 2 and domains of higher dimensions we use the following theorem which provides a unique way (up to a constant) to assign a utility function (or prices) for a cyclically monotone function on a polygonally connected domain.[11] This property is called *revenue equivalence* (see also Myerson (1981)).

Theorem 4 (Derived from (Rockafellar, 1970)) *Let f be an allocation rule on a domain D. f is cyclically monotone on D if and only if there exists a real valued function U on D such that*[12]

$$U(v_2) - U(v_1) \geq \langle f(v_1), v_2 - v_1 \rangle, \quad \forall v_1, v_2 \in D. \tag{4}$$

Let $[w, v] \subseteq D$. Every function U satisfying (4) *satisfies:*

$$U(v) - U(w) = \int_0^1 \phi(t) dt, \tag{5}$$

where

$$\phi(t) = \langle f(w + t(v - w)), v - w \rangle.^{13} \tag{6}$$

Consequently, if D is polygonally connected, any two functions satisfying (4) differ by a constant.

Suppose f is defined on a polygonally connected domain D. By Theorem 4, had f been cyclically monotone one can define a utility function U by choosing a single valuation v, fixing $U(v)$, and for any valuation w, $U(w)$ is defined by just taking the integral of f over some polygonal path from v to w. Hence, if one can provide a pair of valuations v and w and two different polygonal paths from v to w in the domain such that the integral of f over these paths are not equal, or alternatively the integral of f over the polygon that is formed by the two paths is not zero, then f is not cyclically monotone. In the proof for domains of dimension $k = 2$ we essentially constructed a monotone function f such that its integral over a polygon (the triangle) in the domain is not zero. We next explain why such an approach may fail in higher dimensions.

To construct a monotone function f which is not implementable it is useful to first identify polygons for which the above process cannot work, i.e. the integral of f over the polygon must be 0. First any polygon for which its convex hull belongs to D can be excluded, since one can consider the domain to be exactly the convex hull of the polygon. Furthermore any polygon that can be contracted through the domain to a point on the domain can be excluded (and in particular if the domain is simply connected[14] no polygon which is defined on the domain

[11] A domain D is called polygonally connected if for every two values $v, w \in D$ there is a polygonal path in D from v to w.

[12] $U(v)$ can be interpreted as the utility function of the agent when her valuation is v.

[13] Note that if f is monotone, ϕ is necessarily non-decreasing and therefore it is in particular Riemann integrable.

[14] In a simply connected domain every polygon can be contracted to a point.

can be chosen): to see this, suppose for example that the polygons $\Gamma_1 =< v_1, \ldots, v_5, v_1 >$, $\Gamma_2 =< w_5, \ldots w_1, w_5 >$ and the triangles (including their interiors) $\Delta_1, \ldots, \Delta_{10}$ in Figure 2 are contained in D. Since f is monotone on D it is monotone on each Δ_i, and therefore it

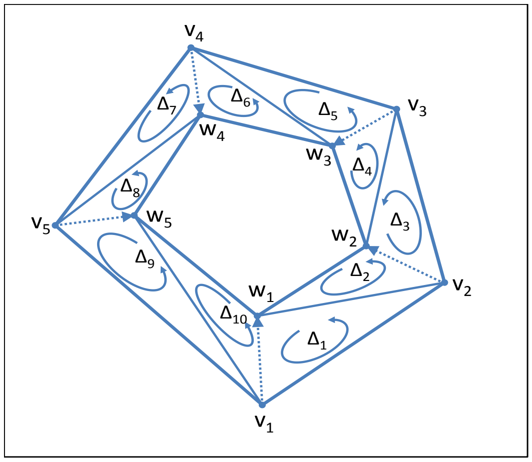

Figure 2.

is cyclically monotone on each Δ_i (by convexity). Thus the integral of f on the boundary on each triangle Δ_i is 0. This implies that the integral of f on Γ_1 equals the integral of f on Γ_2 (by adding to the integral of f on Γ_1 the integrals of f over all triangles $\Delta_1, \ldots, \Delta_{10}$ in the directions as in Figure 2). If Γ_2 can be further contracted in this way to a point then we obtain that integral of f over Γ_1 is 0.

The proof for domain D with dimension $k \geq 3$ has a similar idea; first it is shown how to create a monotone function over the domain R^k excluding the relative interior of convex polytope[15](here the polytope will be a k dimensional triangular prism). Then we show how to embed the construction in any domain with a non-convex closure of dimension k.

However, since the domain we use in the first step for $k \geq 3$ is simply connected we use a slightly different approach (than for $k = 2$) in constructing the function. First note that implementability of a function implies that for every two valuations v, w such that $f(v) = f(w)$ the price in v and w must be the same. Thus for such v and w even if the entire segment between (v, w) is not included in the domain one can still calculate $U(w)$ given the utility $U(v)$. This fact allows to deal also with polygons that part of them is not defined on the domain. This circumvents the contraction problem, i.e. a polygon that part of it is not defined on the domain cannot be contracted through the domain to a point.[16].

[15]A convex polytope is a convex hull of a finite set of points

[16]Another way to see that the direct extension of the first step of $k = 2$ does not work for $k = 3$ is the following; we wish to construct monotone function on the faces of a polytope with vertices, v_0, v_1, v_2 and v_4 (i.e., a tetrahedron), which violates cyclic monotonicity on the vertices. Note that f is cyclically

The following lemma will be useful in our proof; it enables us to embed the constructions in various domains. The first part of the lemma provides that a proper monotonicity domain can be equivalently defined using functions that do not output necessarily sub-probability vectors, i.e. by replacing $f : D \to \bar{Z}(A)$ with $f : D \to R^A$. The second part asserts that monotonicity and cyclic monotonicity are invariant under rotations and dilations. Hence when assessing whether a set $D \subseteq R^A$ is a proper monotonicity domain, we can choose the coordinates in any convenient way. The proof is given in the Appendix.

Lemma 5 1. *Let $D \subseteq R^A$. If there exists a monotone finite-valued function $f : D \to R^A$ which is not cyclically monotone then there also exists a function $\tilde{f} : D \to \bar{Z}(A)$ with the same properties.*

2. *A domain $D \subseteq R^A$ is a proper monotonicity domain if and only if $L(D)$ is a proper monotonicity domain, where $L(D)$ is a rotation, affine shift or contraction of D.*

3.1 Domains of Dimension $k = 2$

In this section we prove Theorem 3 for domains of dimension $k = 2$. We begin by showing that if D is the plane R^2 excluding an interior of a triangle one can define a monotone finite-valued function on D which is not cyclically monotone.

3.1.1 Preparations: the plane excluding a triangle

A set $L = \{v, w, z\}$ is called affine independent if its dimension is 2. The convex hull of an affine independent set $L = \{v, w, z\}$ is a simplex (triangle) denoted by $\Delta(L)$ and its relative interior is denoted by $\Delta^0(L)$.

Let $\alpha > 0, \beta > 0$ be any non-negative reals and let

$$S = \{(0,0), (1,0), (\frac{1}{1+\alpha\beta}, \frac{\alpha}{1+\alpha\beta})\}. \tag{7}$$

Note that S is affine independent. The complement of $\Delta^0(S)$ is the union of the following regions (see Figure 3):

$$U_0 = \{v \in R^2 : v_1 \geq 1 - \beta v_2 \quad \text{and} \quad v_2 \geq 0\},$$

monotone on every face of the polytope since each one of the faces is convex. Therefore, for every three vertices v_i,v_j and v_k there exists a real-valued function $U_{i,j,k}$ satisfying (4) on the convex hull of v_i, v_j and v_k. Note that one can shift $U_{0,1,2}$ and $U_{1,2,3}$ so that $U_{0,1,2}(v_1) = U_{1,2,3}(v_1) = U_{0,1,3}(v_1)$. By (5) it must be that $U_{0,1,2}(v_2) = U_{1,2,3}(v_2)$. Note that the function $U : \{v_0, v_1, v_2, v_3\} \to R$ defined by $U(v_i) = U_{0,1,3}(v_i)$ for $i = 0, 1, 3$ and $U(v_2) = U_{1,2,3}(v_2)$ satisfies (4) and therefore f is cyclically monotone on the vertices of the polytope D.

$$U_1 = \{v \in R^2 : v_1 \leq 1 - \beta v_2 \quad \text{and} \quad v_2 \geq \alpha v_1\},$$

and

$$U_2 = \{v \in R^2 : v_2 \leq \alpha v_1 \quad \text{and} \quad v_2 \leq 0\}.$$

For every $0 \leq i, j \leq 2$ let $U_{i,j} = U_i \cap U_j$. In the next proposition we construct a monotone finite-valued function on $R^2 \setminus \Delta^0(S)$. Furthermore this function will violate the cyclic monotonicity condition for every three points v, w, z that each one is on an extension of a different edge of the triangle (see Figure 3). Our parametrization will provide such a construction for any triangle, as we will see later.

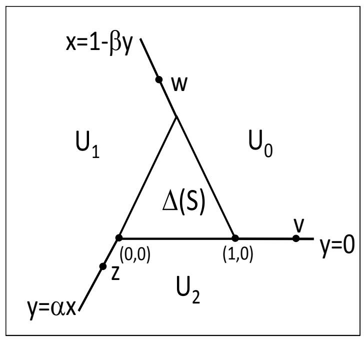

Figure 3.

Proposition 6 *There exists a monotone finite valued function $f : R^2 \setminus \Delta^0(S) \to R^2$ which is not cyclically monotone. Furthermore f can be chosen such that its range contains exactly three vectors $y^0 = (0, 1)$, $y^1 = (-\frac{\alpha}{1+\alpha\beta}, \frac{1}{1+\alpha\beta})$,$y^2 = (0, 0)$ and the following hold:*

1. *For $i = 0, 1, 2$, $f(v) = y^i$ for every $v \in U_i \setminus U_{i,i+1}$.*[17]

2. *For every three vectors v, w, z such that $v \in U_{0,2}$, $w \in U_{0,1}$ and $z \in U_{1,2}$*

$$\langle v - w, f(v)\rangle + \langle w - z, f(w)\rangle + \langle z - v, f(z)\rangle < 0. \tag{8}$$

Proof: First we show that f is monotone. We need to show that $\langle v - w, f(v) - f(w)\rangle \geq 0$ for every $v, w \in R^2 \setminus \Delta^0(S)$. Three cases should be considered. Assume that $f(v) = y^0$ and $f(w) = y^1$. Thus $v \in U_0$ and $w \in U_1$. Therefore

$$\langle v - w, y^0 - y^1\rangle = (v_1 - w_1)\frac{\alpha}{\alpha\beta + 1} + (v_2 - w_2)\frac{\alpha\beta}{\alpha\beta + 1},$$

which is non-negative if and only if $v_1 - w_1 + \beta(v_2 - w_2) \geq 0$ since α and β are positive. $v \in U_0$ implies that $v_1 + \beta v_2 \geq 1$ and $w \in U_1$ implies that $w_1 + \beta w_2 \leq 1$. Therefore $v_1 - w_1 + \beta(v_2 - w_2) \geq 0$.

[17]As usual $U_{2,3} = U_{2,0}$.

Next assume that $f(v) = y^1$ and $f(w) = y^2$. Thus $v \in U_1$ and $w \in U_2$. Therefore

$$\langle v - w, y^1 - y^2 \rangle = (v_1 - w_1)\frac{-\alpha}{\alpha\beta + 1} + (v_2 - w_2)\frac{1}{\alpha\beta + 1},$$

which is non-negative if and only if $-\alpha(v_1 - w_1) + v_2 - w_2 \geq 0$. This inequality holds since $v \in U_1$ and $w \in U_2$. Finally assume that $f(v) = y^2$ and $f(w) = y^0$. Thus $v \in U_2$ and $w \in U_0$. Therefore

$$\langle v - w, y^2 - y^0 \rangle = w_2 - v_2 \geq 0,$$

where the last inequality follows since $v \in U_2$ and $w \in U_0$.

To complete the proof we show that part 2. holds, which in particular shows that f is not cyclically monotone. Let $v \in U_{0,2}$, $w \in U_{0,1}$ and $z \in U_{1,2}$ (see Figure 3). Then $v = (a, 0)$ for some $a > 0$, $w = (1 - \beta b, b)$ for some $b > 0$ and $z = (-c, -\alpha c)$ for some $c > 0$. Therefore the left-hand side of (8) equals

$$-b - (1 - \beta b + c)\frac{\alpha}{\alpha\beta + 1} + (b + \alpha c)\frac{1}{\alpha\beta + 1} = -\frac{\alpha}{\alpha\beta + 1} < 0. \blacksquare$$

3.1.2 Proof of for $k = 2$

Let $D \subseteq R^A$ be a set of dimension $k = 2$ whose closure $cl(D)$ is non-convex. Observe that if one can define a monotone function on a set $D' \supseteq D$ and find a finite sequence of valuations $v_1, \ldots, v_k \in D$ which violates cyclic monotonicity then D is not a proper monotonicity domain (the restriction $f|_D$ is monotone but not cyclically monotone). By Proposition 6 and Lemma 5, D is not a proper monotonicity domain if there exist affine independent valuations, v, w, z, in D, such that the relative interior of the simplex generated by them does not intersect D. For example, such valuations can be easily detected for the non-convex ring at Figure 4a, by choosing the vertices shown in this figure. However, it is not true that such valuations exist for every non-convex set, even if it is closed. For example, if D is the

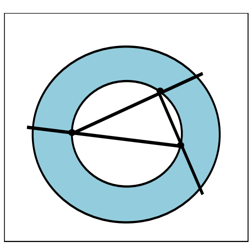

Figure 4a.

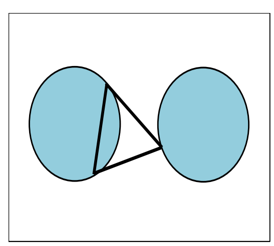

Figure 4b.

union of two disjoint closed disks shown in Figure 4b, then, as demonstrated in that figure, for every three valuations in D, not on the same line, the relative interior of the triangle generated by them intersects D. Therefore we need a more delicate procedure, which uses the following claim.

Claim 1 *Let D be a set of dimension $k = 2$ whose closure is non-convex. There exist 3 affine independent valuations in D such that the relative interior of the simplex Δ generated by them contains a point, say d for which there exists $\eta > 0$ such that $B(d, \eta) \cap D = \emptyset$, where $B(d, \eta) = \{v \in \Delta^0 | \ ||v - d|| < \eta\}$.*

The proof of Claim 1 is postponed to the end of this proof. Without loss of generality we can assume that $D \subseteq R^2$. Let v, w, z be affine independent valuations in D such that there exist d and η as in Claim 1. We choose η to be small enough such that $B(d, \eta) \subset \Delta^0(\{v, w, z\})$ (see Figure 5). By rotating and shifting the plane we can assume without loss of generality

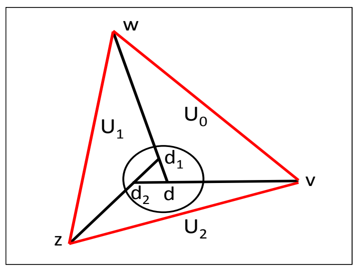

Figure 5: the red triangle is as in Figure 4b (v, w and z belong to the domain). A ball which does not intersect the domain is located in the interior of the red triangle centered at d. A triangle is located in the ball such that v,w and z are each on an extension of a different edge of the triangle.

that $d = (0, 0)$ and $v = (x, 0)$ for some $x > 0$.[18] Consider the line $z + t(d - z - (\epsilon, 0))$ for $\epsilon > 0$. Let d_1 and d_2 be the points in which this line intersects the lines $w + t(d - w)$ and the x-axis respectively. There exists a small enough ϵ such that $d_1 \in B(d, \eta)$ and $d_2 \in B(d, \eta)$ since for $\epsilon = 0$ all three lines intersect in d.

To finish the proof note that it is possible to rotate shift and scale the plane such that $S = \{d, d_1, d_2\}$ (see (7)) and v, w, z are vectors as in part two of Proposition 6. We can now apply Proposition 6 to show that D is not a proper monotonicity domain.■

To complete the proof of the theorem it remains to prove Claim 1.

Proof of Claim 1: The proof is by contradiction. Assume that the claim does not hold. Therefore, for every 3 affine independent valuations in D, the interior of the simplex Δ generated by them is contained in $cl(D)$. Therefore the simplex itself is contained in $cl(D)$.

[18]Lemma 5 provides that any shift, rotation or scaling of D preserves the monotonicity domain property. Hence these operations can be done alternatively on the space itself.

As the dimension of D is 2, for every $v_0 \neq v_1$ in D there exists $v_2 \in D$ such that v_0, v_1, v_2 are affine independent, and therefore the simplex generated by these valuation is contained in $cl(D)$, and therefore the interval $[v_0, v_1] \subseteq cl(D)$. Let w_0, w_1 be in $cl(D)$. There exist sequences v_0^n, v_1^n in D such that $v_i^n \to w_i$, $i = 0, 1$. Therefore, every valuation in $[w_0, w_1]$ is a limit of valuations in $cl(D)$, and hence it belongs to $cl(D)$. This implies that $cl(D)$ is convex contradicting the assumption of the claim. Hence, Claim 1 holds, which completes the proof of the theorem for $k = 2$. ■

3.2 Domains of Dimension $k \geq 3$

In this section we prove Theorem 3 for domains of dimension $k \geq 3$. We need the following definition. A domain D is called *good*, if for every $v, w \in D$ the projection of D onto $I = [v, w]$ is dense in $[v, w]$. The proof of Theorem 3 for dimensions $k \geq 3$ distinguishes between two cases, namely whether D is a good domain or not. For domains which are not good we apply the result for domains of dimension 2 by first projecting the domain to a plane, then finding on the projection a monotone finite-valued function which is not cyclically monotone, and finally extending the function to be monotone on the entire domain. For good domains the proof uses a similar idea as for domains of dimension $k = 2$, but employs a more complicated structure.

Note that if D is not a good domain then its closure is not convex. First we show:

Proposition 7 *If domain D is not good then it is not a proper monotonicity domain.*

Proof: Let $D \subseteq R^A$ be of dimension k where $k \geq 3$. For any closed convex set Q let $\Pi_Q(D)$ denote the projection of D on Q. Since D is not good there exist $v, w \in D$ and an open interval $(a, b) \subseteq [v, w]$ such that $\Pi_{[v,w]}(D) \cap (a, b) = \emptyset$. Let $z \in D$ be a vector in D such that v, w, z are affine independent. Rotate and shift the space such that v, w, z are in the XY plane and $[v, w]$ lies on the X axis. There exists $a < d < b$ and $\epsilon > 0$ such that $B((d, 0), \epsilon) \cap \Pi_{XY}(D) = \emptyset$ where Π_{XY} is the projection to the XY plane. Therefore, by our proof for dimension 2, $\Pi_{XY}(D)$ is not a proper monotonicity domain. In particular there exists a monotone finite-valued function $f : \Pi_{XY}(D) \to R^2$ that is not cyclically monotone. Let $\tilde{f} : D \to R^A$ be the function defined by $\tilde{f}(x_1, \ldots, x_A) = (f_1(x_1, x_2), f_2(x_1, x_2), 0, \ldots, 0)$ for every $x \in D$. Clearly $\tilde{f}$ is finite-valued, monotone and not cyclically monotone.■

By Proposition 7 it remains to deal only with good domains. These are studied in the next two subsections. One example of a good domain which is not convex is the unit sphere in R^3.

3.2.1 Preparations for the proof for good domains

Remark: throughout this section we will denote by v_l the l-th coordinate of v and indices of vectors will be denoted by superscripts.

Let $k \geq 2$ be some integer. Let S^k be the following hyperplane in R^{k+1}:

$$S^k = \{v \in R^{k+1} : \sum_{i=1}^{k} v_i = 0\}. \tag{9}$$

For every $\alpha > 0$, $\varepsilon_1, \varepsilon_2 > 0$, and $\delta > 0$, and $i = 1, \ldots, k$ we define the following regions in S^k:

$$U_i^k(\varepsilon_1, \alpha, \delta) = \left\{ v \in S^k : v_{k+1} \geq \alpha, \quad v_i = \max_{j=1}^{k} v_j, \quad v_{k+1} \geq \alpha + \frac{1 - \delta - v_i}{\varepsilon_1} \right\},$$

$$M_i^k(\alpha) = \left\{ v \in S^k : -\alpha \leq v_{k+1} \leq \alpha, \quad v_i = \max_{j=1}^{k} v_j \geq 1 \right\},$$

and

$$D_i^k(\varepsilon_2, \alpha) = \left\{ v \in S^k : v_{k+1} \leq -\alpha, \quad v_i = \max_{j=1}^{k} v_j, \quad v_{k+1} \leq -\alpha - \frac{(1 - v_i)}{\varepsilon_2} \right\}.$$

Let

$$P_u^k(\varepsilon_1, \alpha, \delta) = \left\{ v \in S^k : v_{k+1} \geq \alpha, \quad \text{for every } i \leq k \quad v_{k+1} \leq \alpha + \frac{(1 - \delta - v_i)}{\varepsilon_1} \right\}$$

and

$$P_d^k(\varepsilon_2, \alpha) = \left\{ v \in S^k : v_{k+1} \leq -\alpha, \quad \text{for every } i \leq k \quad v_{k+1} \geq -\alpha - \frac{(1 - v_i)}{\varepsilon_2} \right\}.$$

Note that if $v \in P_u$ or $v \in P_d$, then $v_i \leq 1$ for every $i \leq k$. Finally, let

$$T(\alpha) = \{v \in S^k : \text{for every } i \leq k \quad v_i < 1, \quad \text{and } -\alpha < v_{k+1} < \alpha\}. \tag{10}$$

Figure 6 illustrates the hyperplane S^2 and the regions defined above for $k = 2$.

The superscript k and the arguments $\varepsilon_1, \varepsilon_2, \alpha, \delta$ will be dropped whenever these are clear from the context. Let $\Omega = \{U_1, \ldots, U_k, M_1, \ldots, M_k, D_1, \ldots, D_k, P_d, P_u\}$, and let $G = \bigcup_{Q \in \Omega} Q$. Observe that $S^k = G \cup T$. For every set L we denote by $ri(L)$ the relative interior of L with respect to S^k. In the following proposition we show that G is not a proper monotonicity domain. Let u, w be the peaks of P_u and P_d respectively as illustrated in Figure 6. For any $i, 1 \leq i \leq k$, we show how to construct a monotone function on G such that for every two points $c^1 \in U_i \cap M_i$ and $c^2 \in M_i \cap D_i$ the sequence of points u, w, c^1, c^2 (see Figure 6) violates the cyclic monotonicity condition (3). This construction will be a key tool in our main proof.

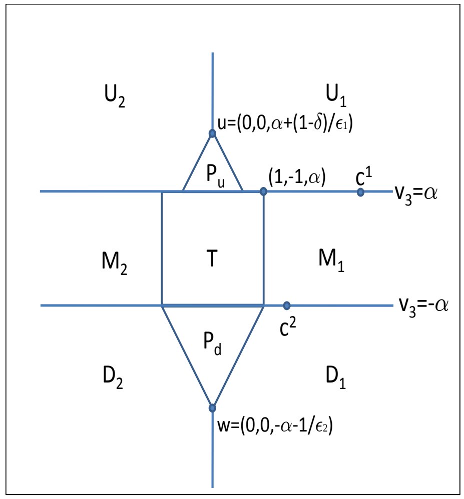

Figure 6.

Proposition 8 *Let $\alpha, \varepsilon_1, \varepsilon_2, \delta$ be positive reals such that $2\alpha\varepsilon_1 > \delta$. There exists a monotone finite-valued function $f : G \to R^{k+1}$, which is not cyclically monotone. Moreover f can be chosen such that its range contains exactly $3k+1$ distinct vectors $y^{U_1}, \ldots, y^{U_k}, y^{M_1}, \ldots, y^{M_k}$, $y^{D_1}, \ldots, y^{D_k}, y^P$ and such that for some fixed $i, 1 \leq i \leq k$ the following hold:*

1. *For every set $Q \in \Omega$, $f(v) = y^Q$ for all $v \in ri(Q)$ where $y^{P_u} = y^{P_d} = y^P$.*[19]
2. *$f(v) = y^{U_i}$ for all $v \in U_i \setminus M_i$, $f(v) = y^{M_i}$ for all $v \in M_i \setminus D_i$, $f(v) = y^{D_i}$ for all $v \in D_i \setminus P_d$ and $f(v) = y^P$ for all $v \in P_d$.*
3. *For every $v \in G$ other than in 1. and 2., let $f(v) = y^Q$ for an arbitrary Q in which $v \in Q$.*
4. *For every two distinct vectors c^1 and c^2 in which $c^1 \in U_i \cap M_i$ and $c^2 \in M_i \cap D_i$,*

$$\langle w - u, f(w) \rangle + \langle u - c^1, f(u) \rangle + \langle c^1 - c^2, f(c^1) \rangle + \langle c^2 - w, f(c^2) \rangle < 0. \tag{11}$$

where $w = (0, \ldots, 0, -\alpha - \frac{1}{\varepsilon_2})$ and $u = (0, \ldots, 0, \alpha + \frac{1-\delta}{\varepsilon_1})$.

The proof of Proposition 8 is straightforward and is given in the Appendix. By Lemma 5 and Proposition 8 we obtain:

Corollary 9 *For any $\alpha, \varepsilon_1, \varepsilon_2, \delta$ such that $2\alpha\varepsilon_1 > \delta$, G is not a proper monotonicity domain.*

[19]Since all the sets in Ω are defined with equalities we first define the function on the interior of every set in Ω, and then break ties on the boundaries.

3.2.2 Proof of Theorem 3 for good domains with $k \geq 3$

Proposition 10 *Let D be a good domain with $dim(D) = k \geq 3$ with a non-convex closure. Then D is not a proper monotonicity domain.*

The technique of this proof resembles the technique of the proof for $k = 2$, as we embed the structure in the previous section in a way that allows us to apply Proposition 8. In particular we show that for any good domain D there exist parameters $\alpha, \varepsilon_1, \varepsilon_2$ and δ as in Proposition 8 such that the set T and the sets in Ω (see Section 3.2.1) can be embedded in the space so that T is in the relative interior of $ConvexHull(D) \setminus D$ (in the proof of $k = 2$ we located a triangle to be in the relative interior). The peaks of the simplexes $w \in P_d$ and $u \in P_u$ can both be located in D, and there exist c^1 and c^2 as in Proposition 8 that also belong to D. See Figure 7 for an illustration of this construction when D is a sphere (the colored regions in Figure 7 represent P_u and P_d).

Proof of Proposition 10:

Since $cl(D)$ is not convex there exist $w, u \in D$, $z \in I := [w, u]$ and $r > 0$ such that $B(z, r) \cap cl(D) = \emptyset$, where $B(z, r) = \{v \in R^k : ||v - z|| < r\}$. We can assume without loss of generality that D is embedded in R^{k+1}. Rotate the space so that the positive x_{k+1} axis is from w to u. All other coordinates are parameterized by $x_1, x_2, \ldots, x_k$ such that $\sum_{i=1}^{n} x_i = 0$.

Let $I_z = I_z(r_1)$ be the interval of length $r_1 > 0$ centered in z on I. There exists $r_1 > 0$ such that for every $a = (a_1, \ldots, a_k, a_{k+1})$ with $a_{k+1} \in I_z(r_1)$ and $a_i \leq r_1$ for every $i \leq k$, $a \notin cl(D)$. We scale D by $\frac{1}{r_1}$. Thus, if $a_{k+1} \in I_z$ and $a_i \leq 1$ for every $i \leq k$ then $a \notin cl(D)$.

Let $a^1, a^2, \ldots, a^{k+1}$ be $k+1$ equally spaced vectors in I_z, i.e. there exists $d > 0$ such that for every $i, 1 \leq i \leq k$, $a^{i+1}_{k+1} - a^i_{k+1} = d$.

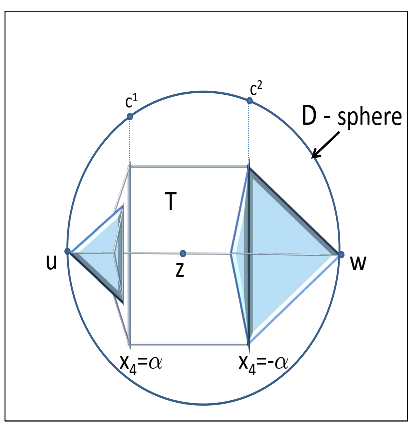

Figure 7.

Let Π_{wu} be the projection operator on the interval $I = [w, u]$. Since D is a good set there exist $k + 1$ distinct vectors $b^1, \ldots, b^k, b^{k+1} \in D$ such that $\Pi_{wu}(b^i) \in (a^{i-1}, a^i)$. For any vector $b = (b_1, \ldots, b_k)$ we define $indmax(b)$ to be some arbitrary index in $\arg\max_{i=1}^k b_i$. That is $b_{indmax(b)} = \max_{i=1}^k b_i$. Note that there exist $i, j \leq k+1$ such that $i < j$ and $indmax(b^i) = indmax(b^j)$. Define $t := indmax(b^i)$.

Let $c = \frac{\Pi_{wu}(b^j)_{k+1} + \Pi_{wu}(b^i)_{k+1}}{2}$. We now shift the set so that c moves to $(0, 0, \ldots, 0)$. Therefore $\Pi_{wu}(b^j) = -\Pi_{wu}(b^i)$. In particular there exists $\alpha > 0$ such that $\Pi_{wu}(b^j) = (0, \ldots, 0, \alpha)$ and $\Pi_{wu}(b^i) = (0, \ldots, 0, -\alpha)$. We obtained that

$$w = (0, \ldots, 0, -w_{k+1}), u = (0, \ldots, 0, u_{k+1}), b^i = (., \ldots, -\alpha), b^j = (., \ldots, \alpha) \quad \text{where} \quad w_{k+1}, u_{k+1} > \alpha.$$

Define ε_1,ε_2 and δ as follows:

$$\varepsilon_2 = \frac{1}{w_{k+1} - \alpha}; \delta < \min\left(\frac{\alpha}{u_{k+1} - \alpha}, \frac{1}{2}\right); \varepsilon_1 = \frac{1 - \delta}{u_{k+1} - \alpha}.$$

Therefore

$$\varepsilon_1 > \frac{1}{2(u_{k+1} - \alpha)}; \quad 2\alpha\varepsilon_1 > \frac{\alpha}{u_{k+1} - \alpha} > \delta.$$

Recall that $T(\alpha)$ is the k dimensional prism (see (10)). Therefore $T(\alpha) \cap D = \emptyset$. We can now apply Proposition 8 with $i = t$, where w, u are as in part 4 of the proposition and $c^1 = b^j$, $c^2 = b^i$. This implies that D is not a proper monotonicity domain.$\blacksquare$

4 Monotonicity Domains - Characterization

In this section we complete our characterization of monotonicity domains.

Recall that a domain D is a *monotonicity domain* if every monotone finite-valued allocation rule is also cyclically monotone. Note that every proper monotonicity domain is a monotonicity domain. We characterize monotonicity domains via proper monotonicity domains.

Let $H^A = \{v \in R^A : \sum_{a \in A} v_a = 0\}$ be the hyperplane which is orthogonal to the vector $(1, \ldots, 1) \in R^A$. Denote by $\Pi : R^A \to H^A$ the projection onto the hyperplane H^A.

Theorem 11 *Let $D \subseteq R^A$. D is a monotonicity domain if and only if $\Pi(D)$ is a proper monotonicity domain.*

Proof: We first prove that if D is a monotonicity domain then $\Pi(D)$ is a proper monotonicity domain. Assume for contradiction that $\Pi(D)$ is not a proper monotonicity domain, i.e. there exists a function $f^0 : \Pi(D) \to \bar{Z}(A)$ which is monotone but not cyclically monotone. Let f^1 be an allocation rule obtained from f^0 by adding an appropriate multiple of $(1,\ldots,1)$ to each value of f^0:

$$f^1(v) := f^0(v) + \frac{1 - \sum_{a\in A} f^0_a(v)}{|A|}(1,\ldots,1).$$

Let f^2 be the natural extension of f^1 to D. That is $f^2(v) = f^1(\Pi(v))$ for every $v \in D$. Thus f^2 is also a finite-valued allocation rule. We claim that f^2 is monotone but not cyclically monotone. To see this it is enough to show that for any $v, w, z \in D$, $\langle v, f^2(w) - f^2(z)\rangle = \langle \Pi(v), f^0(\Pi(w)) - f^0(\Pi(z))\rangle$. Let $v, w, z \in D$.

$$\langle v, f^2(w) - f^2(z)\rangle = \langle \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\rangle + \langle v - \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\rangle.$$

Since $f^1(\Pi(w)) - f^1(\Pi(z)) \in H^A$, we have that $\langle v - \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\rangle = 0$. Therefore,

$$\langle v, f^2(w) - f^2(z)\rangle = \langle \Pi(v), f^0(\Pi(w)) - f^0(\Pi(z))\rangle + \langle \Pi(v), c(1,\ldots,1)\rangle,$$

where c is some real number. Since $\langle \Pi(v), c(1,\ldots,1)\rangle = 0$ we are done.

We proceed to prove the other direction. Assume $\Pi(D)$ is a proper monotonicity domain and suppose that D is not a monotonicity domain, i.e. there exists an allocation rule f^0 on D that is monotone but not cyclically monotone. Let $v_1,\ldots,v_k$ be a shortest sequence of valuations which violates the cyclic monotonicity condition:

$$\sum_{i=1}^{k} \langle v_i, f^0(v_i) - f^0(v_{i-1})\rangle < 0. \tag{12}$$

We have that (12)=

$$\sum_{i=1}^{k} \langle \Pi(v_i), f^0(v_i) - f^0(v_{i-1})\rangle + \sum_{i=1}^{k} \langle v_i - \Pi(v_i), f^0(v_i) - f^0(v_{i-1})\rangle = \sum_{i=1}^{k} \langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i)\rangle, \tag{13}$$

where the second equality follows since $v_i - \Pi(v_i)$ is orthogonal to $f^0(v_i) - f^0(v_{i-1})$.

Claim 2 *For any $i \neq j$, $\Pi(v_i) \neq \Pi(v_j)$.*

Proof: Suppose $i < j$ and $\Pi(v_i) = \Pi(v_j)$. Taking all indices modulo k we have $\sum_{l=1}^{k} \langle \Pi(v_l) - \Pi(v_{l+1}), f^0(v_l) \rangle =$

$$\langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i) \rangle + \cdots + \langle \Pi(v_{j-1}) - \Pi(v_j), f^0(v_{j-1}) \rangle +$$

$$\langle \Pi(v_j) - \Pi(v_{j+1}), f^0(v_j) \rangle + \cdots + \langle \Pi(v_{i-1}) - \Pi(v_i), f^0(v_{i-1}) \rangle =$$

$$\langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i) \rangle + \cdots + \langle \Pi(v_{j-1}) - \Pi(v_i), f^0(v_{j-1}) \rangle + \tag{14}$$

$$\langle \Pi(v_j) - \Pi(v_{j+1}), f^0(v_j) \rangle + \cdots + \langle \Pi(v_{i-1}) - \Pi(v_j), f^0(v_{i-1}) \rangle] < 0. \tag{15}$$

Clearly at least one of (14) or (15) is negative contradicting the minimality of k. ■

Next we say that f^1 on $\Pi(D)$ is a projection of f^0 if for any $v \in \Pi(D)$, $f^1(v) \in f^0(\Pi^{-1}(v))$.

Claim 3 *Any projection f^1 of f^0 is monotone.*

Proof: For any $v, w \in \Pi(D)$ there is $\tilde{v}, \tilde{w} \in D$ such that $\Pi(\tilde{v}) = v$, $\Pi(\tilde{w}) = w$, $f^0(\tilde{v}) = f^1(v)$ and $f^0(\tilde{w}) = f^1(w)$. We have

$$\langle v - w, f^1(v) - f^1(w) \rangle = \langle v - w, f^0(\tilde{v}) - f^0(\tilde{w}) \rangle = \langle \tilde{v} - \tilde{w}, f^0(\tilde{v}) - f^0(\tilde{w}) \rangle. \blacksquare$$

Finally, since by Claim 2 all the $\Pi(v_i)$'s are distinct we can select a projection f^1 of f^0 such that $f^1(\Pi(v_i)) = f^0(v_i)$ for all $i = 1, \ldots, k$. Therefore f^1 is monotone but not cyclically monotone:

$$\sum_{i=1}^{k} \langle \Pi(v_i), f^1(\Pi(v_i)) - f^1(\Pi(v_{i-1})) \rangle = \sum_{i=1}^{k} \langle v_i, f^1(\Pi(v_i)) - f^1(\Pi(v_{i-1})) \rangle =$$

$$\sum_{i=1}^{k} \langle v_i, f^0(v_i) - f^0(v_{i-1}) \rangle < 0.$$

This contradicts that $\Pi(D)$ is a proper monotonicity domain. ■

5 Appendix

5.1 Proof of Lemma 5

We begin with the first part. Let D be a domain and let $f : D \to R^A$ be a monotone finite-valued function which is not cyclically monotone. Let $y^1, \ldots, y^m$ be the distinct values of f. There exist $\alpha > 0$ and $y \in R^A$ such that for every $i = 1, \ldots, m$, $\tilde{y}^i = \alpha(y^i + y) \in \bar{Z}(A)$. Let $\tilde{f}$ be the function defined by $\tilde{f}(v) = \tilde{y}^i$ if and only if $f(v) = y^i$. Thus for every $v, w \in D$,

$$\langle v, \tilde{f}(v) - \tilde{f}(w) \rangle = \alpha \langle v, f(v) - f(w) \rangle. \tag{16}$$

Therefore all inner products in (16) are multiplied by the same positive factor, implying that $\tilde{f}$ is monotone and not cyclically monotone.

To prove the second part we first notice that by the first part we do not need to restrict ourselves to functions that output only sub-probability vectors. Assume that D is not a proper monotonicity domain and let $f : D \to R^A$ be a monotone function which is not cyclically monotone. We show that there exits a monotone function $\tilde{f} : L(D) \to R^A$ which is not cyclically monotone.

Suppose $L(D)$ is a rotation. Thus, there exists a unitary matrix U such that for every $y \in L(D)$ there exists $x \in D$ such that $Ux = y$. For all $x \in L(D)$ let $\tilde{f}(x) = Uf(U^{-1}x)$. For every three points $x, y, z \in D$, we have

$$\langle x - y, f(z)\rangle = \langle Ux - Uy, Uf(z)\rangle = \langle Ux - Uy, \tilde{f}(Uz)\rangle$$

as U is unitary. Since all the monotonicity and cyclic monotonicity constraints are defined via inner products, $\tilde{f}$ is monotone but not cyclic monotone over $L(D)$. Suppose now that $L(D)$ is an affine shift by some fixed vector $\vec{t}$. For every $x \in L(D)$, let $\tilde{f}(x) = f(x - \vec{t})$. Therefore $\langle x-y, f(z)\rangle = \langle (x-\vec{t})-(y-\vec{t}), f(z-\vec{t})\rangle$ which implies the result. Finally, suppose $L(D)$ is a contraction by a constant $c > 0$. For every $x \in L(D)$ let $\tilde{f}(x) = f(cx)$. In this case, all the inner products are multiplied by $c > 0$, and the result follows.■

5.2 Proof of Proposition 8

In order to define the range of f we make use of the following notation. Let $e^j(\gamma) \in R^{k+1}$ denote the sum $e^j + (0, \ldots, 0, \gamma)$ where both vectors are in R^{k+1}. The range of f is defined as follows:

$$y^Q = \begin{cases} e^j(\varepsilon_1) & Q = U_j, \\ e^j & Q = M_j, \\ e^j(-\varepsilon_2) & Q = D_j, \\ \bar{0} & Q = P_d \quad \text{or} \quad Q = P_u. \end{cases} \tag{17}$$

We first show that f is not cyclically monotone. To see this it is enough to verify that (11) holds. Let w, v, c^1, c^2 be as in part 4 of the proposition. Since $f(w) = \bar{0}$, $\langle w - u, f(w)\rangle = 0$. Since $c^1 \in U_i \cap M_i$ it has the form $c^1 = (c^1_1, \ldots, c^1_k, \alpha)$. Similarly $c^2 = (c^2_1, \ldots, c^2_k, -\alpha)$. Therefore

$$\langle u - c^1, f(u)\rangle = -c^1_i + \frac{1-\delta}{\varepsilon_1} \cdot \varepsilon_1 = -c^1_i + 1 - \delta,$$

$$\langle c^1 - c^2, f(c^1)\rangle = c^1_i - c^2_i, \quad \text{and} \quad \langle c^2 - w, f(c^2)\rangle = c^2_i + \frac{1}{\varepsilon_2} \cdot (-\varepsilon_2) = c^2_i - 1.$$

Summing up all the terms we obtain that (11)$=-\delta < 0$.

To complete the proof we need to show that f is monotone on G. Let $v = (v_1, \dots, v_{k+1}), w = (w_1, \dots, w_{k+1})$ be any two vectors in G. Let e^0 denote the zero vector. We distinguish between the following cases (i will be used now as an arbitrary index):

1. $f(v) = y^{U_i}$ and $f(w) = y^{U_j}$. Thus, $v \in U_i, w \in U_j$. Therefore

   $$\langle v-w, f(v)-f(w)\rangle = \langle v-w, e^i - e^j\rangle = (v_i - w_i) + (v_j - w_j) = (v_i - v_j) + (w_j - w_i) \geq 0$$

   where the last inequality follows since $v_i \geq v_j$ and $w_j \geq w_i$.

2. $f(v) = y^{U_i}$ and $f(w) = y^{M_i}$. Thus, $v \in U_i, w \in M_i$, implying that

   $$\langle v-w, f(v)-f(w)\rangle = \langle v-w, e^0(\varepsilon_1)\rangle = (v_{k+1} - w_{k+1}) \cdot \varepsilon_1 \geq 0,$$

   since $v_{k+1} \geq \alpha, w_{k+1} \leq \alpha$, and $\varepsilon_1 > 0$.

3. $f(v) = y^{U_i}$ and $f(w) = y^{M_j}$ for $i \neq j$. Thus $v \in U_i, w \in M_j$. Therefore

   $$\langle v-w, f(v)-f(w)\rangle = \langle v-w, e^i(\varepsilon_1) - e^j\rangle = (v_i - w_i) + (v_j - w_j) + (v_{k+1} - w_{k+1}) \cdot \varepsilon_1 \geq 0$$

   where the last inequality follows since $v_i \geq w_i$, $v_j \geq w_j$ and $(v_{k+1} - w_{k+1})\varepsilon_1 \geq 0$.

4. $f(v) = y^{U_i}$ and $f(w) = y^{D_i}$. Thus $v \in U_i, w \in D_i$. Therefore

   $$\langle v-w, f(v)-f(w)\rangle = \langle v-w, e_0(\varepsilon_1) - e^0(-\varepsilon_2)\rangle = (v_{k+1} - w_{k+1}) \cdot (\varepsilon_1 + \varepsilon_2) \geq 0$$

   where the last inequality follows since $v_{k+1} \geq \alpha, w_{k+1} \leq -\alpha$, and $\varepsilon_1, \varepsilon_2 > 0$.

5. $f(v) = y^{U_i}$ and $f(w) = y^{D_j}$. $x \in U_i,\ w \in D_j$. Then

   $$\langle v-w, f(v)-f(w)\rangle = \langle v-w, e^i(\varepsilon_1) - e^j(-\varepsilon_2)\rangle = (v_i - w_i) + (v_j - w_j) + (v_{k+1} - w_{k+1}) \cdot (\varepsilon_1 + \varepsilon_2) \geq 0$$

   where the last inequality follows since $v_i \geq w_i$, $v_j \geq w_j$ and $v_{k+1} \geq w_{k+1}$.

6. $f(v) = y^{M_i}$ and $f(w) = y^{M_j}$. Thus $x \in M_i, y \in M_j$. Therefore

   $$\langle v-w, f(v)-f(w)\rangle = \langle v-w, e^i - e^j\rangle = (v_i - w_i) + (v_j - w_j) = (v_i - v_j) + (w_j - w_i) \geq 0$$

   where the last inequality follows since $v_i \geq v_j$, $w_j \geq w_i$.

7. $f(v) = y^{U_i}$ and $f(w) = y^P$. Thus $v \in U_i, w \in P_u \cup P_d$. Then

   $$\langle v-w, f(v)-f(w)\rangle = \langle v-w, e^i(\varepsilon_1)\rangle = (v_i - w_i) + \varepsilon_1 \cdot (v_{k+1} - w_{k+1}) \geq$$

   $$v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - \varepsilon_1 w_{k+1}. \tag{18}$$

where the last inequality follows since $v_{k+1} \geq \alpha + \frac{1-\delta-v_i}{\varepsilon_1}$. If $w \in P_u$ then $w_{k+1} \leq \alpha + \frac{1-\delta-w_i}{\varepsilon_1}$, and therefore

$$(18) \geq v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - (\alpha \cdot \varepsilon_1 + 1 - \delta - w_i) = 0.$$

If $w \in P_d$ then $w_{k+1} \leq -\alpha$ and $w_i \leq 1$. Therefore, since $2\alpha\varepsilon_1 \geq \delta$,

$$(18) \geq v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - (-\alpha \cdot \varepsilon_1) = 1 - w_i - \delta + 2\alpha\varepsilon_1 \geq 0.$$

8. $f(v) = y^{M_i}$ and $f(w) = y^P$. Thus $v \in M_i$, and $w \in P_u \cup P_d$. Therefore

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^i\rangle = v_i - w_i \geq 0,$$

where the last inequality follows since $v_i \geq 1, w_i \leq 1$.

9. $f(v) = y^{D_i}$ and $f(w) = y^P$. Thus $v \in D_i, w \in P_u \cup P_d$. Then

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^i(-\varepsilon_2)\rangle =$$

$$(v_i - w_i) + \varepsilon_2 \cdot (w_{k+1} - v_{k+1}). \tag{19}$$

If $w \in P_u$ then $w_{k+1} \geq \alpha > -\alpha - \frac{1-w_i}{\varepsilon_2}$, since $w_i \leq 1$. If $w \in P_d$, then by definition $w_{k+1} \geq -\alpha - \frac{1-w_i}{\varepsilon_2}$. In either case, since $v \in D_i$, $v_{k+1} \leq -\alpha - \frac{1-v_i}{\varepsilon_2}$. Therefore

$$(19) \geq v_i - w_i + (\alpha \cdot \varepsilon_2 + 1 - v_i) - (\alpha \cdot \varepsilon_2 + 1 - w_i) = 0.$$

The other cases in which $f(v) = y^{D_i}$ are very similar to those in which $f(v) = y^{U_i}$. In fact it is easier for monotonicity to hold in these cases since P_d is a larger set than P_u. ■

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