Sum of Squares Lower Bounds for Constraint Satisfaction

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Joint work with Boaz Barak and Siu On Chan.
Constraint Satisfaction

n Boolean variables.
m k-ary clauses

Predicate
$P : \{0, 1\}^k \rightarrow \{0, 1\}$
on literals on clauses.

CSP($P$)

$P(x_1, \neg x_2, x_3) = 1$, $P(\neg x_3, \neg x_4, x_5) = 1$...
Constraint Satisfaction

\[ \text{n Boolean variables.} \]

\[ \text{m k-ary clauses} \]

\[ \text{find assignment to satisfy maximum number of constraints.} \]

\( \text{CSP}(P) \)

\( P : \{0, 1\} \rightarrow \{0, 1\} \)

on literals on clauses.

\[ \ell_i = x_i \text{ or } \neg x_i \]

MAX-3SAT, MAX-3XOR, ...
Random Assignment

assign a uniform, independent bit to each variable.
satisfies $\frac{|P^{-1}(1)|}{2^k}$ fraction of constraints.
Random Assignment

assign a uniform, independent bit to each variable.
satisfies $\frac{|P^{-1}(1)|}{2^k}$ fraction of constraints.

Can we do better?
Random Assignment

MAX CUT  Yes!  [GW95]
MAX 2LIN  Yes!  [AEH01]
MAX 3MAJ  Yes!  [Zwi98]
Random Assignment

MAX CUT  Yes! [GW95]
MAX 2LIN  Yes! [AEH01]
MAX 3MAJ  Yes! [Zwi98]
MAX k-SAT*  No! [Hås99]
MAX k-XOR*  No! [Hås99]
MAX TSA  No! [KH01]

*k = 3 and above
Random Assignment

\[x_1 \ x_2 \ x_3\]
\[x_4\]
\[x_5 \ x_8 \ x_6 \ x_7 \ x_{13}\]
\[x_9\]
\[x_{10} \ x_{11} \ x_{12}\]

MAX CUT Yes! [GW95]
MAX 2LIN Yes! [AEH01]
MAX 3MAJ Yes! [Zwi98]
MAX k-SAT No! [Hås99]
MAX k-XOR No! [Hås99]
MAX TSA No! [KH01]

“approximation resistance”
Green = can efficiently beat the random assignment.
Red = approximation resistant.

can beat random on **most** predicates with sparsity

\[ O\left(\frac{k^2}{\log(k)}\right) \]
Green = can efficiently beat the random assignment.
Red = approximation resistant.

[Hadamard is approx. resistant.]

Most predicates with \( \Omega\left(\frac{2^k}{\sqrt{k}}\right) \) sat. assignments are implied by Hadamard predicate.

\[ O\left(\frac{2^k}{\sqrt{k}}\right) \]

\[ \Omega(k^2/\log(k)) \]
Green $=$ can efficiently beat the random assignment.
Red $=$ approximation resistant.

$\frac{k^2}{\log(k)}$  

$O\left(\frac{2^k}{\sqrt{k}}\right)$  

$\Omega\left(\frac{k^2}{\log(k)}\right)$

# satisfying assignments.

[Cha13]+[Has07]

[AH09,AM09]
Random Assignment

“pairwise independent”

- $x_1$, $x_2$, $x_3$
- $x_4$
- $x_5$, $x_8$, $x_6$, $x_7$, $x_{13}$
- $x_9$
- $x_{10}$, $x_{11}$, $x_{12}$

**MAX CUT**  Yes!  [GW95]
**MAX 2LIN**  Yes!  [AEH01]
**MAX 3MAJ**  Yes!  [Zwi98]
**MAX k-SAT**  No!  [Hås99]
**MAX k-XOR**  No!  [Hås99]
**MAX TSA**  No!  [KH01]
Pairwise Independence

[Austrin-Mossel’08]

A CSP(P) is \( P : \{0, 1\}^k \rightarrow \{0, 1\} \) pairwise independent if there is a prob. dist. \( \gamma \) on \( \{0, 1\}^k \)

1. supported on \( P^{-1}(1) \)
2. for any \( a, b \in \{0, 1\} \)

\[
\Pr_{x \sim \gamma} [x_i = a, x_j = b] = \frac{1}{4}
\]
Green = can efficiently beat the random assignment.
Red = approximation resistant.

\[ O\left(\frac{k^2}{\log (k)}\right) \]

*most* predicates with sparsity \( \Omega(k^2) \) are pairwise independent

\[ 1 - o_k(1) \]

\[ O(k^2) \]

\[ \Omega(k^2 / \log (k)) \]
Green = can efficiently beat the random assignment. Red = approximation resistant.

pairwise independent CSPs are approximation resistant.

\[ O(k^2) \]
\[ \Omega(k^2/\log(k)) \]
Green = can efficiently beat the random assignment.
Red = approximation resistant.

**basic sdp**
cannot beat random

[Rag08] **UGC**

[AM09]

pairwise independent CSPs are approximation resistant.

O\left(k^2\right)
\Omega\left(k^2 / \log (k)\right)

[AH09,AM09]

# satisfying assignments.
Green = can efficiently beat the random assignment.
Red = approximation resistant.

$\Omega(n)$ round Sherali Adams/Lovász Schrijver

basic sdp
cannot
beat random

$O(k^2)$
$\Omega(k^2 / \log(k))$

[AG09,AM09]

[BGMT09,TW13]
Green = can efficiently beat the random assignment.
Red = approximation resistant.

Sum of Squares?

+ basic sdp
cannot beat random

[BGMT09, TW13]

# satisfying assignments.

[AH09, AM09]

\[ O(k^2) \]
\[ \Omega(k^2 / \log(k)) \]
Sum of Squares SDP

[Grigoriev-Vorobjov99, Shor87, Nesterov99, Parillo00, Lasserre01]

hierarchical strengthening of the basic SDP/spectral algo.

d: “degree” in the hierarchy.

\(n^{O(d)}\): running time
Sum of Squares SDP

[Grigoriev-Vorobjov99, Shor87, Nesterov99, Parillo00, Lasserre01]

... highly successful convex relaxation for comb. optimization.

SPARSEST CUT [ARV04]

UNIQUE GAMES [ABS10, BRS12, GS12]

... breaks known integrality gap instances for weaker hierarchies in \textit{polynomial} time for

UNIQUE GAMES, BALANCED SEPARATOR, MAX CUT
[BBHKSZ12, OZ13, DMN13]

... optimal amongst all poly size SDPs for CSPs.

[LRS15]
SOS vs CSP?
SOS vs CSP?

\[ \text{?} \quad \text{VS} \quad \text{?} \]

[Gri01, Sch08]  k-XOR requires exponential SOS.

[Tul09, Cha13]  Predicates supporting a pairwise independent subgroup* require exponential SOS.

* \( \gamma \) is uniform on a subgroup inside \( P^{-1}(1) \)

SoS lower bounds known only for approx. resistant predicates.
SOS vs CSP?

[Gri01, Sch08] k-XOR requires

[Tul09, Cha13] Predicates supporting a pairwise independent subgroup* require exponential SOS.

Strong algebraic condition. Almost no sparse predicates satisfy!
This Talk

Exponential time SoS cannot beat random for any pairwise independent CSP.

Not known to be approx. resistant.
Our Result

Previous Proofs

Reduction to **Resolution Width** lower bounds algebraic structure crucial.
Our Result

Previous Proofs

Our Proof

Reduction to Resolution Width lower bounds algebraic structure crucial.

“Local Gram Schmidt”
1. Sum of Squares Pseudodistributions

2. The Instance and the Pseudodistribution

3. Local Gram Schmidt - independence lemma
Talk Plan

1. Sum of Squares Pseudodistributions

2. The Instance and the Pseudodistribution

3. Local Gram Schmidt - independence lemma
CSP(P)

Instance

k-tuple of literals in i\(^{th}\) constraint

Constraints

\[ \mathcal{I} : C_1, C_2, ..., C_m \]

\[ x_{C_i} \]

\[ \{P(x_{C_i}) = 1\} \quad 1 \leq i \leq m \]

What’s the value of this poly opt program?

What’s the value of \( \mathcal{I} \)?
CSP(P)

Instance

k-tuple of literals in i\(^{th}\) constraint

Constraints

maximize value of \( \mathcal{I}(x) \)
x is the Boolean assignment.

What’s the value of \( \mathcal{I} \)?

\[
\mathcal{I} : C_1, C_2, \ldots, C_m
\]

\( x C_i \)

\{ \( P(x C_i) = 1 \) \} 1 \leq i \leq m

What’s the value of this poly opt program?

\[
\max \mathcal{I}(x) = \frac{1}{m} \sum_{i=1}^{m} P(x C_i)
\]

\[
x_i^2 = x_i \quad \forall \ i
\]
Sum of Squares Algorithm

What’s the value of this poly opt program?

$$\max \mathcal{I}(x) = \frac{1}{m} \sum_{i=1}^{m} P(x_{C_i}) \quad x_i^2 = x_i \; \forall \; i$$

$\mu$: “a prob. dist. on solutions.”

$\mu$: “knowledge” of best assignments for $\mathcal{I}$. 
Sum of Squares Algorithm

\[ \mu: \text{prob. dist. on solutions.} \]

\[ \mathbb{E} = \mathbb{E}_\mu, \text{ q: any polynomial.} \]

value \quad \mathbb{E}[\mathcal{I}(x)]

\( \mu \) satisfies constraints. \quad \mathbb{E}[q(x)(x_i^2 - x_i)] = 0 \quad \forall i
Sum of Squares Algorithm

\( \mu \) : prob. dist. on solutions.

\( \mathbb{E} = \mathbb{E}_\mu \), q: any polynomial.

- \( \mathbb{E} \) is positive semidefinite.
- \( \mathbb{E} \) satisfies constraints.
- \( \mathbb{E} \) value \( \mathbb{E}[\mathcal{I}(x)] \)

\[ \mathbb{E}[q(x)(x_i^2 - x_i)] = 0 \quad \forall i \]

\[ \mathbb{E}[q^2] \geq 0 \]
Sum of Squares Algorithm

$\mu$ : prob. dist. on solutions.

$\mathbb{E} = \mathbb{E}_\mu$ , q: any polynomial.

$\mathbb{E}[\mathcal{I}(x)]$

$\mu$ satisfies constraints.

$\mathbb{E}[q(x)(x_i^2 - x_i)] = 0 \ \forall i$

$\mathbb{E}$ is positive semidefinite.

$\mathbb{E}[q^2] \geq 0$

$\mathbb{E}$ is normalized.

$\mathbb{E}[1] = 1$
Sum of Squares Algorithm

\[ \mu : \text{prob. dist. on solutions.} \]

\[ \mathbb{E} = \mathbb{E}_\mu, \text{ q: any polynomial.} \]

\[ \mathbb{E} \text{ is linear.} \]

\[ \mu \text{ satisfies constraints.} \]

\[ \mathbb{E} \text{ is positive semidefinite.} \]

\[ \mathbb{E} \text{ is normalized.} \]

\[ \mathbb{E}[\mathcal{I}(x)] \]

\[ \mathbb{E}[\prod_{i \in S} x_i] \]

\[ \mathbb{E}[q(x)(x_i^2 - x_i)] = 0 \quad \forall i \]

\[ \mathbb{E}[q^2] \geq 0 \]

\[ \mathbb{E}[1] = 1 \]
What’s the value of this poly opt program?

\[ \text{max } \mathcal{I}(x) = \frac{1}{m} \sum_{i=1}^{m} P(x_{C_i}) \]

\[ x_i^2 = x_i \ \forall \ i \]

\[ \sum x_i \]

\[ \tilde{\mu} : \text{“a pseudo dist. on solutions.”} \]

\[ \tilde{\mu} : \text{“knowledge” of best assignments for } \mathcal{I} \text{.} \]
Sum of Squares Algorithm

What’s the value of this poly opt program?

\[
\max \mathcal{I}(x) = \frac{1}{m} \sum_{i=1}^{m} P(x_{C_i}) \quad x_i^2 = x_i \ \forall \ i
\]

\[\tilde{\mu} : \text{“a pseudo dist. on solutions.”}\]

of a “computationally bounded” agent

\[\tilde{\mu} : \text{“knowledge” of best assignments for } \mathcal{I} \text{.}\]
Sum of Squares Algorithm

\(\tilde{\mu}:\) pseudo. dist. on solutions.

\(\tilde{\mathbb{E}} = \tilde{\mathbb{E}}\tilde{\mu},\ q: \) any polynomial.

\(\tilde{\mathbb{E}}\) is linear. \textbf{Pseudo Moments} \(\tilde{\mathbb{E}}[\prod_{i \in S} x_i]\)

\(\tilde{\mu}\) satisfies constraints. \(\tilde{\mathbb{E}}[q(x) \left( x_i^2 - x_i \right)] = 0 \ \forall i\)

\(\tilde{\mathbb{E}}\) is positive semidefinite. \(\tilde{\mathbb{E}}[q(x)^2] \geq 0\)

\(\tilde{\mathbb{E}}\) is normalized. \(\tilde{\mathbb{E}}[1] = 1\)
Sum of Squares Algorithm

\[ \tilde{\mu} : \text{pseudo. dist. on solutions.} \]
\[ \tilde{\mathbb{E}} = \tilde{\mathbb{E}}_{\tilde{\mu}}, \text{q: any polynomial.} \]

**Total degree of argument of** \( \tilde{\mathbb{E}} \text{ is at most } d! \)

- \( \tilde{\mathbb{E}} \) is linear.
- **Pseudo Moments** \( \tilde{\mathbb{E}}[\prod_{i \in S} x_i] \)
- \( \tilde{\mu} \) satisfies constraints.
- \( \tilde{\mathbb{E}}[q(x) (x_i^2 - x_i)] = 0 \quad \forall i \)
- \( \tilde{\mathbb{E}} \) is positive semidefinite.
- \( \tilde{\mathbb{E}}[q(x)^2] \geq 0 \)
- \( \tilde{\mathbb{E}} \) is normalized.
- \( \tilde{\mathbb{E}}[1] = 1 \)
Sum of Squares Algorithm

$\hat{\mu}$ : pseudo. dist. on solutions.

$\hat{E} = \hat{E}_{\hat{\mu}}$, q: any polynomial.

Total degree of argument of $\hat{E}$ is at most \( d \)!

Computed using SDP on \( n^{O(d)} \) variables and constraints!

$\hat{\mu}$ satisfies constraints.

$\mathbb{E}[q(x)(x_i^2 - x_i)] = 0 \ \forall i$

$\hat{E}$ is positive semidefinite.

$\mathbb{E}[q(x)^2] \geq 0$

$\hat{E}$ is normalized.

$\mathbb{E}[1] = 1$
Sum of Squares Algorithm

What’s the value of this poly opt program?

\[
\max \mathcal{I}(x) = \frac{1}{m} \sum_{i=1}^{m} P(x_{C_i}) \quad x_i^2 = x_i \quad \forall \quad i
\]

Algorithm

\[
\max \tilde{\mathbb{E}}[\mathcal{I}(x)]
\]

\[\text{SoS}_d(\mathcal{I}): \quad \tilde{\mathbb{E}} \text{ satisfies } x_i^2 = x_i \quad \forall \quad i \]

\[\tilde{\mathbb{E}}: \text{pseudoexpectation of deg } d\]
Proving Lower Bounds

What’s the value of this poly opt program?

\[
\max \mathcal{I}(x) = \frac{1}{m} \sum_{i=1}^{m} P(x_{C_i}) \quad x_i^2 = x_i \quad \forall i
\]

P: pairwise indep pred with \( \gamma \)
Proving Lower Bounds

What's the value of this poly opt program?

\[
\max \mathcal{I}(x) = \frac{1}{m} \sum_{i=1}^{m} P(x_{C_i}) \quad x_i^2 = x_i \quad \forall i
\]

P: pairwise indep pred with \( \gamma \)

Goal

Show that there is an instance \( \mathcal{I} \) with \( m \) constraints:

1) \( \text{val}(\mathcal{I}) \approx \frac{|P^{-1}(1)|}{2^k} \)

2) \( \text{SoS}_d(\mathcal{I}) = 1 \)

for any \( d = \varepsilon n \)
Proving Lower Bounds

What’s the value of this poly opt program?

\[
\max \mathcal{I}(x) = \frac{1}{m} \sum_{i=1}^{m} P(x_{C_i}) \quad x_i^2 = x_i \ \forall \ i
\]

P: pairwise indep pred with \( \gamma \)

Goal

Show that there is an instance \( \mathcal{I} \) with \( m \) constraints:

1) \( \text{val}(\mathcal{I}) \approx \frac{|P^{-1}(1)|}{2^k} \)

2) there is \( \tilde{E} \) of deg \( d \) s.t. \( \tilde{E}[\mathcal{I}(x)] = 1 \)

for any \( d = \varepsilon n \)
Proving Lower Bounds

What’s the value of this poly opt program?

Goal

Show that there is an instance $\mathcal{I}$ with $m$ constraints:

1) $\text{val}(\mathcal{I}) \approx \frac{|P^{-1}(1)|}{2^k}$
2) there is $\tilde{E}$ of deg $d$

s.t. $\tilde{E}[\mathcal{I}(x)] = 1$

Deg $d = \varepsilon n$ SOS “thinks” $\mathcal{I}$ is satisfiable but $\mathcal{I}$ is far from it!
Talk Plan

1. Sum of Squares Pseudodistributions
2. The Instance and the Pseudodistribution
3. Local Gram Schmidt - independence lemma
Hard Instances

$I$: random $k$-hypergraph with $m = \Theta(n)$ clauses*.

1. But kill all $o(\log(n))$ cycles.
2. No two clauses intersect in more than 1 variable.

Take random literals in each clause.

*as in all previous works [BGMT09, TW13, ...]
Hard Instances

\( \mathcal{I} \) : random k-hypergraph with \( m = \Theta(n) \) clauses*

1. But kill all \( o(\log(n)) \) cycles.
2. No two clauses intersect in more than 1 variable.

Take random literals in each clause.

\[
\text{val}(\mathcal{I}) \approx \frac{|P^{-1}(1)|}{2^k}
\]

*as in all previous works [BGMT09, TW13, ...]
d is a large constant (instead of $\epsilon n$)
Constructing $\tilde{E}$

**Goal**

Construct $\tilde{E}$ that thinks every constraint is satisfied.

$$\text{Fix } S \subseteq [n], |S| \leq d. \quad \tilde{E}[x_S] = ?$$

same as [BGMT09, TW13, ...]

P: pairwise indep pred with $\gamma$
Sum of Squares Algorithm

\[ \tilde{\mathbb{E}}[\mathcal{I}(x)] \]

\( \tilde{\mathbb{E}} \) is linear.

PseudoMoments \( \tilde{\mathbb{E}}[\prod_{i \in S} x_i] \)

\( \tilde{\mu} \) satisfies constraints.

\[ \tilde{\mathbb{E}}[q(x) (x_i^2 - x_i)] = 0 \quad \forall i \]

\( \tilde{\mathbb{E}} \) is positive semidefinite.

\[ \tilde{\mathbb{E}}[q(x)^2] \geq 0 \]

\( \tilde{\mathbb{E}} \) is normalized.

\[ \tilde{\mathbb{E}}[1] = 1 \]

Corollary

The marginal of \( \tilde{\mu} \) on any \( d \) variables is an actual prob. distribution supported on solutions.
Constructing $\tilde{E}$

Goal

construct $\tilde{E}$ that thinks every constraint is satisfied.

Fix $S \subseteq [n], |S| \leq d$. $\tilde{E}[x_S(x)] = ?$

Suppose $S = \{1, 2, 3\}$

want marginal on $S$ to be supported on solutions.

P: pairwise indep pred with $\gamma$
Constructing $\tilde{E}$

**Goal**

Construct $\tilde{E}$ that thinks every constraint is satisfied.

$\tilde{E}[x_S(x)] = ?$

Fix $S \subseteq [n], |S| \leq d$.

Suppose $S = \{1, 2, 3\}$

want marginal on $S$ to be supported on solutions.

force marginal on $S$ to be $\gamma$!

P: pairwise indep pred with $\gamma$
Constructing $\tilde{E}$

**Goal**

construct $\tilde{E}$ that thinks every constraint is satisfied.

Fix $S \subseteq [n], |S| \leq d$. $\tilde{E}[x_S(x)] = \?$

What should be the local marginal of $\tilde{\mu}$ on $S$?

expand $S$ into a set $\text{cl}(S)$ that collects variables in all clauses that “affect” $S$

P: pairwise indep pred with $\gamma$
Constructing $\tilde{E}$

**Goal**: construct $\tilde{E}$ that thinks every constraint is satisfied.

- expand $S$ into a set $\text{cl}(S)$ that collects variables in all clauses that “affect” $S$.

**Closure**

1. Start from $S$. Repeat till impossible:

2. Add all variables in any clause that touches at least two vertices in $S$.

$P$: pairwise indep pred with $\gamma$
Constructing $\tilde{E}$

Goal

construct $\tilde{E}$ that thinks every constraint is satisfied.

expand $S$ into a set $\text{cl}(S)$ that collects variables in all clauses that “affect” $S$

Closure

1. Start from $S$. Repeat till impossible:
2. Add all variables in any clause that touches at least two vertices in $S$.

no small cycle can not do more than $\sim |S|$ steps.

P: pairwise indep pred with $\gamma$
Constructing $\tilde{E}$

**Goal**

construct $\tilde{E}$ that thinks every constraint is satisfied.

expand $S$ into a set $\text{cl}(S)$ that collects variables in all clauses that “affect” $S$

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**Closure**

1. Start from $S$. Repeat till impossible:
2. Add all variables in any clause that touches at least two vertices in $S$. 

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Constructing $\tilde{E}$

goal construct $\tilde{E}$ that thinks every constraint is satisfied.

expand $S$ into a set $cl(S)$ that collects variables in all clauses that “affect” $S$

Closure

1. Start from $S$. Repeat till impossible:
2. Add all variables in any clause that touches at least two vertices in $S$. 
Constructing $\tilde{E}$

construct $\tilde{E}$ that thinks every constraint is satisfied.

expand $S$ into a set $\text{cl}(S)$ that collects variables in all clauses that “affect” $S$

How to choose a local marginal on $\text{cl}(S)$ so that it induces $\gamma$ on every clause in it?

P: pairwise indep pred with $\gamma$
Constructing $\tilde{E}$

Construct $\tilde{E}$ that thinks every constraint is satisfied.

How to choose a local marginal on $cl(S)$ so that it induces $\gamma$ on every clause in it?

root the tree
Constructing $\tilde{\mathcal{E}}$

construct $\tilde{\mathcal{E}}$ that thinks every constraint is satisfied.

How to choose a *local marginal* on $\text{cl}(S)$ so that it induces $\gamma$ on every clause in it?

sample root clause from $\gamma$
Constructing $\tilde{\mathcal{E}}$

Goal: construct $\tilde{\mathcal{E}}$ that thinks every constraint is satisfied.

How to choose a *local marginal* on $\text{cl}(S)$ so that it induces $\gamma$ on every clause in it?

- Sample root clause from $\gamma$.
- Sample child clauses from $\gamma$ conditioned on root being sampled!
Constructing $\tilde{E}$

Construct $\tilde{E}$ that thinks every constraint is satisfied.

How to choose a *local marginal* on $cl(S)$ so that it induces $\gamma$ on every clause in it?

- Sample root clause from $\gamma$
- Sample child clauses from $\gamma$ conditioned on root being sampled!
Constructing $\tilde{\Theta}$

can construct $\tilde{\Theta}$ that thinks every constraint is satisfied.

How to choose a \textit{local marginal} on $\text{cl}(S)$ so that it induces $\gamma$ on every clause in it?

sample root clause from $\gamma$

sample child clauses from $\gamma$ conditioned on root being sampled!
Constructing $\tilde{\mathcal{E}}$

**Goal**

construct $\tilde{\mathcal{E}}$ that thinks every constraint is satisfied.

How to choose a *local marginal* on $\text{cl}(S)$ so that it induces $\gamma$ on every clause in it?

- sample root clause from $\gamma$
- sample child clauses from $\gamma$ conditioned on root being sampled!
Constructing $\tilde{E}$

Goal

construct $\tilde{E}$ that thinks every constraint is satisfied.

How to choose a
that it induces $\gamma$ on every clause in it?

sample root clause from $\gamma$
sample child clauses from $\gamma$ conditioned on root being sampled!

can be done because marginals on each variable are \textbf{balanced}!
Constructing $\tilde{E}$

**Goal**

construct $\tilde{E}$ that thinks every constraint is satisfied.

How to choose a
that it induces $\gamma$ on every clause in it?

- sample root clause from $\gamma$
- sample child clauses from $\gamma$ conditioned on root being sampled!
- can be done because marginals on each variable are **balanced**!

\[
\tilde{E}[x_{S(x)}] = \mathbb{E}_{\mu_{cl(S)}}[x_{S(x)}]
\]
Talk Plan

1. Sum of Squares Pseudodistributions

2. The Instance and the Pseudodistribution

3. Local Gram Schmidt - independence lemma
Proof Outline

Goal

1) \( \text{val}(I) \sim \frac{|P^{-1}(1)|}{2^k} \)

2) \( \tilde{E}[I(x)] = 1 \)

3) \( \tilde{E}[p^2] \geq 0 \ \forall \ p \ \text{of deg} \ d = \varepsilon n \)
Proof Outline

1) $\text{val}(I) \sim \frac{|P^{-1}(1)|}{2^k}$

2) $\mathbb{E}[\mathcal{I}(x)] = 1$

3) $\mathbb{E}[p^2] \geq 0 \; \forall \; p \; \text{of deg} \; d = \varepsilon n$
An actual probability dist. satisfies \((3)\) for polynomials of all degrees up to \(n\).
Proof Outline

1) $\text{val}(\mathbf{I}) \sim \frac{|P^{-1}(1)|}{2^k}$

2) $\mathbb{E}[\mathcal{I}(x)] = 1$

3) $\mathbb{E}[p^2] \geq 0 \ \forall \ p \text{ of deg } d = \Theta(1)$
PSDness

Goal

1) $\text{val}(\mathbf{0}) \sim \frac{|P^{-1}(1)|}{2^k}$

2) $\tilde{E}[\mathcal{I}(x)] = 1$

3) $\tilde{E}[p^2] \geq 0 \ \forall \ p \ \text{of deg} \ d = \Theta(1)$

Find $\chi_1, \chi_2, \cdots, \chi_N$ such that:

1) their span is degree $d$ polys.

2) $\tilde{E}[\chi_i^2] \geq 0 \ \forall i$

3) $\tilde{E}[\chi_i \cdot \chi_j] = 0 \ \forall i \neq j$
PSDness

Goal

1) \( \text{val}(I) \sim \frac{|P^{-1}(1)|}{2^k} \)

2) \( \tilde{E}[\mathcal{I}(x)] = 1 \)

3) \( \tilde{E}[p^2] \geq 0 \quad \forall \ p \text{ of deg } d = \Theta(1) \)

Find \( \chi_1, \chi_2, \ldots, \chi_N \) such that:

1) their span is degree \( d \) polys.

2) \( \tilde{E}[^{2}x_i] \geq 0 \ \forall i \)

3) \( \tilde{E}[\chi_i \cdot \chi_j] = 0 \ \forall i \neq j \)

\[
\tilde{E}[p^2] = \tilde{E}[(\sum_{i=1}^{N} p_i \chi_i)^2] = \sum_{i=1}^{n} p_i^2 \tilde{E}[\chi_i^2] \geq 0
\]
PSDness

Goal

1) \[ \text{val}(I) \sim \frac{|P^{-1}(1)|}{2^k} \]

2) \[ \tilde{E}[\mathcal{I}(x)] = 1 \]

3) \[ \tilde{E}[p^2] \geq 0 \quad \forall \ p \text{ of deg } d = \Theta(1) \]

Find \( I \) such that:

1) their span is degree \( d \) polys.

2) \[ \tilde{E}[\chi_i^2] \geq 0 \quad \forall i \]

3) \[ \tilde{E}[\chi_i \cdot \chi_j] = 0 \quad \forall i \neq j \]

local distributions!
1) start from the monomials in some order.
2) run Gram-Schmidt using $\tilde{E}$ inner product.

Find $\chi_1, \chi_2, \ldots, \chi_N$ such that:

1) their span is degree $d$ polys.
2) $\tilde{E}[\chi_i^2] \geq 0 \ \forall i$
3) $\tilde{E}[\chi_i \cdot \chi_j] = 0 \ \forall i \neq j$
PSDness

Find $\chi_1, \chi_2, \ldots \chi_N$ such that:

1) their span is degree d polys.
2) $\tilde{E}[\chi_i^2] \geq 0 \ \forall i$
3) $\tilde{E}[\chi_i \cdot \chi_j] = 0 \ \forall i \neq j$

1) start from the monomials in some order.
2) run Gram-Schmidt using $\tilde{E}$ inner product.

$$\chi_k = x_{S_k} - \sum_{i < k} \chi_i \tilde{E}[x_{S_k} \cdot \chi_i]$$

$\tilde{E}$ corresponds to an actual expectation associated with a local prob. dist.
PSDness

1) start from the monomials in some order.

2) Gram Schmidt

Circular argument!
Key Idea: “Independence Lemma”

\(|S| \leq d\)

\(|T| \leq d\)
Key Idea: “Independence Lemma”

Local distributions on $S$ and $T$ given by $\tilde{E}$ are “independent”.

$S$, $T$ “far enough”

$|S| \leq d$  
$|T| \leq d$

$O(d)$
Key Idea: “Independence Lemma”

\[ |S| \leq d \quad \Rightarrow \quad \text{cl}(S) \supseteq O(d) \]

\[ |T| \leq d \quad \Rightarrow \quad \text{cl}(T) \supseteq O(d) \]

Intuition:

Ind. Lemma \( \tilde{E} \) captures no long range correlations!

BUT!

Any \( \sim \varepsilon n \) constraints in the instance are \textit{highly} satisfiable!
Key Idea: “Independence Lemma”

\(|S| \leq d\)

\(|T| \leq d\)

\(O(d)\)

\(\text{cl}(S)\)

\(\text{cl}(T)\)

**Intuition**

Ind. Lemma $\Rightarrow$ $\tilde{E}$ captures no long range correlations! BUT!

Any $\sim \varepsilon n$ constraints in the instance are

“Hypergraph Expansion”

Small sets of constraints is almost disjoint. + pairwise independence
Key Idea: “Independence Lemma”

$|S| \leq d$

$|T| \leq d$

\[ \widetilde{E} \]

Local distributions on $S$ and $T$ given by $\widetilde{E}$ are “independent”.

Orthogonalize only in a local neighborhood.

$S$, $T$ “far enough”
PSDness

Find $\chi_1, \chi_2, \ldots, \chi_N$ such that:

1) their span is degree $d$ polys.
2) $\tilde{E}[\chi_i^2] \geq 0 \ \forall i$
3) $\tilde{E}[\chi_i \cdot \chi_j] = 0 \ \forall i \neq j$

1) start: monomials in a **carefully chosen** order.
PSDness

Find $\chi_1, \chi_2, \ldots, \chi_N$ such that:

1) their span is degree $d$ polys.
2) $\widetilde{E}[\chi_i^2] \geq 0 \ \forall i$
3) $\widetilde{E}[\chi_i \cdot \chi_j] = 0 \ \forall i \neq j$

1) start: monomials in a **carefully chosen** order.

2) run a step of Gram-Schmidt using $\widetilde{E}$ inner product

....but only subtract components on monomials that belong to the local neighborhood.
PSDness

Find $\chi_1, \chi_2, \ldots \chi_N$ such that:

1) their span is degree d polys.
2) $\mathbb{E}[(\chi_i)^2] \geq 0 \ \forall i$
3) $\mathbb{E}[\chi_i \cdot \chi_j] = 0 \ \forall i \neq j$

1) start: monomials in a **carefully chosen** order.

2) run a step of Gram-Schmidt using $\tilde{\mathbb{E}}$ inner product

    ....but only subtract components on monomials that belong to the local neighborhood.

3) Hope for the best.
PSDness

Find $\chi_1, \chi_2, \ldots, \chi_N$ such that:

1) their span is degree $d$ polys.

2) $\tilde{E}[\chi_i^2] \geq 0 \ \forall i$

3) $\tilde{E}[\chi_i \cdot \chi_j] = 0 \ \forall i \neq j$

1) start: monomials in a **carefully chosen** order.

2) run a step of Gram-Schmidt using $\tilde{E}$ inner product

....but only subtract components on monomials that belong to the local neighborhood.

3) every little thing is gonna be alright!
Exponential time SoS cannot beat random for any pairwise independent CSP.
What next?

1. Simpler proof that avoids painful combinatorics?
2. Do without pruning random instances - SoS lower bounds for refuting random CSPs.
3. Prove NP hardness?
Questions?
A CSP(P) is **pairwise independent** if there is a prob. dist. $\mu$

1) pairwise independent.
2) supported on $P^{-1}(1)$

[AH09] random predicates with $\Omega(k^2)$ satisfying assignments are pairwise independent.
Pairwise Independence

A CSP(P) is **pairwise independent** if there is a prob. dist. $\mu$

1) pairwise independent.
2) supported on $P^{-1}(1)$

[AH09] sparse p.i. preds are abundant.

[AM09]

UGC pairwise independent CSPs are approximation resistant*.

*can’t beat random assignment.
Pairwise Independence

\[ x_1 \ x_2 \ x_3 \]
\[ x_4 \]
\[ x_5 \ x_8 \ x_6 \ x_7 \ x_{13} \]
\[ x_9 \]
\[ x_{10} \ x_{11} \ x_{12} \]

[**Austrin-Mossel09**]

**UGC** \[\rightarrow\] p.i. CSPs are hard.

[**Raghavendra08**] For CSPs

**UGC**

Hardness \[\leftrightarrow\]

integrality gaps for "basic" SDP
Pairwise Independence

[AM09]
Basic SDP can't beat random assignment for p.i. CSPs.
Pairwise Independence

[AM09]
Basic SDP doesn’t beat random assignment for p.i. CSPs.

[BGMT09] [TW13]
Sherali Adams/Lovasz-Schrjiver SDPs of exponential size do not beat random assignment for p.i. CSPs.
Sum of Squares SDP .....in a slide

Pseudoexpectations

A linear operator $\tilde{E}$ on degree $d$ polynomials on $\{0, 1\}^n$ satisfying:

$$\tilde{E}[p^2] \geq 0$$

for every degree $d/2$ polynomial $p$. 
Sum of Squares SDP

.....in a slide

Pseudo expectations

A linear operator \( \tilde{E} \) on degree d polynomials on \( \{0, 1\}^n \) satisfying:

\[ \tilde{E} [p^2] = 0, \]

for every degree d/2 polynomial p.

Value of CSP(\( P \)) at an assignment x