Discrete-Time Signal Design for Maximizing Separation in Amplitude

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***Abstract*—Given a discrete-time, linear, shift-invariant channel with finite impulse response, the problem of designing finite-length input signals with bounded amplitude (l_∞ norm) such that the corresponding output signals are maximally separated in amplitude (l_∞ sense) is considered. In general, this is a non-convex optimization problem, and appears to be computationally difficult. An optimization algorithm that seems to perform well is described. Optimized signal sets and associated minimum distances (minimum l_∞ separation between two distinct channel outputs) are presented for some example impulse responses. A conjectured upper bound on the minimum distance is given that is easily computed given the impulse response of the channel, the number of inputs, and the input length. This upper bound is shown to be valid for a limited class of impulse response functions.**

***Index Terms*—Signal design, amplitude constraint, intersymbol interference.**

I. Introduction

SIGNAL design in digital communications typically refers to the way in which source bits are mapped to the channel input. The objective is typically to optimize some performance criterion, such as probability of error, given a description of the channel, and subject to certain constraints on the receiver and on the channel inputs. For discrete-time, real-valued channels (i.e., channels that allow real-valued inputs and produce real-valued outputs), a typical performance criterion is minimum Euclidean distance between two distinct channel output sequences. This criterion is especially appropriate when the channel and receiver noise is assumed to be additive and Gaussian.

Here we consider signal design for a different class of linear, discrete-time channels than are normally considered. Namely, we assume that the channel and receiver noise is additive, and is bounded in *amplitude,* say, by $d/2$. This implies that distinct channel outputs can be distinguished at the receiver provided that their minimum pairwise separation in amplitude is at least d. In addition, we assume that the channel inputs are bounded in amplitude, and are time-limited to the interval $[0, K-1]$. The problem considered is therefore the design of N *finite-length input sequences* to a given discrete-time, linear, shift-invariant channel with finite impulse response so that the corresponding N outputs are maximally separated in l_∞ norm. Because the inputs are assumed to be bounded in the l_∞ sense, and the outputs are to be separated in the l_∞ sense, we refer to this problem as the l_∞/l_∞ *signal design problem.* The continuous-time version of this problem has previously been considered in [1].

This signal design problem was motivated by situations in which system performance is limited primarily by the precision with which the receiver can measure the channel output (i.e., the precision of the A/D converter). An application for which this type of signal design may be especially appropriate is the "Asymmetric Digital Subscriber Line" [2]. In this case, the channel is copper wire, which has a very low ambient noise level. Since transmission is half-duplex, the only type of crosstalk present is far-end crosstalk, which is highly attenuated. It is therefore likely that measurement error at the receiver will pose a major limitation on achievable data rate.

In general, it seems difficult to prove the optimality of a given signal set for a specific channel. Consequently, the approach taken here is to view this signal design problem as a nonconvex optimization problem, and attempt to solve it numerically. We present a numerical algorithm for finding locally optimal solutions which appears to work well on relatively small problems. We find that the algorithm typically finds many local optima, and that the number of these local optima grows extremely rapidly as the size of the problem (i.e., input length and number of inputs) increases. Some examples are presented in which the computer generated solutions are conjectured to be optimal.

Before presenting the numerical optimization algorithm we first consider the continuous-time version of the l_∞/l_∞ signal design problem, and state a conjectured upper bound on the asymptotic information rate for which a specified minimum (L_∞) distance between outputs can be achieved. This conjectured upper bound is uniformly better than a previous upper bound that was presented in [3]. As supporting evidence for the conjecture, the upper bound is shown to apply to a specific class of channel impulse response functions. A discrete-time version of the bound is also presented, and is consistent with the results obtained from the numerical search algorithm.

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II. Continuous-Time L_∞/L_∞ Signal Design

Consider a linear, time-invariant channel with continuous-time impulse response $h(t)$, where $h(t) = 0$ for $t \notin [0, \tau]$, and

$$\int_0^\tau |h(t)|dt < \infty \qquad (\text{i.e., } h(t) \in L_1).$$

All inputs to the channel satisfy $u(t) = 0$ for $t \notin [0, T)$, and it assumed that $|u(t)| \leq 1$ for all t. Given a rate R (bits per second), and input length T, we wish to find $N = \lfloor 2^{RT} \rfloor$ inputs $u_1(\cdot), \cdots, u_N(\cdot)$ to

$$\max\left\{ d = \min_{i \neq j} \| y_i(\cdot) - y_j(\cdot) \|_\infty \right\} \tag{1}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x

$$y_i(t) = \int_0^T h(t-s)u_i(s)ds$$

and the L_∞ norm of any continuous function f is

$$\| f \|_\infty = \sup |f(t)|.$$

This problem is called L_∞/L_∞ signal design since the inputs are restricted in L_∞ norm, and the outputs are to be separated in the L_∞ sense.

As mentioned previously, this problem pertains to communications systems in which the noise may be amplitude-limited (i.e., noise due to the A/D converter). This type of signal design also seems appropriate when the receiver estimates each transmitted symbol independently by sampling the channel output at a particular time and using a simple threshold device. In this case, the receiver is less likely to make a detection error if the received samples corresponding to different source sequences are widely separated.

For fixed input length T and minimum distance d, we define the maximum number of inputs that can be separated by d at the channel output as

$$N_{\max}(T, d) = \max\left\{ N: \min_{i \neq j} \| y_i - y_j \|_\infty \geq d \right\} \tag{2}$$

and the *Maximum Channel Throughput* (MCT) is defined as

$$\text{MCT}(d) = \lim_{T \to \infty} \frac{\log_2 N_{\max}(T, d)}{T} \quad (b/s). \tag{3}$$

The MCT is therefore the maximum asymptotic information rate for which the minimum distance between outputs is at least d. Consider an additive noise channel consisting of the original channel with impulse response $h(t)$ followed by an additive noise source $n(t)$ where $|n(t)| < d/2$, but otherwise $n(t)$ has arbitrary statistics. Clearly, by using a signal set with minimum distance d as the set of channel inputs, it is possible to design the receiver so as to achieve zero error probability. MCT (d) for the channel $h(t)$ is therefore a lower bound for the zero-error capacity of the corresponding additive noise channel.

III. Conjectured Upper Bound on the MCT

Upper and lower bounds on the MCT have been presented previously in [3], and it is shown in [1] and [3] that these bounds are tight for the impulse response $h(t) = e^{-\alpha u}$, $\alpha > 0$. The upper bound, however, tends to be quite loose when the impulse response is highly oscillatory, and in fact diverges as the frequency of oscillation tends to infinity. However, other considerations based on bounding the volume of the region in signal space defined by the input constraints indicate that the MCT remains finite. We now present a conjectured upper bound which is uniformly better than the upper bound presented in [3]. This conjectured bound is also tight for the impulse response $h(t) = e^{-\alpha u}$, and in addition, is finite for all $h(t) \in L_1$.

For a given $c > 0$, define the set $A(c)$ so that if $t \in A(c)$ then $|h(t)| \geq c$, and if $t \notin A(c)$, then $|h(t)| \leq c$. Now for a given d, consider all sets $A[c(d)]$ for which

$$\int_{A[c(d)]} |h(t)|\, dt \geq \frac{d}{2},$$

and define

$$c^*(d) = \sup\left\{ c: \int_{A[c(d)]} |h(t)|\, dt \geq \frac{d}{2} \right\}. \tag{4}$$

Furthermore, let $\tau(d) = \text{meas}\, A[c^*(d)]$. We will refer to sets $A[c^*(d)]$ as "minimal" sets, since $\tau(d) \leq \text{meas}\, A[c(d)]$ for any set $A[c(d)]$ defined previously. Note that $h(t) = c(d)$ for any t in the symmetric difference between two minimal sets. For a given d, if $h(t)$ is continuous, then $c(d)$ is uniquely defined. Furthermore, if in addition the set of t for which $h(t) = c(d)$ has measure zero, then the symmetric difference between any two minimal sets has measure zero.

Conjecture:

$$\text{MCT}(d) \leq \frac{1}{\tau(d)}. \tag{5}$$

To see why the conjecture might be true, first consider the case of designing $N = 2$ inputs, u_1 and u_2, to maximize the minimum L_∞ distance between outputs. The two inputs must satisfy $u_1 = -u_2$, and the corresponding minimum distance is then $d = 2|y_1(t_0)|$, where t_0 is the time at which $|y_1|$ achieves its maximum value. If the inputs have support on a set of measure T where $T \leq t_0$, then

$$\left| \int_0^{t_0} h(t_0 - s)u_1(s)ds \right| \leq \int_{A(c)} |h(t)|dt \tag{6}$$

where c is selected so that meas $A(c) = T$. Equality is achieved if $u_1(s) = \text{sgn}\, h(t_0 - s)$ for $s \in A(c)$, and $u_1(s) = 0$ for $s \notin A(c)$. Since meas $A(c) \geq \tau(d)$, it therefore takes at least time $\tau(d)$ to distinguish one bit, given that only one bit is being transmitted.

The conjectured upper bound has the following properties, which seem intuitively reasonable. First, if the impulse response is nonincreasing and nonnegative, then this upper bound can be achieved by "bit-by-bit" signaling [1]. That is, the transmitted signal is the square wave associated with the

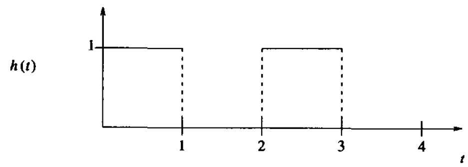

Fig. 1. Example impulse response.

sequence of source bits. The duration of each bit τ is selected so that

$$\int_0^{\tau(d)} h(t)dt = \frac{d}{2}.$$

In this case a minimal set is $A[c^*(d)] = [0, \tau(d)]$. This signaling scheme has been proven optimal (i.e., achieves the MCT) for $h(t) = e^{-\alpha u}$, $\alpha > 0$. The second property is that the upper bound is invariant to time translations of $h(t)$, and third, the upper bound is the same for $h(t)$ and $|h(t)|$. Finally, the discrete-time version of this conjecture, which will be given in the next section, is consistent with the numerical results in Section VI.

Although there currently is no proof of the conjecture for arbitrary $h(t) \in L_1$, the following theorem specifies a class of impulse response functions for which the conjecture is true.

Theorem: Let $h(t) \geq 0$ for all t, $d > 0$, and suppose that $c^*(d) > 0$. Let

$$\overline{h}(t) = \max[h(t), c^*(d)]. \tag{7}$$

Then $\mathrm{MCT}(\overline{h}; d) \leq 1/[\tau(d)]$.

The proof is given in the Appendix. It is also indicated there how this theorem might be generalized so as to apply to both positive and negative $h(t)$.

If it were the case that $\mathrm{MCT}(h_1; d) \geq \mathrm{MCT}(h_2; d)$ whenever $|h_1(t)| \geq |h_2(t)|$ for all t, then the theorem would imply that the conjecture is true for all $c(d) \geq 0$. The following example, however, suggests that this "monotone" property is unlikely to be true. Consider the impulse response shown in Fig. 1. For $d = 4$, the conjecture says that MCT(4) $\leq 1/2$. In this case signal sets that achieve this MCT can be constructed explicitly. Specifically, for each sequence of source bits $\{b_k\}$, where $b_k \in \{\pm 1\}$, the corresponding transmitted waveform is

$$s(t) = \sum_k (b_{2k}p(t-4k) + b_{2k+1}p(t-4k-1)) \tag{8}$$

where

$$p(t) = \chi_{[0,1)} + \chi_{[2,3)} \tag{9}$$

and $\chi_{[a,b)}$ is the characteristic function for the interval $[a, b)$. It is easily verified that two distinct channel outputs can separate by $d = 4$ only at times $4k$, $4k+1$, $k = 1, 2, \cdots$.

Consider now the impulse response $h_a(t) = \max[h(t), a]$, where $h(t)$ is shown in Fig. 1. If the MCT satisfies the monotone property, then we would have that MCT $(h_a; 4) \geq 1/2$. However, it seems unlikely that this rate of 1/2 can be achieved when $0 < a \leq 1$. That is, at a rate of 1/2 b/s the sections where $h(t) = a$ appear to *decrease* the minimum distance below what it would be for $a = 0$. Note that for $0 < a \leq 1$ the conjecture states that MCT $(h_a; 4) \leq 1/2$, and that as a increases beyond one, the conjectured upper bound increases beyond one half. This discussion will be developed further in Section VI where a discrete-time version of this impulse response (i.e., $h(D) = 1 + aD + D^2$) will be used to generate numerical results.

IV. Discrete-Time (l_∞/l_∞) Signal Design

In this case the set of inputs can be represented by the vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N$. Assuming that each channel input sequence is zero outside of the time interval $[0, K-1]$ implies that each input vector has K components. Letting $\boldsymbol{y}_i$ denote the output vector corresponding to $\boldsymbol{u}_i$, then

$$\boldsymbol{y}_i = \boldsymbol{H}\boldsymbol{u}_i \tag{10}$$

where $\boldsymbol{H}$ is the appropriate $(K+m-1) \times K$ lower triangular Toeplitz convolution matrix, m is the length of the channel impulse response, which is assumed to be finite, and $\boldsymbol{y}_i$ has $K + m - 1$ components. The problem of maximizing the minimum pairwise separation in l_∞ can be expressed as finding values for the $K \cdot N$ components of $\boldsymbol{u}_i$, $i = 1, \cdots, N$, to

$$\max\left\{d = \min_{i \neq j} \max_{0 \leq k \leq K+m-1} |y_i[k] - y_j[k]|\right\} \tag{P}$$

subject to

$$|u_i[k]| \leq 1 \tag{C}$$

for all i and k, where $u_i[k]$ $(y_i[k])$ is the kth component of $\boldsymbol{u}_i$ $(\boldsymbol{y}_i)$. The maximum value of d for given R and K will be denoted as $d_{\max}(K)$, and will be referred to as "max–min" distance.

The l_∞/l_∞ signal design problem has the following geometric interpretation. The amplitude constraint means that the input vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N$ must lie within the unit cube in $\boldsymbol{R}^K$. The corresponding output vectors $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_N$ must therefore lie within a parallelopiped in $\boldsymbol{R}^{K+m-1}$ defined by the vectors $\boldsymbol{H}\boldsymbol{e}_i$, $i = 1, \ldots, K$, where $\boldsymbol{e}_i$ is the ith unit vector in $\boldsymbol{R}^K$. Since the output points must be separated in the l_∞ sense by d, this means that N nonintersecting cubes in $\boldsymbol{R}^{K+m-1}$ of length d on a side can be placed so that an output point is at the center of each cube. Of course, cubes corresponding to points on the boundary region are not entirely contained within the boundary region.

The l_∞/l_∞ signal design problem is therefore closely related to packing cubes within a parallelepiped. Although there is an extensive mathematical literature on packing problems, much of this literature is concerned with packing spheres, instead of cubes. References [4]–[7] are concerned with packing parallelepipeds into larger cubes ([6] and [7] only consider the problem in $\mathbb{R}^2$). However, the emphasis in all of this work is on proving the existence of packings that cover the entire region within the boundary, and on estimating the amount of "wasted space," or portion of the region which is not covered. It appears that the packing problem considered here has not yet received significant attention.

The objective function d is a piecewise linear function defined over the unit cube in $K \cdot N$-dimensional space. In general (P-C) is a nonconvex optimization problem, and may have local optima that are not globally optimal. The discrete-time MCT can be defined exactly as in the continuous-time case, i.e., by (2) and (3), where for any bounded vector $\boldsymbol{v}$ its l-infinity norm is $\| \boldsymbol{v} \|_\infty = \max |v_i|$, and where T is replaced by K.

To state the discrete-time version of the conjectured upper bound on MCT, assume that the rate $R = (\log_2 N)/K = 2^I$, where I is a finite integer (can be positive or negative). Generalization to arbitrary R is straightforward, but is omitted for simplicity. Let the sequence $\tilde{h}[k]$, $k = 0, \cdots, m-1$, be the magnitudes of the channel impulse response coefficients arranged in *nonincreasing* order. That is

$$\tilde{h}[0] = \max_{0 \le k \le m-1} |h[k]|$$

and

$$\tilde{h}[0] \ge \tilde{h}[1] \ge \cdots \ge \tilde{h}[m-1]$$

where $h[k]$ is the channel impulse response, and $h[k] = 0$, $k \notin [0, m-1]$.

Conjecture:

$$d \le \begin{cases} 2 \sum_{i=0}^{R^{-1}-1} \tilde{h}[i], & \text{if } R \le 1 \\ \frac{2\tilde{h}[0]}{2^R - 1}, & \text{if } R > 1. \end{cases} \tag{11}$$

The plausibility arguments and theorem presented in support of the continuous-time conjecture are easily modified so as to apply to the discrete-time case. If $R \le 1$, then the conjecture states that the minimum distance can be no greater than twice the largest magnitude of the output which results from selecting the input $u[k] = \operatorname{sgn} h[i-k]$ for k such that $|h[i-k]| \ge c^*(d)$ and $u[k] = 0$ elsewhere, where $c^*(d)$ is defined analogously to the continuous-time quantity, and i is the time at which $|y[i]|$ is a maximum. If $R > 1$, then the conjectured upper bound depends only on $\max_k |h[k]|$. In this case, the conjectured upper bound is achieved by simple multilevel signaling in which n incoming source bits are mapped to one of 2^n levels between -1 and 1.

V. Computational Algorithms

We now describe a heuristic computational approach to obtaining solutions to (P-C). The algorithms to be described do not guarantee global optimality, but have been run on problems of moderate size with consistent results, and appear to yield very good, if not optimal, solutions for the examples tried.

We need a few definitions. An *extremal dimension* k_{ij} between an ordered pair of output vectors $\boldsymbol{y}_i$ and $\boldsymbol{y}_j$ is any time for which

$$|y_i[k_{ij}] - y_j[k_{ij}]| = d \tag{12}$$

and the *extremal sense* associated with this extremal dimension is

$$\sigma(k_{ij}) = \operatorname{sgn}(y_i[k_{ij}] - y_j[k_{ij}]). \tag{13}$$

The *shape* of a set of N vectors $\boldsymbol{y}_i$, $i = 1, \cdots, N$, is a set of extremal dimensions and associated senses for each pair i, j, $1 \le i < j \le N$. Loosely speaking, for each pair of output vectors the shape tells us the dimension in which they are maximally separated, and the sense in which they are ordered along that dimension.

Once the shape of a solution is known, the optimal inputs can be determined by solving a linear program. Specifically, for each pair of signals we enforce the constraint $\sigma(k_{ij})(y_i[k_{ij}] - y_j[k_{ij}]) \ge d$. These $N(N-1)/2$ constraints imply that the minimum distance is at least d. We therefore wish to maximize d subject to these constraints and the amplitude constraints on the input signals $\boldsymbol{u}_i$. The linear program is then

$$\max d \tag{14a}$$

such that

$$\sigma(k_{ij})(y_i[k_{ij}] - y_j[k_{ij}]) \ge d, \qquad i, j, 1 \le i < j \le N \tag{14b}$$

and

$$|u_i[k]| \le 1, \qquad i = 1, \cdots, N, \quad k = 0, \cdots, K-1. \tag{14c}$$

The general approach consists of two stages:

1) Search for a reasonably good solution by using an iterative ascent algorithm based on either steepest ascent or random perturbations.
2) Find the shape of that solution, and use the linear program (14) to find the optimal solution with that shape.

In practice we found it effective to run the algorithm many times with different initial inputs chosen at random, and to stop the search in step 1) well before convergence to go on to step 2). The idea is that the shape of the solution can be found relatively quickly via an iterative ascent method, to be explained shortly, and that the optimal solution (for that shape) can be computed quickly from the linear program. The simplex method was used to solve the linear program.

The iterative ascent method in step 1) requires an expression for the gradient of the objective function (d) with respect to the components of the input vectors. This assumes, of course, that the objective is differentiable at this point, which may not be true. For each fixed $i = 1, \cdots, N$ one of the following two cases must apply:

$$\min_{j \ne i} \max_k |y_i[k] - y_j[k]| > d \tag{15a}$$

in which case $\nabla_{\boldsymbol{u}_i} d = 0$, or

$$\min_{j \ne 1} \max_k |y_i[k] - y_j[k]| = d. \tag{15b}$$

In the latter case, suppose that the minimum occurs only for $j = j_0$. Let K_{i,j_0} denote the set of extremal dimensions,

$$K_{i,j_0} = \{k: |y_i[k] - y_j[k]| = d\}. \tag{16}$$

In this case

$$\frac{\partial d}{\partial u_i[k]} = \min_{k \in K_{i,j_0}} H_{ik} \sigma(k_{i,j_0}) \tag{17}$$

where H_{ik} is the ikth component of $\boldsymbol{H}$, and $\sigma(k_{ij})$ is defined by (13). If the minimum in (16) occurs at two different values of k, then in general the derivative does not exist. Furthermore, the gradient $\nabla_{\boldsymbol{u}_i} d$ does not exist in general if the minimum in (15b) occurs for two different values of j. Of course, perturbing $\boldsymbol{u}_i$ in any direction by a very small distance ϵ generally eliminates this problem. The gradient portion of the algorithm, which was implemented, computed the gradient under the assumption that the minima in (15b) and (17) occur at unique values of j and k, respectively. This is followed by a line search along that descent direction to find the optimal step length.

Clearly, an optimal packing of many signal vectors must have the property that a particular vector $\boldsymbol{u}_i$ has more than one nearest neighbor. In this case, the minimum in (15b) holds for more than one j. The gradient $\nabla_{\boldsymbol{u}_i} d$ therefore does not exist in general when the signal vectors are optimally packed. This is one motivation for using random perturbations, instead of a gradient type of search, to find a good shape for the signal set.

The two iterative ascent methods tried in step 1) yielded similar results. The random search method requires many more steps to achieve the same increase in minimum distance as the steepest ascent method, but each step is much less expensive computationally. The practical limits of the algorithm are reached not because of time but memory, because the linear program has $N(N-1)/2 + KN$ constraints.

VI. Numerical Results

In this section we present the results obtained from running the previous optimization algorithms for a few partial response channels, namely the $1-D$ and $1+aD+D^2$ channels, where D is the delay operator and a is a constant. Consider first the $1-D$ channel with $K = 2$ and $R = 1/2$ b/symbol so that $N = 2$. That is, two input vectors of length two are to be chosen to maximize the minimum l_∞ distance between the two corresponding outputs. The solution in this case is clearly

$$\boldsymbol{u}_1' = [-11] \qquad \boldsymbol{u}_2 = -\boldsymbol{u}_1 \tag{18}$$

which gives $d = 4$. Note also that this solution achieves the conjectured upper bound (11). For a fixed information rate $R = 1/2$ b/symbol, longer code sequences can be constructed by taking the Cartesian product of the previous signal set in (18) with itself. That is, denoting the preceding signal set as $U = [\boldsymbol{u}_1, \boldsymbol{u}_2]$, then a signal set of length K containing $2^{K/2}$ vectors that achieves $d = 4$ is $U^{K/2}$ (assuming K is even). In this case, the conjectured upper bound (11) is achieved for each even K, and the conjecture furthermore implies that MCT (4) $= 1/2$ b/symbol.

The preceding simple example illustrates an important property of signal design using the l_∞ criterion, which is apparently quite different from more conventional signal design using the l_2 distance criterion. Namely, for fixed R the max–min distance is a nondecreasing function of input length K, and seems to have a *finite* asymptote which is achieved for *finite* K. That is, for the channels considered our results indicate that there exists an input length K_0 for which $d_{\max}(K_0) \geq d_{\max}(K)$ for all K. For many impulse response sequences and information rates, this follows from the conjectured upper bound (11). That is, in many cases, the conjectured upper bound (11) can be achieved by an explicit construction of signal sets, which have the preceding property. In contrast, if the channel inputs are power-constrained, and are selected to separate the outputs in the l_2 (rather than l_∞) sense, then as the input length becomes large, the analogous l_2 max–min distance increases linearly with input length.

The numerical algorithms of Section V were also run for the $1-D$ channel for the case $R = 1$ b/symbol, $K = 2$, and $N = 4$, and the best solution found was

$$\boldsymbol{u}_1' = [1\ 1] \qquad \boldsymbol{u}_2' = [1\ -1] \qquad \boldsymbol{u}_3 = -\boldsymbol{u}_1 \qquad \text{and} \quad \boldsymbol{u}_4 = -\boldsymbol{u}_2 \tag{19}$$

which gives $d = 2$. That is, binary signaling was the best solution found. This solution also satisfies the conjectured upper bound (11) with equality. This suggests that $d_{\max}(K)$ achieves its maximum for $K = 1$, and MCT (2) $= 1$ b/symbol.

Consider now the $1 + D^2$ channel with $R = 1/2$ b/symbol, and $K = 4$. This channel is analogous to the continuous-time channel shown in Fig. 1. The best solution found was

$$\boldsymbol{u}_1' = [1\ 1\ 1\ 1] \qquad \boldsymbol{u}_2' = [1\ -1\ 1\ -1] \qquad \boldsymbol{u}_3 = \boldsymbol{u}_1 \qquad \boldsymbol{u}_4 = -\boldsymbol{u}_2 \tag{20}$$

which gives $d = 4$ and achieves the conjectured upper bound (11). It therefore appears that optimal signal sets of length $4m$, where m is a positive integer, can be constructed by taking Cartesian products of this signal set with itself. If this is true, then $d_{\max}(K)$ achieves its maximum for $K = 4$, and MCT (4) $= 1/2$ b/symbol.

Finally, consider the channel $h(D) = 1 + aD + D^2$ for $R = 1/2$ and $K = 4$. Fig. 2 shows a plot of the max–min distance found by the optimization programs versus a for specific values of a between $a = 0$ and $a = 2$. As an example, for $a = 0.4$ one of the best solutions found is

$$\boldsymbol{u}_1' = [1\ 1\ 1\ 1] \qquad \boldsymbol{u}_3' = \left[1\ -\frac{5}{9}\ 1\ -1\right] \qquad \boldsymbol{u}_3 = -\boldsymbol{u}_1 \qquad \boldsymbol{u}_4 = -\boldsymbol{u}_2 \tag{21}$$

which gives $d = 32/9$. It is easily shown that if the shape of the solution, as defined in the preceding section, is the same as for the solution shown in (21) for $a = 0.4$, then the max–min distance is $d = 3 + (1 + 2a)^{-1}$. Fig. 2 indicates that this shape does yield an optimal solution for small a; however, the discontinuity in the derivative suggested in Fig. 2 is due to the fact that the shape of the best solution changes at $a = 1/\sqrt{2}$. For any $a > 1/\sqrt{2}$ the best solution found is

$$\boldsymbol{u}_1' = [1\ 1\ 1\ 1] \qquad \boldsymbol{u}_2' = [1\ 1\ -1\ -1] \qquad \boldsymbol{u}_3 = -\boldsymbol{u}_1 \qquad \boldsymbol{u}_4 = -\boldsymbol{u}_2 \tag{22}$$

which gives $d = 2(1 + a)$. Notice that $d_{\max}$ in this region would be the same for the impulse response $h(D) = 1 + aD$, and that these results are consistent with the conjectured upper bound (11).

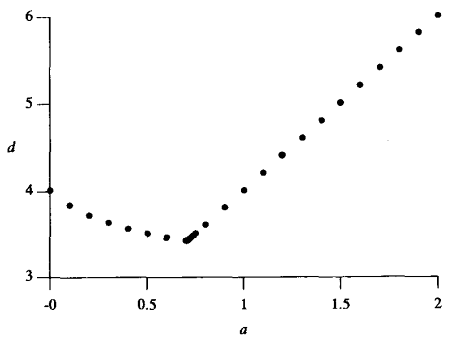

Fig. 2. Max–min distance found by the numerical search algorithms for the channel $h(D) = 1 + aD + D^2$.

VII. Conclusions

A class of numerical optimization algorithms have been proposed for the l_∞/l_∞ signal design problem that seem to perform well for small input lengths and moderate information rates. The performance of the proposed algorithms degrades for larger sized problems because the number of possible solution shapes grows extremely fast with input length. Furthermore, the number of local optima also appears to become quite large, greatly increasing the chance that the algorithm will yield a locally optimal solution that is not globally optimal. In contrast to signal design using the l_2 criterion (i.e., the l_2/l_2 signal design problem [8]), in principle the globally optimal solution for the l_∞/l_∞ problem can always be found by an exhaustive search over all solution shapes, and by solving the associated linear programs.

There are, of course, many remaining unanswered questions suggested by this work. In addition to establishing the conjectures (5) and (11), it is also of interest to determine, for a given channel and information rate, the minimum input length K for which $d_{\max}(K)$ achieves its asymptotic value.

Finally, another interesting question is what is the computational complexity class of the l_∞/l_∞ signal design problem? A related question has been posed in [9] and [10], where the l_2 criterion is used to separate channel outputs. It is conjectured in [9] that this related signal design problem using the l_2 norm (for two inputs) is NP-hard. The l_∞ signal design problems considered here are easier to solve in some cases than the analogous problems using the l_2 criterion (i.e., explicit solutions for the l_∞/l_∞ problem are easily obtained for any impulse response when $N = 2$). However, the question of whether or not this class of problems, or a subset of this class of problems, is NP-hard is open.

Appendix
Proof of Theorem

For any function $f(t)$, let

$$[f(t)]^+ = \begin{cases} f(t), & \text{if } f(t) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{A1}$$

and let $[f(t)]^-$ be defined in the analogous way. The theorem relies on the following Lemma, which is a generalization of [1, Lemma A.8]:

Lemma: Let $h(t)$ be an integrable function, and suppose $d > 0$ is chosen so that $\int |h| \geq \frac{d}{2}$.

Let the set $A[c^*(d)]$ be defined as in Section III, and suppose that

$$\int_{A[c^*(d)]} |h| = \frac{d}{2}.$$

If u is an integrable function, $|u| \leq 1$, and

$$\int_0^T h(T-s)u(s)\,ds \geq \frac{d}{2}$$

at some time T, then

$$\int_0^T [u(s)\,\mathrm{sgn}\,h(T-s)]^+ ds \geq \tau(d) \tag{A2}$$

where $\tau(d) = \mathrm{meas}\,A[c^*(d)]$.

Proof: By assumption,

$$\begin{aligned} \frac{d}{2} &\leq \int_0^T h(T-s)u(s)\,ds \\ &= \int_0^T |h(T-s)|[u(s)\,\mathrm{sgn}\,h(T-s)]\,ds \\ &\leq \int_0^T |h(T-s)|[u(s)\,\mathrm{sgn}\,h(T-s)]^+\,ds. \end{aligned} \tag{A3}$$

Since $|h(T-s)|$ is nonnegative, and the term in brackets is between zero and one, we can apply [1, Lemma A.8], which gives the result. ■

To prove the theorem, let the *sampling time* t_{ij}, $1 \leq i \leq N$, $1 \leq j \leq N$, be defined as

$$t_{ij} = \inf\{t: |y_i(t_{ij}) - y_j(t_{ij})| \geq d\}. \tag{A4}$$

Then

$$\int_0^{t_{ij}} \overline{h}(t_{ij}-s)[u_i(s) - u_j(s)]\,ds \geq d \tag{A5}$$

which implies that

$$\begin{aligned} &\int_0^{t_{ij}} \left(\overline{h}(t_{ij}-s)[u_i(s) - u_j(s)]\right)^+ ds \geq d \\ &+ \int_0^{t_{ij}} \left(\overline{h}(t_{ij}-s)[u_i(s) - u_j(s)]\right)^- ds \geq d + c(d)\kappa \end{aligned} \tag{A6}$$

where

$$\kappa = \int_0^{t_{ij}} [(\mathrm{sgn}\ \overline{h}(t_{ij}-s))(u_i(s) - u_j(s))]^-\ ds \tag{A7}$$

since $\overline{h}(t) \geq c(d)$ for all t. Let $\Delta(s) = \max\{0, 1/2[u_i(s) - u_j(s)]\}$. Since $h(t) \geq 0$ for all t, (A6) implies that

$$\int_0^{t_{ij}} \overline{h}(t_{ij}-s)\Delta(s)\,ds \geq \frac{d}{2} + c(d)\frac{\kappa}{2} \tag{A8}$$

and since $|\Delta(s)| \leq 1$, we can apply the Lemma to obtain

$$\int_0^{t_{ij}} ([\mathrm{sgn}\,\overline{h}(t_{ij}-s)]\Delta(s))^+\,ds \geq \tau[d + c(d)\kappa]. \tag{A9}$$

By definition of $\overline{h}(t)$ and $A[c^*(d)]$, it follows that $\tau[d + c(d)\kappa] = \tau(d) + \kappa$. Furthermore, $\Delta(s)$ in (A9) can be replaced by $1/2[u_i(s) - u_j(s)]$ so that (A7) and (A9) imply that

$$\frac{1}{2}\int_0^{t_{ij}} (\operatorname{sgn}\overline{h}(t_{ij} - s))\,(u_i(s) - u_j(s))\,ds \geq \tau(d). \quad \text{(A10)}$$

If the set of inputs $\{u_i\}$ is applied to the channel with impulse response $1/2\operatorname{sgn}\overline{h}$, then the outputs y_i and y_j are therefore separated by at least $\tau(d)$ at time t_{ij}. Now $h(t) > 0$ implies that $1/2\operatorname{sgn}\overline{h}(t) = 1/2$, and it has been shown in [1] that for a channel with constant impulse response $h(t) = 1/2$, the minimum time to separate N outputs by d, assuming the inputs are bounded in magnitude by one, satisfies

$$T \geq (\log_2 N)d. \quad \text{(A11)}$$

Since in this case d is replaced by $\tau(d)$, it follows that $(\log_2 N)/T \leq 1/\tau(d)$ for all T, and hence this must be true in the limit as $T \to \infty$. ■

We remark that if $h(t)$ is not nonnegative, then $\Delta(s)$ can still be defined so that (A8) remains valid, and furthermore, so that (A9) and (A10) still hold. Consequently, if the bound (A11) applies more generally to an impulse response of the form $1/2\operatorname{sgn}h(t)$, where $h(t) \in L_1$, then the assumption that $h(t) \geq 0$ could be dropped. Whether or not this is true is currently an open question.

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