Encoding of Analog Signals for Binary Symmetric Channels

A. J. BERNSTEIN, MEMBER, IEEE, K. STEIGLITZ, MEMBER, IEEE, AND J. E. HOPCROFT, MEMBER, IEEE

Abstract—Various encoding schemes are examined from the point of view of minimizing the mean magnitude error of a signal caused by transmission through a binary symmetric channel. A necessary property is developed for optimal codes for any binary symmetric channel and any set of quantization levels. The class of optimal codes is found for the case where the probability of error is small but realistic. This class of codes includes the natural numbering and some unit distance codes, among which are the Gray codes.

I. Introduction

With the introduction of PCM systems for telemetry and voice communication, the problem of transmitting samples of an analog signal over a binary channel has become increasingly important. The natural way to code 2^n signal levels for an n-bit binary code is to use the binary expansion of i-1 for the ith signal level (the natural numbering). This is usually the code used, and it is often tacitly assumed [1]–[3] that this is optimum in some sense, with respect to the signal error caused by bit errors in the binary channel. On the other hand, it is argued that the unit distance codes [4], with their adjacency properties, are particularly well suited for coding analog signals. It is not clear which, if either, of these is best for different channel error rates and for different arrangements of signal levels.

It is the purpose of this paper to establish the optimality of a class of codes which includes the natural codes and some unit distance codes (among which are the Gray codes) under practical levels of error rate in the binary channel and for a mean magnitude error criterion. This complements recent results [5],[1] [6] which show the optimality of these codes when only single errors in a word are considered.

II. Derivation of the Mean Magnitude Error

We will assume that the 2^n quantization levels of the signal

$$k_1 \leq k_2 \leq \cdots \leq k_{2^n}$$

are equally probable, and that an n-bit binary word is used to code each of these levels. This is a reasonable assumption in many cases and ensures that the source has maximum entropy. We will call the 2^n levels, k_i, a k-vector.

Now consider the process of assigning each number k_i to a vertex of the n-cube, starting with k_1 and working sequentially to k_{2^n}. Call the vertex assigned k_i the ith vertex. Let r_{ij} be the number of vertex assigned to signal levels which are j-distant neighbors of the ith vertex when only the first i signal levels have been assigned. This r_{ij} matrix ($1 \leq i \leq 2^n$, $1 \leq j \leq n$) characterizes the code completely for our purposes, since, as we shall see, it can be used to calculate the mean magnitude error.

Since each of the previous i-1 vertex enters into the ith row of the r_{ij} matrix by adding a 1 to exactly one column, it follows that

$$\sum_{j=1}^{n} r_{ij} = i - 1.$$

It also follows that the sum of the jth column of the r_{ij} matrix will be the total number of j-connections on the n-cube:

$$\sum_{i=1}^{2^n} r_{ij} = 2^{n-1}\binom{n}{j}.$$

Now consider the component of the mean magnitude error caused by j-errors. In the final assignment, the ith vertex has r_{ij} j-neighbors which are not greater than k_i, and $\left[\binom{n}{j} - r_{ij}\right]$ j-neighbors which are not smaller than k_i. It follows that, in the computation of the mean magnitude error due to j-errors, k_i will have the coefficient

$$r_{ij} - \left[\binom{n}{j} - r_{ij}\right] = 2r_{ij} - \binom{n}{j}.$$

Hence the average value of a j-error will be the sum of all the k_i with this weight, divided by the total number of j-connections on the n-cube:

$$\frac{1}{2^{n-1}\binom{n}{j}} \sum_{i=1}^{2^n} \left[2r_{ij} - \binom{n}{j}\right] k_i.$$

Assuming that the channel is a binary symmetric channel with probability of error p, we see that the probability of a j-error occurring in an n-bit word is

$$\binom{n}{j} p^j (1-p)^{n-j}.$$

Manuscript received April 4, 1965; revised December 1, 1965. This study made use of computer facilities sponsored, in part, by the National Science Foundation Grant NSF-GP-579.

A. J. Bernstein is with the General Electric Research and Development Center, Schenectady, N. Y.

K. Steiglitz and J. E. Hopcroft are with the Department of Electrical Engineering, Princeton University, Princeton, N. J.

[1] See [7] for a corrected version of the results in [5].

Hence the mean magnitude error caused by all errors is

$$E = \sum_{j=1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \frac{1}{2^{n-1}\binom{n}{j}} \sum_{i=1}^{2^n} \left[2r_{ij} - \binom{n}{j} \right] k_i \quad (1)$$

which reduces to

$$E = \frac{1}{2^{n-2}} \sum_{i=1}^{2^n} \sum_{j=1}^{n} p^j (1-p)^{n-j} r_{ij} k_i - \frac{1}{2^{n-1}} [1 - (1-p)^n] \sum_{i=1}^{2^n} k_i. \quad (2)$$

When $p = 0$, the mean magnitude error is zero. Expanding E in a power series in p about $p = 0$ yields

$$E = \left[\frac{1}{2^{n-1}} \sum_{i=1}^{2^n} (2r_{i1} - n) k_i \right] p + \quad (3)$$

higher order terms in p.

For sufficiently small p, the first-order term determines the mean magnitude error. Since this linear term involves only the r_{i1}, one can think of this as the component of the error caused by single errors only.

III. Harper Codes

Consider the following algorithm for numbering the vertexes of the n-cube.

Algorithm

Let the first vertex be arbitrary. Having numbered vertexes $1, 2, \cdots, l$, let the $l + 1$st vertex be an unnumbered vertex which has the most numbered one distant neighbors.

Those codes generated by assigning k_i to the ith vertex numbered by the algorithm will be called Harper codes. These include the natural code mentioned in the Introduction. The following results concerning this algorithm will be needed.

Lemma 1

An array consisting of the first l vertexes numbered by the algorithm contains more one connections between vertexes within the array than any other array of l vertexes, where $l < 2^n$.

Proof: The proof is exactly the same as in [5], except that a strict inequality is verified at each step.

Lemma 2

All Harper codes have the same r_{ij} matrix, and hence the same mean magnitude error.

Proof: This is obvious for the first column of the r_{ij} matrix by the nature of the algorithm. Let l $(1 \leq l \leq 2^n)$ have the binary expansion

$$l = 2^{s_1} + 2^{s_2} + \cdots + 2^{s_q} \qquad (s_1 > s_2 > \cdots > s_q).$$

Then Harper [5] has shown that the array consisting of the first l vertexes numbered by the algorithm consists of a sequence of subcubes of dimension s_q to s_1, such that each subcube is in the shadow of every larger subcube. A shadow of a subcube is any adjacent subcube. Since this structure determines the r_{ij} matrix, the lemma follows.

Theorem

Harper codes minimize the linear coefficient of (3).

Proof: See [5] and [6].

From the above theorem one can draw the conclusion that, for a given n, the Harper codes have minimum mean magnitude error for all $p \leq \epsilon$ for some $\epsilon > 0$. Let $\hat{p}(n)$ be the largest such value of ϵ for a code of n dimensions. $\hat{p}(n)$ will then be the largest value of p for which there does not exist a code C, and a k-vector such that C has a smaller mean magnitude error than a Harper code for the given k-vector. One of the main purposes of this paper is to find a lower bound on $\hat{p}(n)$ that corresponds to practical error rates, and hence to justify the use of Harper codes for p that are not vanishingly small.

IV. Properties of Optimal Codes

Consider an array V of l vertexes $v_1, v_2, \cdots, v_l$ of the n-cube. The origin A of such an array of vertexes is that vertex whose ith coordinate a_i is zero if the majority of points have a zero in that coordinate, or one if the majority of points have a one in that coordinate. If the number of zeros and ones are equal, a_i will arbitrarily be chosen to be zero. Let the ith dimension x_i divide the n-cube into two half-cubes. Then the array is said to be one-dimensionally stable if, and only if, for each vertex v_j in the half-cube $x_i = \bar{a}_i$, the corresponding vertex in the half-cube $x_i = a_i$ (i.e., that vertex whose coordinates differ from those of v_j in only the ith position) is also a member of the array where $1 \leq i \leq n$. A code is said to be one-dimensionally stable if its first l vertexes form a one-dimensionally stable array, where $1 \leq l \leq 2^n$. It is easily shown that the origin is always contained in a one-dimensionally stable array. Since we are interested only in the mutual distances between points, and since these distances remain the same if the ith coordinate of every point in the array is complemented, it is clear that, by appropriate complementations, we can cause the origin of the array to be specified by $a_i = 0$ for $1 \leq i \leq n$. We will denote this vertex by A^0. In this special case, the structure of a one-dimensionally stable set of points is easily visualized. If v_j is a member of such an array, then every vertex obtained by setting ones to zeros in the coordinates of v_j is also a member of the array.

Consider a one-dimensionally stable array V of l vertexes $v_1, v_2, \cdots, v_l$ with origin A. Let dimensions i and j divide the n-cube into quadrants. Let V' be the subset of V consisting of those vertexes in the quadrant $x_i = a_i$, $x_j = \bar{a}_j$, and let V'' be the subset of V consisting of those vertexes in the quadrant $x_j = a_j$, $x_i = \bar{a}_i$. Then V is two-dimensionally stable in coordinates i and j if, and only if, either the subset obtained by complementing the ith and jth coordinate of each vertex in V' is contained in V'', or vice versa. V is two-dimensionally stable if, and only if, it is two-dimensionally stable in every pair of coordinates. A code is said to be two-dimensionally stable if its first l vertexes form a two-dimensionally stable array, where $1 \leq l \leq 2^n$.

If the origin of V is A^0, the above condition can be modified so that the structure of a two-dimensionally stable array is easily visualized. In this case, if v_i is a member of such an array, then every vertex obtained by shifting ones to the right in the coordinates of v_i is also a member of V. Thus, if a two-dimensionally stable array contains the vertex (1 0 0 1 0), then the following set of points must also be in the array. It should be remembered that a two-dimensionally stable array is, by definition, also one-dimensionally stable.

$$
\begin{array}{ll}
(0\ 1\ 0\ 1\ 0) & (1\ 0\ 0\ 0\ 0) \\
(0\ 0\ 1\ 1\ 0) & (0\ 1\ 0\ 0\ 0) \\
(1\ 0\ 0\ 0\ 1) & (0\ 0\ 1\ 0\ 0) \\
(0\ 1\ 0\ 0\ 1) & (0\ 0\ 0\ 1\ 0) \\
(0\ 0\ 1\ 0\ 1) & (0\ 0\ 0\ 0\ 1) \\
(0\ 0\ 0\ 1\ 1) & (0\ 0\ 0\ 0\ 0).
\end{array}
$$

For any code which is not two-dimensionally stable, it can be shown[2] that there exists a code with smaller mean magnitude error for any k-vector and any p, $0 < p < 0.5$. Thus the code which gives the smallest mean magnitude error for a given k-vector and a given p must be a two-dimensionally stable code. It can be shown that every Harper code is two-dimensionally stable.

An example of a code which is two-dimensionally stable but which is not a Harper code is the code which assigns the vertexes of the n-cube to the k-vector as follows: k_1 is assigned to the vertex of weight zero, k_2 through k_{n+1} are assigned to the vertexes of weight one in numerical order (i.e., to 1, 2, 4, $\cdots$, 2^{n-1}, respectively), k_{n+2} through $k_{\binom{n}{2}+n+1}$ are assigned to the vertexes of weight two in numerical order (i.e., to 3, 5, 6, 9, $\cdots$, $2^{n-1} + 2^{n-2}$, respectively), etc. This code has been called the star code. In addition to the star code, there exists a large class of two-dimensionally stable codes which essentially fall between the star and the Harper codes in their encoding schemes.

It has been shown that, for p small enough, the Harper codes minimize the mean magnitude error for all k-vectors. It is not true, however, that the Harper codes are optimal in all situations. For example, consider the k-vector with 256 elements satisfying

$$
\begin{aligned}
k_i &= 0 \qquad 1 \leq i \leq 9 \\
k_i &= 1 \qquad 10 \leq i \leq 256.
\end{aligned}
$$

It can be shown that, for $p \geq 0.37$, the star encoding gives a smaller mean magnitude error. Although the above k-vector is rather degenerate, it points to the existence of many other nondegenerate k-vectors for which the star code is better than the Harper code for large p. It should be noted that the authors have found no counter-example to the optimality of the Harper codes for n less than 8. Harper codes have been proven optimal for all p and all k-vectors for $n \leq 4$.

[2] See Appendix.

V. A Lower Bound on $\hat{p}(n)$

We will now determine a lower bound on $\hat{p}(n)$. For all p less than this bound, the Harper codes will minimize the mean magnitude error for any k-vector. In order to obtain the bound, the expression for the mean magnitude error using a code C is rewritten as

$$
E^C = \frac{1}{2^{n-2}} \sum_{m=1}^{2^n} \Delta k_m \left[\sum_{i=m}^{2^n} \sum_{j=1}^{n} r_{ij}^C p^j (1-p)^{n-j} - \tfrac{1}{2}(2^n - m + 1)(1-(1-p)^n) \right] \tag{4}
$$

where

$$
\begin{aligned}
\Delta k_i &= k_i - k_{i-1} \qquad i \neq 1 \\
\Delta k_1 &= k_1.
\end{aligned} \tag{5}
$$

A typical term in this summation is

$$
\frac{1}{2^{n-2}} \sum_{j=1}^{n} p^j (1-p)^{n-j} \sum_{i=m+1}^{2^n} r_{ij}^C - \frac{1}{2^{n-1}} (2^n - m)(1-(1-p)^n), \tag{6}
$$

and this is seen to be the mean magnitude error which results when C is used to encode the k-vector

$$
\begin{aligned}
k_i &= 0 \qquad 1 \leq i \leq m \\
k_i &= 1 \qquad m < i \leq 2^n.
\end{aligned} \tag{7}
$$

Such a k-vector with m in the range $1 \leq m \leq 2^n$ will be called a (0, 1)-vector. It is seen that the expansion (4) is a weighted sum of the errors resulting from using C to encode all possible (0, 1)-vectors. The weights in the expansion are the Δk_i of (5).

It has been shown that, for p sufficiently small, the Harper codes provide the best encoding of each such (0, 1)-vector. Let $E^C(K)$ be the mean magnitude error which results when C is used to encode the k-vector K. Let H denote a Harper code. Let $\hat{p}(n)$ be the largest value of p for which there does not exist a code C and a (0, 1)-vector M such that $E^C(M)$ is less than $E^H(M)$ when $p \leq \hat{p}(n)$. It then follows from (4) that, for all p less than or equal $\hat{p}(n)$, the Harper codes are best for arbitrary k-vectors. Let (7) be the vector M. Since the sum of the elements in the jth column of the r_{ij} matrix is the same constant for all codes, it follows that the error (6) depends only on $\sum_{i=1}^{m} r_{ij}$ where $1 \leq j \leq n$. Since $E^C(M) \neq E^H(M)$ for some p, it follows that

$$
\sum_{i=1}^{m} r_{ij}^C \neq \sum_{i=1}^{m} r_{ij}^H \tag{8}
$$

for some j. If these sums were equal for $j = 1$, then $\sum_{i=1}^{m} r_{i1}^C$ would be maximum by Lemma 1. There is essentially only one configuration of m vertexes with this property—the configuration of m vertexes obtained using the algorithm. From this it would follow that equality holds in (8) for all j, which is impossible, since $E^C(M) \neq E^H(M)$. Hence,

$$\sum_{i=1}^{m} r_{i1}^C < \sum_{i=1}^{m} r_{i1}^H. \tag{9}$$

For p less than or equal to $\hat{p}(n)$, we have that $E^C(M) - E^H(M) \geq 0$, or, from (6)

$$\sum_{i=m+1}^{2^n} \sum_{j=1}^{n} (r_{ij}^C - r_{ij}^H)p^j(1-p)^{n-j} \geq 0. \tag{10}$$

Separating the single from the multiple errors, we have

$$p(1-p)^{n-1} \sum_{i=m+1}^{2^n} (r_{i1}^C - r_{i1}^H) \geq \sum_{j=2}^{n} p^j(1-p)^{n-j} \sum_{i=m+1}^{2^n} (r_{ij}^H - r_{ij}^C). \tag{11}$$

But, from (9),

$$p(1-p)^{n-1} \sum_{i=m+1}^{2^n} (r_{i1}^C - r_{i1}^H) \geq p(1-p)^{n-1} \tag{12}$$

which puts a lower bound on the left-hand side of (11). We can bound the right-hand side of (11) from above by obtaining

$$\max_m \left\{ \sum_{i=m+1}^{2^n} (r_{ij}^H - r_{ij}^C) \right\}.$$

But

$$\max_m \left\{ \sum_{i=m+1}^{2^n} (r_{ij}^H - r_{ij}^C) \right\} = \max_m \left\{ \sum_{i=1}^{m} (r_{ij}^C - r_{ij}^H) \right\}. \tag{13}$$

Since $\sum_{i=1}^{m} r_{ij}^H$ is fixed for any m, and since

$$\sum_{i=1}^{m} r_{ij}^C < \left[\frac{m}{2} \binom{n}{j} \right] \tag{14}$$

where $[x]$ is the largest integer smaller than x, we have that

$$\max_m \left\{ \sum_{i=m+1}^{2^n} (r_{ij}^H - r_{ij}^C) \right\} \leq \max_m \left\{ \left[\frac{m}{2} \binom{n}{j} \right] - \sum_{i=1}^{m} r_{ij}^H \right\}. \tag{15}$$

The right-hand side of (15) can be obtained for all j, $1 < j \leq n$ directly from the r_{ij} matrix for the Harper codes. Let

$$B_j = \max_m \left\{ \left[\frac{m}{2} \binom{n}{j} \right] - \sum_{i=1}^{m} r_{ij}^H \right\}. \tag{16}$$

Then, $\hat{p}(n)$ is greater than the smallest positive value of p satisfying

$$p(1-p)^{n-1} = \sum_{j=2}^{n} B_j p^j (1-p)^{n-j}. \tag{17}$$

In Table I, the lower bound on $\hat{p}(n)$ is given for values of n up to $n = 10$. It should be noted that, for $n = 3$, 4, and 5, a bound on $\sum_{i=1}^{m} r_{ij}^C$ was used which is slightly tighter than (14), to give better values of this bound. Since most physical channels, when approximated by a binary symmetric model, have a value of p less than 10^{-4}, and since signals are not generally quantized into more than 2^8 levels, it follows from Table I that, for most practical situations, Harper codes are optimal.

By noting that $B_j < 2^{n-1}\binom{n}{j}$ and using just the first term in the summation of (17), we can approximate the lower bound on $\hat{p}(n)$ by $1/n^2 2^{n-2}$ for large n.

VI. Optimal Unit Distance Codes

Since unit distance codes appear to be well suited for coding analog signals [4], it is interesting to determine the class of unit distance codes which are Harper codes. It will be shown below that every one-dimensionally stable unit distance code is a Harper code. The proof is by induction. Consider the assignment for the first 2^l vertexes. For $l = 1$, the assignment for any one-dimensionally stable unit distance code satisfies the algorithm for a Harper code. Assume that the assignment of the first 2^l vertexes for any one-dimensionally stable unit distance code satisfies the algorithm for a Harper code. This implies that the first 2^l vertexes fill an l subcube S_0. The next vertex assigned must be in a subcube S_1 adjacent to S_0. The first vertex i, $i > 2^l + 1$, which is not assigned in S_1 must be in a shadow of S_1. But, in order that the code be one-dimensionally stable, all vertexes in S_1 must have been previously assigned. This implies that the first 2^{l+1} vertexes fill an $l + 1$ subcube. Since the order in which the vertexes in S_1 are assigned can be any order which is allowed for S_0, and since each of these orders satisfies the algorithm for a Harper code, it follows that every one-dimensionally stable unit distance code is a Harper code.

An example of a one-dimensionally stable unit distance code is the Gray code shown in Table II for $n = 3$.

TABLE I

n	Lower bound on $\hat{p}(n)$
3	0.333
4	0.116
5	0.028
6	0.011
7	4.4×10^{-3}
8	1.9×10^{-3}
9	0.82×10^{-3}
10	0.36×10^{-3}

TABLE II

k_1	000
k_2	001
k_3	011
k_4	010
k_5	110
k_6	111
k_7	101
k_8	100

VII. Conclusion

In this paper, we have examined the effectiveness of various encoding schemes in minimizing the mean magnitude error of a signal caused by transmission through a binary symmetric channel. A necessary property, called two-dimensional stability, that all optimal codes for any binary symmetric channel must possess has been developed. The binary symmetric channel with small error probability has been considered in detail. It has been shown that, for this channel, Harper codes are optimal. This is a class of codes which is essentially equivalent to the natural numbering system, and which includes certain types of unit distance codes. A bound on the probability of error in the channel has been developed such that, for all channels with smaller error probability, the Harper codes are optimal. This bound applies to most practical situations.

It should be pointed out that there are a large number of problems connected with this work which have not as yet been considered. Best codes for other error criteria are not known. For example, it can be shown that Harper codes do not necessarily minimize the mean square error, even when p is arbitrarily small. The use of redundancy in signal encoding has not been considered. More generally, the problem of finding the best code word length n, given the statistics of the signal and the bit rate of the channel, has not been considered. The solution of this general problem would certainly involve taking into account quantization error as well as error caused by channel noise.

Appendix

Consider a typical term (6) of the expansion (4) for the mean magnitude error, which results when C is used to encode an arbitrary k-vector. It has been pointed out that such a term represents the mean magnitude error which results when C is used to encode the (0, 1)-vector (7). A series of transformations will now be described which change C to a two-dimensionally stable code. It will be shown that each of these transformations decreases (6), and as a result decreases E^C [except in certain degenerate situations where (6) and E^C are unchanged].

Two vertexes v_r and v_s will be said to form an i-pair if they agree in all but the ith coordinate. Consider the transformation T_i on C

$$T_i : C \rightarrow C' \tag{18}$$

where C and C' are both encodings of the (0, 1)-vector (7). The transformation is defined by the following rule. Let v_r and v_s form an i-pair, and let k_r and k_s be the numbers assigned to them by C. Let v_r and v_s have $\bar{a}_i$ and a_i, respectively, as their ith coordinate. Then,

1) if $k_r > k_s$, the coding of these numbers in C' is unchanged;
2) if $k_r \leq k_s$, the coding of these numbers in C' is reversed. That is, k_r is reassigned to v_s and k_s to v_r.

It follows that the code which results when the n transformations T_i, $1 \leq i \leq n$ are applied to C is one-dimensionally stable with the origin A defined by the coordinates $(a_1, a_2, \cdots, a_n)$.

Theorem 1

Let r_{ij} and r'_{ij} be the parameters describing C and C', respectively, where C and C' are related by (18). Then,

$$\sum_{j=1}^{k} \sum_{i=1}^{l} r_{ij} \leq \sum_{j=1}^{k} \sum_{i=1}^{l} r'_{ij} \tag{19}$$

for all k and l where $1 \leq k \leq n$ and $1 \leq l \leq 2^n$.

Proof: Since we are summing up to l, only the locations of the first l vertexes in C and C' are involved in (19). Let these sets be denoted by V and V', respectively. V can be divided into the four disjoint subsets R_1, R_2, S, and U, as shown in Fig. 1. Here, the n-cube has been split into the two half-cubes $x_i = a_i$ and $x_i = \bar{a}_i$.

The subsets R_1 and R_2 are equinumerous and correspond to those vertexes in V which form i-pairs. Vertexes in V which form i-pairs with vertexes not in V are contained in subsets S and U. Since the k_i assigned to vertexes not in V are not less than the k_i assigned to vertexes in V, the first l vertexes of C', which will be denoted V', are as shown in Fig. 2. U' has been obtained by complementing the ith coordinate of all the vertexes in U. Note that the distances between vertexes within U are equal to the distances between corresponding vertexes in U'. Also, every j-connection between a vertex in U and one in R_2 is replaced by a corresponding connection between a vertex in U' and one in R_1. A similar statement holds concerning connections between vertexes in U and R_1, on one hand, and U' and R_2 on the other. Thus the difference between the connectivities of V and V' is that connections between vertexes in U and vertexes in S have been replaced by connections between U' and S. Since each vertex in U' is one unit closer to every vertex in S than its image in U, it follows that (19) holds.

Let C be a one-dimensionally stable code with origin A. Two vertexes v_r and v_s will be said to form an i, j-pair if they agree in all but the ith and jth coordinates, and if v_r has $x_i = a_i$, $x_j = \bar{a}_j$ and v_s has $x_i = \bar{a}_i$, $x_j = a_j$. Consider the transformation T_{ij} on C;

$$T_{ij} : C \rightarrow C' \tag{20}$$

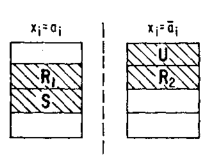

Fig. 1.

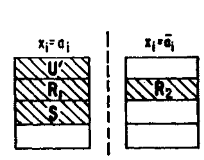

Fig. 2.

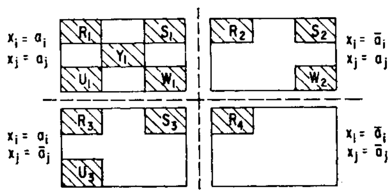

Fig. 3.

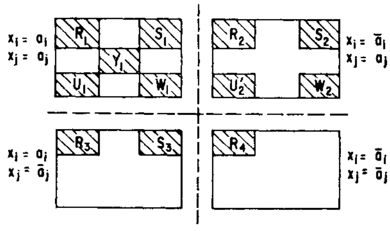

Fig. 4.

where C and C' are both encodings of the (0, 1)-vector (7). The transformation is defined by the following rule. Let v_r and v_s form an i, j-pair with coordinates in those dimensions as stated above, and let k_r and k_s be the numbers assigned to them by C.

1) If $k_r > k_s$, the coding of the numbers in C' is unchanged;
2) if $k_r \leq k_s$, the coding of these numbers in C' is reversed. That is, k_r is reassigned to v_s and k_s to v_r.

It follows that the code which results when T_{ij} is performed for each pair i, j is two-dimensionally stable.

Theorem 2

Let r_{ij} and r'_{ij} be the parameters describing C and C', respectively, where C and C' are related by (20). Then,

$$\sum_{j=1}^{k} \sum_{i=1}^{l} r_{ij} \leq \sum_{j=1}^{k} \sum_{i=1}^{l} r'_{ij} \tag{21}$$

for all k and l where $1 \leq k \leq n$, $1 \leq l \leq 2^n$.

Proof: Since we are summing up to l, only the locations of the first l vertexes in C and C' are involved in (21). Let these sets be denoted by V and V', respectively. Since V is a one-dimensionally stable array, it can be broken into disjoint subsets, as shown in Fig. 3. Here, the n-cube has been split into four quadrants by coordinates x_i and x_j, with the origin A in the quadrant $x_i = a_i$, $x_j = a_j$. Vertexes in R_4 form i-pairs and j-pairs with vertexes in R_3 and R_2, respectively. Similarly, vertexes in R_1 form i-pairs and j-pairs with vertexes in R_2 and R_3, respectively. Similar statements hold with respect to S_1, S_2, and S_3, W_1 and W_2, and U_1 and U_3. The distribution of vertexes in V', the first l vertexes of C', is shown in Fig. 4. U'_2 has been obtained by complementing coordinates i and j of all vertexes in U_3. By an argument similar to that used in Theorem 1, it follows that the only difference between the connectivities of V and V' is that connections between vertexes in U_3 and W_2 have been replaced by connections between U'_2 and W_2. Since each vertex in U'_2 is two units closer to every vertex in W_2 than its image in U_3, we have shown that (21) holds.

It follows from Theorems 1 and 2 that, if C' is obtained from C by a transformation of the form T_i or T_{ij}, then

$$\sum_{j=1}^{k} \sum_{i=l+1}^{2^n} r_{ij} \geq \sum_{j=1}^{k} \sum_{i=l+1}^{2^n} r'_{ij}; \quad 1 \leq j \leq n, \quad 1 \leq l \leq 2^n. \tag{22}$$

Theorem 3

Let E^C and $E^{C'}$ be the mean magnitude error when codes C and C' are used to encode the (0, 1)-vector (7), where C and C' are related by either (18) or (20). Then $E^{C'} \leq E^C$.

Proof: Let P_i represent $p^i(1-p)^{n-i}$, where $p \leq \frac{1}{2}$. Then,

$$P_{w-1} - P_w \geq 0$$

and, from (22), we have that the following set of inequalities hold.

$$P_n \sum_{j=1}^{n} \sum_{i=m+1}^{2^n} r_{ij} \geq P_n \sum_{j=1}^{n} \sum_{i=m+1}^{2^n} r'_{ij}$$

$$(P_{w-1} - P_w) \sum_{j=1}^{w-1} \sum_{i=m+1}^{2^n} r_{ij}$$

$$\geq (P_{w-1} - P_w) \sum_{j=1}^{w-1} \sum_{i=m+1}^{2^n} r'_{ij} \qquad 1 < w \leq n. \tag{23}$$

Adding the n inequalities (23), we get

$$\sum_{j=1}^{n} P_j \sum_{i=m+1}^{2^n} r_{ij} \geq \sum_{j=1}^{n} P_j \sum_{i=m+1}^{2^n} r'_{ij}.$$

The theorem follows from (6).

Acknowledgment

The authors would like to thank Professor A. W. Lo for suggesting this problem.

References

[1] E. Bedrosian, "Weighted PCM," *IRE Trans. on Information Theory*, vol. IT-4, pp. 45–49, March 1958.
[2] J. J. Downing, *Modulation Systems and Noise*. Englewood Cliffs, N. J.: Prentice-Hall, 1964, p. 168.
[3] S. Karp, "Noise in digital-to-analog conversion due to bit errors," *IEEE Trans. on Space Electronics and Telemetry* (*Correspondence*), vol. SET-10, p. 124, September 1964.
[4] A. Susskind, Ed., *Notes on Analog-Digital Conversion Techniques*. Cambridge, Mass.: M.I.T. Press, 1957, ch. 3.
[5] L. H. Harper, "Optimal assignments of numbers to vertices," *J. SIAM*, vol. 12, pp. 131–135, March 1964.
[6] K. Steiglitz and A. J. Bernstein, "Optimal binary coding of ordered numbers," *J. SIAM*, vol. 13, pp. 441–443, June 1965.
[7] A. J. Bernstein, "Maximally connected arrays on the n-cube," General Electric Co., Schenectady, N. Y., Research Rept. 66-C-353, May 1966.