$$E[I(\sigma; s)] = P(ts) \log LP(ts) + P(fs) \log LP(fs) + P(fs) \sum_{\substack{j=1 \\ j \neq i}}^{L} a_{j-i} \log a_{j-i}, \qquad (1)$$

where $a_{j-i+L} = a_{j-i}$, from which it is concluded that *the average amount of information that a group synchronizer can provide will be determined completely by its probability of false sync* (complement of probability of true sync) *and by the distribution of that probability among the $L - 1$ non-sync positions*. Furthermore, the average information is a monotonically increasing function of the probability of true sync for any particular distribution of the probability of false sync, as can be seen by differentiating (1) with respect to $P(ts)$.

For a given probability of true or of false sync there are distinct limits imposed by the distribution of false sync on the amount of information that can be provided. The upper limit occurs when the probability of false sync is all concentrated in a single position and the lower limit when it is distributed equally among the $L - 1$ non-sync positions. At the upper limit the summation in (1) rises to its maximum value of zero and at the lower limit falls to its minimum value of $-\log (L - 1)$. Thus it is concluded that the upper and lower limits on the average information that any group synchronization process can provide for a given probability of true or of false sync and for a given number of possible locations L of the sync position are

$$E_{\max}[I(\sigma; s)] = P(ts) \log LP(ts) + P(fs) \log LP(fs) \qquad (2)$$

and

$$E_{\min}[I(\sigma; s)] = P(ts) \log LP(ts) + P(fs) \log \left[\frac{L}{L-1} P(fs)\right]. \qquad (3)$$

They are expressed in bits when the logarithmic base is two. The upper limit is of no practical interest because it can occur only for a pathological case of a noiseless signal with only a single data position and a sync pattern occupying no less than half of the group. But the lower limit is a good approximation to many, if not most, of the cases occurring in practice. It represents exactly all cases in which the sync pattern does not overlap adjoining positions (*i.e.*, is contained wholly within a single position) and in which the statistics of the noisy signal samples are the same for all data (non-sync) positions.

When the probability of false sync approaches zero (probability of true sync approaches unity) it is seen that both the upper and lower limits converge to $\log L$, the absolute maximum average information available. Thus the absolute best that any group synchronization process whatsoever can achieve will occur when its probability of false sync vanishes, as would be expected. At the other extreme all positions become indistinguishable because of complete masking by noise. Since all positions would be equally likely to be the sync position the selection would consist simply of a random guess. The probability of true sync would then be $1/L$ and the lower limit would be zero, again as one would expect. Thus the general formulation for the limits on average information is clearly consistent with the two known absolute limits. A plot of the upper and lower limits is shown in Fig. 1 as a function of probability of false sync for the simplest possible (but not practically useful) case of a two-position group, and for a practical case of an eight-position group.

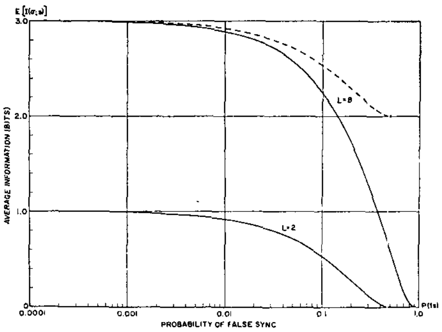

Fig. 1—The upper (dashed) and lower (solid) limits on the average mutual information provided by any group synchronization process for a two- and an eight-position group. The upper and lower limits coincide for the two-position group.

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Crosstalk in Time Division Multiplex Systems*

Summary

An analysis of the interchannel crosstalk in time division multiplex systems is given. The results apply to systems with a general linear transmission channel and with arbitrary sampling waveforms.

The general result is obtained in closed form. This is then applied to the particular case of multiplexing with rectangular waveforms and a single pole transmission channel. A normalized plot of crosstalk vs system parameters for such systems is presented.

* Received July 10, 1962.

Introduction

Several investigators have considered the problem of interchannel crosstalk in time division multiplexing. It is usually necessary to invoke various restrictions on the nature of the transmission channel to simplify the mathematics of the problem. The present approach, which is similar in some respects to that of W. R. Bennett,[1] has the virtue of presenting a closed form solution to the problem for a quite general transmission channel.

The important components of a time division multiplex system are illustrated in Fig. 1. The input sampler repetitively presents samples of the inputs to the transmission channel. Usually, the sampling function is rectangular, but it is possible in the present analysis to generalize the shape of the sampling pulse to any function of time expressible as a Fourier series.

The transmission channel is assumed to be linear with a transfer function $\tilde{G}(j\omega)$. Bandwidth limitations in this channel have important effects on the crosstalk.

The output sampler is synchronized with the input sampler. Again, the sampling function need not be rectangular and aside from being repetitive with the same repetition frequency as the input sampler, the output sampling waveform need not bear any relation to the input sampling waveform.

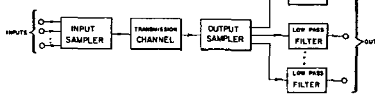

Fig. 1—The components of a time division multiplex system.

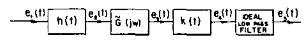

Fig. 2—A model characterizing the time division multiplex system of Fig. 1.

Analysis

The multiplex system shown in Fig. 1 may be studied by applying an input signal to one of the input leads and investigating the signals appearing at any of the output leads. When this is done, the system illustrated in Fig. 1 may be characterized by the model shown in Fig. 2. In the case of rectangular sampling, the delays of the waveforms $h(t)$ and $k(t)$ from some reference time determine which input and output channels are being considered.

In this model, $e_1(t)$ is the information bearing input signal to one of the channels. It may be represented as

$$e_1(t) = E_1 \cos (\omega_0 t + \Phi) = \Re\{\tilde{E}_1 \epsilon^{j\omega_0 t}\}. \qquad (1)$$

[1] W. R. Bennett, "Time division multiplex systems," *Bell Sys. Tech. J.*, vol. 20, pp. 199–221, April, 1941.

This input signal is multiplied by the time-varying transmission of the input switch to produce the voltage $e_2(t)$, so that

$$e_2(t) = h(t)e_1(t). \tag{2}$$

Eq. (2) may be written as

$$\tilde{E}_2(t) = h(t)\tilde{E}_1, \tag{3}$$

where the following substitution has been made:

$$e_2(t) = \Re\{\tilde{E}_2(t)\epsilon^{j\omega_0 t}\}. \tag{4}$$

This notation which will be adhered to in the remainder of this communication uses a tilde over an upper case letter to represent a complex quantity. Time dependence of the complex quantity is explicitly indicated.

The time-varying transmission $h(t)$ has a period T and may be represented by the Fourier expansion

$$h(t) = \sum_{n=-\infty}^{\infty} \tilde{H}_n \epsilon^{jn\omega_s t} \tag{5}$$

where

$$\tilde{H}_n = \frac{1}{T}\int_0^T h(t)\epsilon^{-jn\omega_s t}\,dt \quad \text{and} \quad \omega_s = \frac{2\pi}{T}. \tag{6}$$

Thus,

$$\tilde{E}_2(t) = \tilde{E}_1 \sum_{n=-\infty}^{\infty} \tilde{H}_n \epsilon^{jn\omega_s t}. \tag{7}$$

The voltage $e_2(t) = \Re\{\tilde{E}_2(t)\epsilon^{j\omega_0 t}\}$ consists of sidebands displaced by ω_0 from the harmonics of the sampling frequency ω_s. Each of these sidebands is multiplied by the transmission channel transfer function $\tilde{G}(j\omega)$ evaluated at the particular sideband frequency $(j\omega_0 + jn\omega_s)$, resulting in

$$\tilde{E}_3(t) = \tilde{E}_1 \sum_{n=-\infty}^{\infty} \tilde{H}_n \tilde{G}(j\omega_0 + jn\omega_s)\epsilon^{jn\omega_s t}. \tag{8}$$

The transfer function of the transmission channel $\tilde{G}(j\omega)$ may be written as a partial fraction expansion in terms of the poles of the network as

$$\tilde{G}(j\omega) = \sum_{i=1}^{q}\sum_{k=1}^{r_i} \frac{\tilde{A}_{ik}}{(j\omega + \alpha_i)^k} \tag{9}$$

where it has been assumed that the network has q poles each of multiplicity r_i. This must be evaluated at the frequencies of interest, namely $j\omega = j\omega_0 + jn\omega_s$, so that

$$\tilde{G}(j\omega_0 + jn\omega_s) = \sum_{i=1}^{q}\sum_{k=1}^{r_i} \frac{\tilde{A}_{ik}}{(j\omega_0 + jn\omega_s + \alpha_i)^k}. \tag{10}$$

Introducing the complex frequency $s_i = \alpha_i + j\omega_0$ and using the identity

$$\frac{1}{(s_i + jn\omega_s)^k} = \frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{d^{(k-1)}}{d(s_i)^{(k-1)}}\left[\frac{1}{s_i + jn\omega_s}\right], \tag{11}$$

we may write

$$\tilde{G}(j\omega_0 + jn\omega_s) = \sum_{i=1}^{q}\sum_{k=1}^{r_i} \tilde{A}_{ik}\frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{d^{(k-1)}}{d(s_i)^{k-1}}\left[\frac{1}{s_i + jn\omega_s}\right]. \tag{12}$$

The symbol s is usually reserved for the variable in the Laplace transform of a function. It was with this in mind that $(\alpha_i + j\omega_0)$ was replaced by s_i because it is now convenient to write $1/(s_i + jn\omega_s)$ as

$$\frac{1}{s_i + jn\omega_s} = \underset{\tau_i \to s_i}{\mathcal{L}}\{[u(\tau_i)\epsilon^{-jn\omega_s\tau_i}]\} \tag{13}$$

where $u(\tau_i)$ represents the unit step function in some fictitious time domain τ_i. The notation $\tau_i \to s_i$ that appears under the symbol for the Laplace operator emphasizes the fact that the transformation is from the τ_i domain to the s_i domain and is not associated with the real time variable t. Thus (12) becomes

$$\tilde{G}(j\omega_0 + jn\omega_s) = \sum_{i=1}^{q}\sum_{k=1}^{r_i} \tilde{A}_{ik}\frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{d^{(k-1)}}{d(s_i)^{k-1}} \underset{\tau_i \to s_i}{\mathcal{L}}\{[u(\tau_i)\epsilon^{-jn\omega_s\tau_i}]\}. \tag{14}$$

Introducing this into (8) and moving the infinite summation inside the Laplace operator results in

$$\tilde{E}_3(t) = \tilde{E}_1 \sum_{i=1}^{q}\sum_{k=1}^{r_i} \tilde{A}_{ik}\frac{(-1)^{(k-1)}}{(k-1)!}\frac{d^{(k-1)}}{d(s_i)^{k-1}} \cdot \underset{\tau_i \to s_i}{\mathcal{L}}\left\{u(\tau_i)\sum_{n=-\infty}^{+\infty}\tilde{H}_n\epsilon^{jn\omega_s t}\epsilon^{-jn\omega_s\tau_i}\right\}. \tag{15}$$

The infinite summation in (15) may be recognized as the Fourier series for $h(t)$ delayed by τ_i so that

$$\sum_{n=-\infty}^{\infty}\tilde{H}_n\epsilon^{jn\omega_s t}\epsilon^{-jn\omega_s\tau_i} = h(t - \tau_i). \tag{16}$$

Therefore, (15) becomes

$$\tilde{E}_3(t) = \tilde{E}_1 \sum_{i=1}^{q}\sum_{k=1}^{r_i} \tilde{A}_{ik}\frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{d^{(k-1)}}{d(s_i)^{k-1}} \underset{\tau_i \to s_i}{\mathcal{L}}\{[h(t - \tau_i)u(\tau_i)\}. \tag{17}$$

The voltage $\tilde{E}_3(t)$ is now multiplied by $k(t)$, the time-varying transfer function of the output sampler, to provide the voltage $\tilde{E}_4(t)$. Thus

$$\tilde{E}_4(t) = \tilde{E}_1 \sum_{i=1}^{q}\sum_{k=1}^{r_i} \tilde{A}_{ik}\frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{d^{(k-1)}}{d(s_i)^{k-1}} k(t) \underset{\tau_i \to s_i}{\mathcal{L}}\{[h(t - \tau_i)u(\tau_i)]\}. \tag{18}$$

In this expression for $\tilde{E}_4(t)$, the phasor $\tilde{E}_1$ contributes a time variation at a frequency ω_0. The remaining factor contributes time variation at frequencies which are harmonics of the sampling frequency ω_s. Thus, $e_4(t) = \Re\{\tilde{E}_4(t)\epsilon^{j\omega_0 t}\}$ consists of sidebands displaced by a frequency ω_0 from the harmonics of the sampling frequency ω_s.

The standard method of recovering the information bearing signal from the voltage $\tilde{E}_4(t)$ is to select, by means of a filter, the sideband of the low-frequency harmonic of the sampling frequency. For this selection process to be possible, it is necessary to limit the frequency ω_0 of the input signal to

$$\omega_0 < \frac{\omega_s}{2}. \tag{19}$$

If this condition is met, there can be no overlapping of the sidebands associated with each of the harmonics of the sampling frequency. Thus, an ideal low-pass filter with cutoff frequency $\omega_s/2$ will pass only the desired sideband of the zero-frequency harmonic of the switching waveform.

In order to evaluate this component of $e_4(t)$, which exists at the frequency ω_0, let us rewrite (18) with the factor periodic with fundamental frequency ω_s replaced by a Fourier series

$$\tilde{E}_4(t) = \tilde{E}_1\left[\tilde{B}_0 + \sum_{m=1}^{\infty}\tilde{B}_n \cdot \cos(m\omega_s t + \Phi_m)\right]. \tag{20}$$

Remembering that the symbol $\tilde{E}_1$ carries with it the implication of a frequency ω_0 makes the frequencies of $e_4(t)$ apparent. Thus subject to the restriction of (19) the output of the filter $\tilde{E}_5$ is

$$\tilde{E}_5 = \tilde{E}_1\tilde{B}_0. \tag{21}$$

$\tilde{B}_0$ is the constant term in the Fourier analysis presented in (20), and is a complex number independent of t. We may, therefore, write (18) averaged over a period to find $\tilde{B}_0$:

$$\tilde{E}_5 = \tilde{E}_1 \sum_{i=1}^{q}\sum_{k=1}^{r_i} \tilde{A}_{ik}\frac{(-1)^{k-1}}{(k-1)!}\frac{d^{(k-1)}}{d(s_i)^{k-1}} \cdot \frac{1}{T}\int_0^T k(t) \underset{\tau_i \to s_i}{\mathcal{L}}\{h(t - \tau_i)u(\tau_i)\}\,dt. \tag{22}$$

Thus, to extract the ω_0 component of $e_4(t)$ we have found the average value of the time-varying phasor $\tilde{E}_4(t)$. This gives us $\tilde{E}_5$, a true nontime-varying phasor with respect to the frequency ω_0, which represents the effect of the entire system on the magnitude and phase of the input signal in the sinusoidal steady-state.

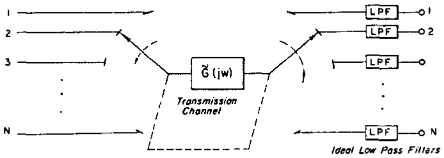

Fig. 3—A model for a time division multiplex system with rectangular sampling waveforms.

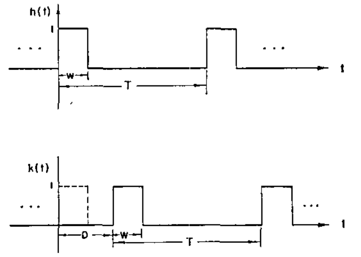

Fig. 4—Switching waveforms $h(t)$ and $k(t)$ for the system of Fig. 3.

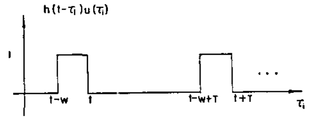

Fig. 5—The waveform $h(t - \tau)u(\tau_i)$ as a function of τ_i.

Eq. (22) presents a general result applicable to a wide variety of transmission channels and to sampling waveforms of arbitrary shape.

Application to Rectangular Switching Waveforms

We shall now apply this general result to the ordinary system where time division is accomplished with synchronized input and output switches. Such a system is shown in Fig. 3.

We shall assume that there is some input to channel 1 and that all of the other inputs are zero. The output of channel 1 will then give the direct transmission, and the outputs of channels 2, 3, $\cdots$, N will represent undesirable crosstalk due to distortion in the transmission channel.

The waveforms $h(t)$ and $k(t)$ that represent this system are shown in Fig. 4. The waveform $k(t)$ is assumed to be delayed by a time D with respect to the switching waveform $h(t)$. For the direct transmission case, $D = 0$.

To evaluate the expression in (22), we must first find

$$\underset{\tau_i \to s_i}{\mathfrak{L}} \{[h(t - \tau_i)u(\tau_i)]\}$$

in the range $0 \leq t \leq T$. We must also be careful to investigate separately the cases where the input and output waveforms do and do not overlap. These two cases are $D \leq W$, and $D \geq W$, respectively. We shall here derive the result for $D \geq W$; and the result for $D \leq W$, which requires a similar but more complicated calculation, will be given without derivation.

Fig. 5 shows $h(t - \tau_i)u(\tau_i)$ as a function of τ_i. Since $k(t) = 0$ when $t < W$, and $k(t)$ is a factor of the integrand in (22), we need only consider values of t greater than W in this calculation. If $D \leq W$, however, there is overlapping of $h(t)$ and $k(t)$, and the range of integration in (22) must be broken up so that $h(t - \tau_i)u(\tau_i)$ can be drawn explicitly. We may now find the Laplace transform required

$$\underset{\tau_i \to s_i}{\mathfrak{L}} \{[h(t - \tau_i)u(\tau_i)]\} = \frac{\epsilon^{-s_i(t-W)} - \epsilon^{-s_i t}}{s_i(1 - \epsilon^{-s_i T})} \quad \text{for} \quad D \geq W, \tag{23}$$

and the integral of (22) becomes

$$\int_0^T k(t) \underset{\tau_i \to s_i}{\mathfrak{L}} \{h(t - \tau_i)u(\tau_i)\}\, dt = \frac{\epsilon^{s_i W} - 1}{s_i(1 - \epsilon^{-s_i T})} \int_D^{D+W} \epsilon^{-s_i t}\, dt = \frac{\epsilon^{-s_i(D-W)}(1 - \epsilon^{-s_i W})^2}{s_i^2(1 - \epsilon^{-s_i T})} \text{for } D \geq W. \tag{24}$$

For the case $D \leq W$, a similar calculation gives

$$\int_0^T k(t) \underset{\tau_i \to s_i}{\mathfrak{L}} \{h(t - \tau_i)u(\tau_i)\}\, dt = \frac{W - D}{s_i} + \frac{1}{s_i^2}\left[1 - \frac{\epsilon^{-s_i D}(2 - \epsilon^{-s_i W} - \epsilon^{-s_i(T-W)})}{1 - \epsilon^{-s_i T}}\right] \quad \text{for} \quad D \leq W. \tag{25}$$

Note the direct transmission term in this last expression due to the overlapping of the input and output waveforms.

The double summation in (22) sums the effects due to the several simple or repeated poles of the transmission channel. We shall now calculate the output signals in a system where the transmission channel has only one simple pole; all the other results can be obtained in terms of a linear combination of this result and its derivatives (with respect to s_i). Let the simple pole be at $-\alpha$ so that

$$\tilde{G}(j\omega) = \frac{\alpha}{j\omega + \alpha}. \tag{26}$$

In most cases of general interest the information-bearing signal can be considered to be dc (when compared with the switching waveforms), so that $s_i = \alpha$ is real, $A_i = \alpha$ and (22) becomes

$$\begin{aligned} \tilde{E}_5 &= \tilde{E}_1 \frac{1}{\alpha T} \frac{\epsilon^{-\alpha(D-W)}(1 - \epsilon^{-\alpha W})^2}{(1 - \epsilon^{-\alpha T})} \quad \text{for} \quad D \geq W \\ &= \tilde{E}_1 \left\{\frac{W - D}{T} + \frac{1}{\alpha T}\left[1 - \frac{\epsilon^{-\alpha D}(2 - \epsilon^{-\alpha W} - \epsilon^{-\alpha(T-W)})}{1 - \epsilon^{-\alpha T}}\right]\right\} \quad \text{for} \quad D \leq W. \end{aligned} \tag{27}$$

When ω_0 cannot be considered to be zero s_i is complex, phase shift is introduced, and (24) and (25) become difficult to use, even if there is only a single pole in the transmission channel.

The crosstalk X_T will be defined as the ratio of the output of the second channel to the output of the first channel. Thus defining D_1 as the time allotted to each channel, the crosstalk is

$$X_T = \frac{\left.\dfrac{E_5}{E_1}\right|_{D=D_1}}{\left.\dfrac{E_5}{E_1}\right|_{D=0}} = \frac{(1 - \epsilon^{-\alpha W})^2 \epsilon^{-\alpha(D_1 - W)}}{\alpha W(1 - \epsilon^{-\alpha T}) - (1 - \epsilon^{-\alpha(T-W)})(1 - \epsilon^{-\alpha W})}. \tag{28}$$

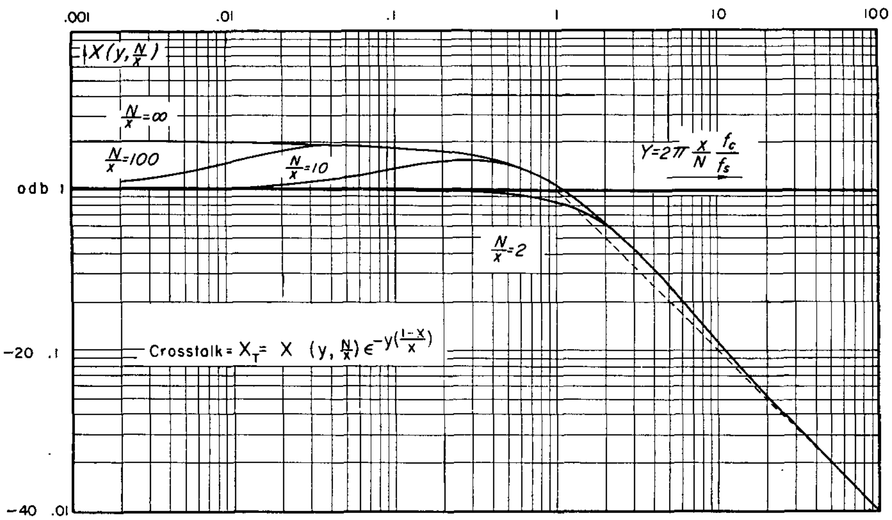

Fig. 6—A plot of the crosstalk factor in terms of the parameters of the time division multiplex system.

The number of channels N is given by

$$\text{number of channels} = N = \frac{T}{D_1}, \quad (29)$$

and the fraction x of time that is utilized for transmission is

$$\text{per unit time used} = x = \frac{W}{D_1}. \quad (30)$$

We may also define the switching frequency f_s as

$$\text{switching frequency} = f_s = \frac{1}{T}, \quad (31)$$

and the 3-db bandwidth of the transmission channel f_c as

$$\text{channel bandwidth} = f_c = \frac{\alpha}{2\pi}. \quad (32)$$

The crosstalk may now be expressed in terms of the parameters of the multiplex system

$$X_T = \frac{(1 - \epsilon^{-2\pi(xf_c/Nf_s)})^2 \epsilon^{-2\pi(x/N)(f_c/f_s)((1-x)/x)}}{2\pi \frac{x}{N}\frac{f_c}{f_s}(1 - \epsilon^{-2\pi(x/N)(f_c/f_s)(N/x)}) - (1 - \epsilon^{-2\pi(x/N)(f_c/f_s)((N/x)-1)})(1 - \epsilon^{-2\pi(x/N)(f_c/f_s)})}. \quad (33)$$

A natural dimensionless parameter is $y = 2\pi x f_c/N f_s$ so that the crosstalk becomes

$$X_T = \frac{(1 - \epsilon^{-y})^2}{y(1 - \epsilon^{-y(N/x)}) - (1 - \epsilon^{-y((N/x)-1)})(1 - \epsilon^{-y})} \cdot \epsilon^{-y((1-x)/x)}. \quad (34)$$

This may be written

$$X_T \doteq X\left(y, \frac{N}{x}\right)\epsilon^{-y((1-x)/x)} \quad (35)$$

where the exponential factor in (35) represents the reduction in crosstalk due to the use of a sampling time less than the alloted channel time, and reduces to unity when $x = 1$, *i.e.*, when the full channel width is used for sampling. The more complicated function $X(y, N/x)$ is plotted vs y in Fig. 6 with N/x as a parameter. In using this graph to compute crosstalks, the value obtained from the graph must be multiplied by $\epsilon^{-y(1-x/x)}$ in accordance with (35). When $y > 5$, the crosstalk may be approximated by

$$X_T \doteq \frac{1}{y-1}\epsilon^{-y((1-x)/x)}. \quad (36)$$

Thus, on log-log paper the factor $X(y, N/x)$ approaches a single straight line asymptote which has a slope of −20 db/dec and passes through the point that represents unity crosstalk and $y = 1$.

Conclusion

The outputs of a generalized time division system have been calculated analytically in closed form. Eq. (22) represents the general result, and (27) gives the outputs when the switching waveforms are rectangular as shown in Fig. 3.

The analysis procedure has consisted essentially of first finding the Fourier series for the desired waveforms, expressing the transmission channel transfer function as a partial fraction expansion, and then putting the result in closed form by the use of a Laplace transform. This amounts to a somewhat unusual method of finding the sinusoidal steady-state in a system where the input sinusoid is applied through a time-varying switch to a linear network.

Finally, the normalized crosstalk ratio in a system with a single pole was calculated and plotted. Whenever the transmission channel can be characterized in this simple manner, the results of (35) and Fig. 6 will enable the designer to assay the effects upon the crosstalk of the channel bandwidth, switching frequency, and the use of guardbands in time around the sampling pulses.

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