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BASES IN HILBERT SPACE RELATED TO THE REPRESENTATION OF STATIONARY OPERATORS*

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1. Introduction. Every complete orthonormal set of functions in $L^2(dt: -\infty, \infty)$ induces an isomorphism from the space $L^2(dt: -\infty, \infty)$ of continuous-time signals onto the space l^2 of discrete-time signals. If the Laguerre set is used, it has been shown [1] that each stationary continuous-time linear filter has an isomorphically equivalent discrete-time linear filter which is also stationary, and vice versa. The first part of this paper deals with the problem of characterizing all those bases in $L^2(dt: -\infty, \infty)$ with this property.

We formulate the problem abstractly as follows: let $B(\Lambda)$ be the space of $L^2(dt: -\infty, \infty)$ functions whose Fourier transforms vanish a.e. outside of the Lebesgue measurable set $\Lambda \subset R^1$. We fix the basis in l^2 to be the standard basis. Each basis in $B(\Lambda)$ establishes an isomorphism μ from l^2 onto $B(\Lambda)$. We define two Banach spaces Σ and $\hat{\Sigma}$ of bounded linear operators which are stationary in their respective domains of definition $B(\Lambda)$ and l^2. It is required to find a necessary and sufficient condition on the isomorphism μ (or equivalently on the bases in $B(\Lambda)$) such that the Banach spaces of operators Σ and $\hat{\Sigma}$ be isomorphically equivalent, i.e., $A \in \Sigma$ implies $\hat{A} = \mu^{-1}A\mu \in \hat{\Sigma}$ and, conversely, $\hat{A} \in \hat{\Sigma}$ implies $A = \mu\hat{A}\mu^{-1} \in \Sigma$.

It is shown that the set $\{e_n(t)\}_{n=-\infty}^{\infty}$ of complete orthonormal functions in $B(\Lambda)$ must have the form

$$E_n(j\omega) = G(j\omega)e^{jn\varphi(\omega)} \tag{1}$$

as their Fourier transforms.

This brings us to the second part of the paper which is concerned with the related question: under what conditions does a set $\{e_n(t)\}$ characterized by (1) constitute a basis in $B(\Lambda)$? For convenience, we take Λ to be a finite or infinite interval (a, b). It is shown, under the assumption $0 \leqq \varphi(\omega) \leqq 2\pi$, that the set $\{e_n(t)\}$ is orthonormal if and only if the set E_y defined by $E_y(\omega) = \{\omega \mid \varphi(\omega) \leqq y\}$ satisfies the measure condition

$$\beta(E_y) = y \quad \text{for all} \quad y \in [0, 2\pi], \tag{2}$$

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where the measure β is given by

$$\beta(\omega) = \int_a^\omega |G(jx)|^2 \, dx.$$

It is shown further that each $g(t) \in B(a,b)$, $\|g(t)\| = 1$, whose Fourier transform is nonzero a.e., generates a complete orthonormal set of functions $\{e_n(t)\}$ in $B(a, b)$ given by

$$(3) \qquad e_n(t) = \text{l.i.m.} \frac{1}{2\pi} \int_a^b G(j\omega) \exp\left(jn \int_a^\omega |G(jx)|^2 \, dx\right) e^{j\omega t} \, d\omega.$$

In particular, the Laguerre functions in $B(-\infty, \infty)$ and the cardinal functions in $B(-2\pi W, 2\pi W)$ belong to the above class. Other bases in $B(a, b)$ can easily be constructed. These bases can be applied to the analysis and synthesis of signals and systems.

2. Preliminaries. Let $B(\Lambda)$ denote the space of $L^2(dt: -\infty, \infty)$, possibly complex, functions whose Fourier transforms vanish a.e. outside of the Lebesgue measurable set Λ, i.e.,

$$(4) \quad B(\Lambda) = \{f(t) \in L^2(dt: -\infty, \infty) \mid F(\omega) = 0 \quad \text{a.e. on} \quad R^1 - \Lambda\}.$$

In particular, we shall consider $\Lambda = (a, b)$. Thus, if $(a, b) = (-\infty, \infty)$, $B(\Lambda)$ is $L^2(dt: -\infty, \infty)$. On the other hand, if (a, b) is a finite interval, $B(\Lambda)$ is the space of square integrable "bandlimited" functions. $B(\Lambda)$ is a Hilbert space.

The space $\mathfrak{L}$ of all bounded linear operators with domain $B(\Lambda)$ and range $L^2(dt: -\infty, \infty)$ is a Banach space [2, p. 161]. Define for every $\tau \in R^1$ the shift operator J_τ by

$$(5) \qquad J_\tau f(t) = f(t + \tau) \quad \text{for all} \quad f(t) \in B(\Lambda).$$

It is clear that J_τ preserves inner product, i.e.,

$$(6) \qquad (J_\tau f(t), J_\tau g(t)) = (f(t), g(t)) \quad \text{for all} \quad f(t), g(t) \in B(\Lambda).$$

Let T denote the space of all such shift operators. Then $T \subset \mathfrak{L}$. In fact, T is a one-parameter group of unitary transformations from $B(\Lambda)$ onto $B(\Lambda)$. The space $\Sigma \subset \mathfrak{L}$ of all bounded linear operators which commute with shift operators is defined by

$$(7) \qquad \Sigma = \{A \in \mathfrak{L} \mid AJ_\tau = J_\tau A \text{ for all } \tau \in R^1\}.$$

Lemma 1. *The space Σ is a Banach space.*

Proof. It is easy to verify that Σ is a normed linear space. We need only to prove completeness. Let $\{A_n\}$ be a Cauchy sequence in Σ. Since $A_n \in \mathfrak{L}$, there exists an $A \in \mathfrak{L}$ such that $A_n \to A$ in norm. We claim that $A \in \Sigma$.

To this end consider

$$\| AJ_\tau f - J_\tau Af \| = \| AJ_\tau f - A_n J_\tau f + A_n J_\tau f - J_\tau Af \|$$
$$\leqq \| A - A_n \| \, \| J_\tau \| \, \| f \| + \| J_\tau \| \, \| A_n - A \| \, \| f \|.$$

From $\| J_\tau \| = 1$ and $\| A_n - A \| \to 0$, we have

$$\| AJ_\tau f - J_\tau Af \| = 0. \tag{8}$$

The result follows after taking supremum over $f(t) \in B(\Lambda)$ with $\| f(t) \| = 1$.

Our next assertion is that the range of each $A \in \Sigma$ is in $B(\Lambda)$. This follows easily from a theorem due to S. Bochner [3, Theorem 72] which we state as a lemma.

LEMMA 2. *Let A be a bounded linear operator from $L^2(dt: -\infty, \infty)$ to $L^2(dt: -\infty, \infty)$ which commutes with shift operators and let $Af = g$, $f(t) \in L^2(dt: -\infty, \infty)$. Then, there exists a bounded measurable function $W(j\omega)$ such that*

$$G(j\omega) = W(j\omega)F(j\omega),$$

where $F(j\omega)$ and $G(j\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$, respectively.

Next we consider the Hilbert space l^2 of all complex-valued square summable sequences $\mathbf{f} = \{f_n\}_{n=-\infty}^{\infty}$ with its appropriate spaces of operators. We give no proofs since they can be carried out in the same manner as for $B(\Lambda)$.

The space $\hat{\mathfrak{L}}$ of all bounded linear operators with domain and range l^2 is a Banach space. Define for every integer r the shift operator $\hat{J}_r$ by

$$\hat{J}_r \mathbf{f} = \{f_{n+r}\}_{n=-\infty}^{\infty} \quad \text{for all} \quad \mathbf{f} \in l^2. \tag{9}$$

Again,

$$(\hat{J}_r \mathbf{f}, \hat{J}_r \mathbf{g}) = (\mathbf{f}, \mathbf{g}) \quad \text{for all} \quad \mathbf{f}, \mathbf{g} \in l^2. \tag{10}$$

Let $\hat{T}$ denote the space of all such shift operators. It follows that $\hat{T} \subset \hat{\mathfrak{L}}$ and $\hat{T}$ is a one-parameter group of unitary transformations from l^2 onto l^2. The space $\hat{\Sigma} \subset \hat{\mathfrak{L}}$ of all bounded linear operators which commute with shift operators is defined by

$$\hat{\Sigma} = \{\hat{A} \in \hat{\mathfrak{L}} \mid \hat{A}\hat{J}_r = \hat{J}_r \hat{A}, \, r \text{ an integer}\}. \tag{11}$$

LEMMA 3. *The space $\hat{\Sigma}$ is a Banach space.*

3. Isomorphic equivalence between the Banach spaces Σ and $\hat{\Sigma}$. Let $\{\mathbf{e}_n\}_{n=-\infty}^{\infty}$ be the standard basis in l^2, i.e., the nth component of $\mathbf{e}_n$ is unity and other components are zero. Denote by μ an isomorphism from l^2 onto

$B(\Lambda)$. Then the set $\{e_n(t) = \mu \mathbf{e}_n\}$ is the corresponding complete orthonormal set in $B(\Lambda)$. Let the Banach spaces of operators $\mathfrak{L}$, $\hat{\mathfrak{L}}$, Σ and $\hat{\Sigma}$ be defined as in §2.

The isomorphism μ induces a relationship between operators. If $A \in \Sigma$, then the operator $\mu^{-1}A\mu$ belongs to $\hat{\mathfrak{L}}$. Conversely, if $\hat{A} \in \hat{\Sigma}$ then $\mu\hat{A}\mu^{-1}$ belongs to $\mathfrak{L}$. In fact, $\mu\hat{A}\mu^{-1}$ has domain and range $B(\Lambda)$, but it may fail to be in Σ.

However, there exists a certain class of isomorphisms μ which induce an isomorphic equivalence [4, §36] between Σ and $\hat{\Sigma}$, i.e., for each $A \in \Sigma$, there corresponds a unique operator $\hat{A} \in \hat{\Sigma}$ given by $\mu^{-1}A\mu$. Conversely, $\mu\hat{A}\mu^{-1}$ belongs to Σ and is the operator corresponding to $\hat{A} \in \hat{\Sigma}$. Thus, isomorphically equivalent operators $A \in \Sigma$ and $\hat{A} \in \hat{\Sigma}$ are related by

$$A = \mu\hat{A}\mu^{-1}. \tag{12}$$

We intend to characterize this class.

THEOREM 1. *A necessary and sufficient condition for the Banach spaces of operators Σ and $\hat{\Sigma}$ to be isomorphically equivalent is that the complete orthonormal set $\{e_n(t)\}_{n=-\infty}^{\infty}$ be of the form*

$$e_n(t) = \text{l.i.m.} \frac{1}{2\pi} \int_\Lambda G(j\omega) e^{jn\varphi(\omega)} e^{j\omega t} \, d\omega \tag{13}$$

for some, possibly complex, $G(j\omega) \in L^2(d\omega : \Lambda)$ and real measurable $\varphi(\omega)$.

Proof. Necessity. Let $\{e_n(t)\}$ be a complete orthonormal set of functions in $B(\Lambda)$. Then

$$f(t) = \text{l.i.m.} \sum_{n=-\infty}^{\infty} f_n e_n(t) \tag{14}$$

with

$$f_n = (f(t), e_n(t)), \tag{15}$$

and the Parseval relation

$$\|f(t)\|^2 = \sum_{n=-\infty}^{\infty} |f_n|^2 \tag{16}$$

holds. Let

$$h(t) = Af(t), \quad A \in \Sigma, \quad f(t) \in B(\Lambda). \tag{17}$$

It was shown in §2 that $h(t) \in B(\Lambda)$. Thus

$$h(t) = \text{l.i.m.} \sum_{k=-\infty}^{\infty} h_k e_k(t). \tag{18}$$

Since every operator in Σ has a matrix representation [4, §26], we rewrite

(18) in the matrix form

$$\mathbf{h} = \hat{A}\mathbf{f}, \tag{19}$$

where $\mathbf{h}$ and $\mathbf{f}$ are column vectors with components h_n and f_n, $n = \cdots, -1, 0, 1, \cdots$, respectively, and $\hat{A}$ is the matrix

$$\hat{A} = [a_{k,n}],$$

where

$$a_{k,n} = (Ae_n(t), e_k(t)). \tag{20}$$

By hypothesis, $\hat{A}$ belongs to $\hat{\Sigma}$. Therefore we must have (in operational form)

$$\hat{J}_r\hat{A}\mathbf{f} = \hat{A}\hat{J}_r\mathbf{f} \quad \text{for all} \quad r,$$

which implies

$$\sum_{n=-\infty}^{\infty} f_n[a_{k+r,n} - a_{k,n-r}] = 0 \quad \text{for all} \quad k, r. \tag{21}$$

Since $\mathbf{f}$ is arbitrary, we conclude that $a_{r,n}$ depends only on the difference of the indices. That is, $\hat{A}$ is a Toeplitz matrix.

From (20), Parseval's theorem and Bochner's theorem (see §2), $a_{k,n}$ is given by

$$a_{k,n} = \frac{1}{2\pi}\int_\Lambda W(j\omega)E_n(j\omega)E_k^*(j\omega)\,d\omega. \tag{22}$$

Since $W(j\omega)$ is arbitrary, the product $E_n(j\omega)E_k^*(j\omega)$ should depend on the difference $n - k$ for all integers n, k and for almost every ω. Consider first $n = k$. $|E_n(j\omega)|^2$ is independent of n. Therefore we can write $E_n(j\omega)$ in the form

$$E_n(j\omega) = B(\omega)e^{j\varphi_n(\omega)}, \tag{23}$$

where

$$B(\omega) = |E_n(j\omega)| \quad \text{for all} \quad n.$$

Next consider $n \neq k$. Then, $E_n(j\omega)E_k^*(j\omega)$ can be written in the form

$$E_n(j\omega)E_k^*(j\omega) = B^2(\omega)e^{j[\varphi_n(\omega)-\varphi_k(\omega)]} = B^2(\omega)e^{j\gamma(n-k,\omega)}$$

for some real measurable function $\gamma(\cdot,\cdot)$. In particular, if we let $n - k = 1$, we get a recursive formula

$$e^{j\varphi_{k+1}(\omega)} = e^{j\gamma(1,\omega)}e^{j\varphi_k(\omega)} \quad \text{for all} \quad k. \tag{24}$$

Therefore

$$e^{j\varphi_k(\omega)} = e^{jk\gamma(1,\omega)}e^{j\varphi_0(\omega)} \quad \text{for all} \quad k. \tag{25}$$

Equations (23) and (25) imply

$$E_n(j\omega) = B(j\omega)e^{j\varphi_0(\omega)}e^{jn\gamma(1,\omega)}$$

which can finally be written as

$$E_n(j\omega) = G(j\omega)e^{jn\varphi(\omega)}, \tag{26}$$

where $\varphi(\omega)$ is a real measurable function and $G(j\omega)$ may be complex. Note that both $G(j\omega)$ and $\varphi(\omega)$ are independent of n.

Sufficiency. It is clear that (26) is sufficient, since the matrix $\hat{A} = [a_{k,n}]$ is then Toeplitz and hence $\hat{A}$ commutes with every shift operator. Similarly, it can be shown that if $\hat{A} \in \hat{\Sigma}$ and (26) holds, then $A \in \Sigma$.

4. Orthonormality and completeness of the set $\{e_n(t)\}$. In the preceding section we found the general form that the desired basis functions in $B(\Lambda)$ should take. We now investigate the problem of orthonormality and completeness of the set of functions characterized by (26). This will give us a class of bases in Hilbert space which have a common functional structure.

Since the Fourier transform is a unitary transformation [4, §37], it suffices to consider the sets $\{E_n(j\omega)\}$ instead of $\{e_n(t)\}$. In the sequel, for matters of convenience, we shall take $\Lambda = (a, b)$, where (a, b) can be a finite or infinite interval.

The orthonormality requirement is

$$\frac{1}{2\pi}\int_a^b |G(j\omega)|^2 e^{jr\varphi(\omega)}\, d\omega = \delta_{0r}, \tag{27}$$

whereas the closure property is

$$\lim_{N\to\infty}\left\| F(j\omega) - \sum_{n=-N}^{N} C_n E_n(j\omega)\right\| = 0, \quad F(j\omega) \in L^2(d\omega: a, b). \tag{28}$$

It follows that $G(j\omega)$ cannot vanish on subsets of Λ of positive measure and that $\| G(j\omega)\| = 1$.

Without loss of generality we assume that $0 \leqq \varphi(\omega) \leqq 2\pi$.

The following lemma will be used later on in the proof of a theorem on orthonormality. It concerns integrals of composite functions.

LEMMA 4 [5, p. 127]. *Let $\varphi(x)$ be a function integrable with respect to the non-decreasing function $\beta(x)$, $a \leqq x \leqq b$. Let $e_y(x)$ be the characteristic function of the set E_y defined by*

$$E_y(x) = \{x \mid \varphi(x) \leqq y\}.$$

Let

$$\alpha(y) = \int_a^b e_y(x)\, d\beta(x), \qquad -\infty < y < \infty.$$

Let $f(y)$ be a function integrable with respect to $\alpha(y)$. Then $f(\varphi(x))$ is integrable with respect to $\beta(x)$ and

$$\int_a^b f(\varphi(x))\, d\beta(x) = \int_{-\infty}^{\infty} f(y)\, d\alpha(y). \tag{29}$$

THEOREM 2. *Let $0 \leqq \varphi(\omega) \leqq 2\pi$. The set of functons $\{E_n(j\omega) = G(j\omega)e^{jn\varphi(\omega)}\}$ is orthonormal if and only if the set $E_y(\omega) = \{\omega \mid \varphi(\omega) \leqq y\}$ satisfies the measure condition*

$$\beta(E_y) = y \quad \textit{for all} \quad y \in [0, 2\pi], \tag{30}$$

where the measure β is defined by

$$\beta(\omega) = \int_a^\omega |G(jx)|^2\, dx.$$

Proof. Necessity. Define

$$\alpha(y) = \int_a^b e_y(\omega)\, d\beta(\omega),$$

where $e_y(\omega)$ is the characteristic function of the set $E_y(\omega)$, $\alpha(y)$ is an increasing function, $0 \leqq y < \infty$.

The orthonormality condition (27) can be written in the form

$$\frac{1}{2\pi}\int_a^b e^{jr\varphi(\omega)}\, d\beta(\omega) = \delta_{0r}. \tag{31}$$

By Lemma 4, we have

$$\delta_{0r} = \frac{1}{2\pi}\int_0^\infty e^{jry}\, d\alpha(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{jry}\, d\alpha(y), \tag{32}$$

where the second equality follows from $\alpha(y) = 2\pi$ on $[2\pi, \infty)$. Now, F. and M. Riesz [6, p. 263] have proved that the equations

$$\frac{1}{2\pi}\int_0^{2\pi} e^{jry}\, d\alpha(y) = 0, \qquad r = 1, 2, \cdots,$$

where $\alpha(y)$ is of bounded variation, imply the absolute continuity of $\alpha(y)$.

It follows from the Lebesgue-Stieltjes formula for integration by parts, from the completeness of the functions e^{jry} in $L^2(dy{:}\,0, 2\pi)$, and from Riesz's theorem, that

$$\alpha(y) = y, \quad y \in [0, 2\pi]. \tag{33}$$

Thus

$$\beta(E_y) = y, \quad y \in [0, 2\pi]. \tag{34}$$

The sufficiency part of the proof is trivial.

In particular, if $\varphi(\omega)$ is taken to be

$$\varphi(\omega) = \int_a^\omega |G(jx)|^2 \, dx,$$

the set $\{E_n(j\omega)\}$ is orthonormal since

$$\frac{1}{2\pi} \int_a^b |G(j\omega)|^2 e^{jr\varphi(\omega)} \, d\omega = \frac{1}{2\pi} \int_0^{2\pi} e^{jry} \, dy = \delta_{0r}.$$

It should be noted, however, that this $\varphi(\omega)$ is *not* the only function which satisfies the measure condition (30). One can easily construct functions $\varphi(\omega)$ which will satisfy the measure condition without being of the above form. On the other hand, by imposing further restrictions on $\varphi(\omega)$, the measure condition may determine $\varphi(\omega)$ uniquely. An example is provided by the following corollary.

COROLLARY. *Suppose, moreover, that $\varphi(\omega)$ is absolutely continuous and monotonically increasing. Then the set $\{E_n(j\omega)\}$ is orthonormal if and only if $\varphi(\omega) = \int_a^\omega |G(jx)|^2 \, dx$.*

Proof. $\varphi^{-1}(\omega)$ exists and is monotonically increasing. Therefore

$$\{\omega \mid \varphi(\omega) \leqq y\} = (\varphi^{-1}(y), a).$$

By the measure condition (30),

$$y = \int_a^{\varphi^{-1}(y)} |G(jx)|^2 \, dx = \beta(\varphi^{-1}(y)).$$

Since $\beta(\omega)$ and $\varphi(\omega)$ have unique inverses, we have

$$\varphi(\omega) = \beta(\omega) = \int_a^\omega |G(jx)|^2 \, dx. \tag{35}$$

Next we consider the closure properties of the set $\{E_n(j\omega)\}$ in $L^2(d\omega\colon a, b)$. While in the orthonormality theorem the only restriction imposed on $\varphi(\omega)$ was $0 \leqq \varphi(\omega) \leqq 2\pi$, it seems inevitable that further assumptions on $\varphi(\omega)$ be made.

THEOREM 3. *Let $\varphi(\omega)$ be absolutely continuous, $0 \leqq \varphi(\omega) \leqq 2\pi$, $\varphi(a) = 0$, $\varphi(b) = 2\pi$. Let the measure condition (30) be satisfied (for orthonormality). Suppose we can write $\varphi'(\omega) = G^*(j\omega)L(j\omega)$ such that $L(j\omega) \in L^2(d\omega\colon a, b)$. Then the orthonormal set of functions $\{E_n(j\omega)\}$ is complete in $L^2(d\omega\colon a, b)$ if and only if $\varphi(\omega) = \int_a^\omega |G(jx)|^2 \, dx$.*

Proof. Necessity. Let $f(\omega) = L(j\omega) - G(j\omega)$. Clearly $f(\omega) \in L^2(d\omega\colon a, b)$. It follows easily that the inner products

$$(f(\omega), E_n(j\omega)) = 0, \qquad n = 0, \pm 1, \pm 2, \cdots.$$

By the completeness hypothesis,

$$f(\omega) = 0 \quad \text{a.e.} \tag{36}$$

Therefore

$$\varphi(\omega) = \int_a^\omega |G(jx)|^2\, dx.$$

Sufficiency. Let $\eta > 0$ be given. Let $F(j\omega)$ be an arbitrary function in $L^2(d\omega: a, b)$.

Let

$$K(j\omega) = \frac{F(j\omega)}{G(j\omega)}. \tag{37}$$

Define

$$K_B(j\omega) = \begin{cases} K(j\omega) & \text{if } |K(j\omega)| < B, \\ B\dfrac{K(j\omega)}{|K(j\omega)|} & \text{if } |K(j\omega)| \geqq B, \end{cases} \tag{38}$$

where B is a positive real number. The sequence $|K_B(j\omega)G(j\omega) - F(j\omega)|^2$ is decreasing everywhere (as $B \to \infty$) and tends to zero a.e. Note that we have used the fact that $G(j\omega)$ does not vanish on sets of positive measure. It follows from the monotone convergence theorem [2, p. 72] that

$$\lim_{B\to\infty} \int_a^b |K_B(j\omega)G(j\omega) - F(j\omega)|^2\, d\omega = 0.$$

Therefore, there exists a positive real number $B = B(\eta)$ such that

$$\int_a^b |K_B(j\omega)G(j\omega) - F(j\omega)|^2\, d\omega < \eta. \tag{39}$$

Next consider

$$\begin{aligned} &\int_a^b \left| K_B(j\omega)G(j\omega) - \sum_{n=-N}^{N} C_n G(j\omega) \exp\left(jn \int_a^\omega |G(jx)|^2\, dx\right)\right|^2 d\omega \\ &= \int_a^b \left| K_B(j\omega) - \sum_{n=-N}^{N} C_n \exp\left(jn \int_a^\omega |G(jx)|^2\, dx\right)\right|^2 |G(j\omega)|^2\, d\omega \\ &= \int_0^{2\pi} \left| K_B(j\omega^{-1}(\lambda)) - \sum_{n=-N}^{N} C_n e^{jn\lambda}\right|^2 d\lambda, \end{aligned} \tag{40}$$

where we have let $\lambda = \int_a^\omega |G(jx)|^2\, dx$. We note that $K_B(j\omega^{-1}(\lambda))$ is a bounded function and thus belongs to $L^2(d\lambda: 0, 2\pi)$. By the completeness of the set $\{e^{jn\lambda}\}_{n=-\infty}^{\infty}$ in $L^2(d\lambda\colon 0, 2\pi)$, there exist numbers N and C_n such that the integral (40) is less than η. It follows from (39), (40) and the Minkowski inequality that

$$(41) \quad \int_a^b \left| F(j\omega) - \sum_{n=-N}^{N} C_n G(j\omega) \exp\left(jn \int_a^\omega |G(jx)|^2\, dx\right)\right|^2 d\omega < 4\eta.$$

5. Summary and examples. We summarize the discussion of the previous section. Each $G(j\omega) \in L^2(d\omega\colon a, b)$ which is nonzero a.e. generates a complete orthonormal set $\{E_n(j\omega)\}$ in $L^2(d\omega\colon a, b)$ given by

$$E_n(j\omega) = G(j\omega) \exp\left(jn \int_a^\omega |G(jx)|^2\, dx\right).$$

Since the Fourier transform is a unitary transformation, the set

$$e_n(t) = \text{l.i.m.}\, \frac{1}{2\pi} \int_a^b E_n(j\omega) e^{j\omega t}\, d\omega$$

is orthonormal and complete in $B(a, b)$.

We give three examples.

Example 1. Let

$$(a, b) = (-2\pi W, 2\pi W),$$
$$G(j\omega) = (1/\sqrt{2W})[u(\omega + 2\pi W) - u(\omega - 2\pi W)],$$

where $u(\cdot)$ is the step function. Then

$$e_n(t) = \sqrt{2W}\, \frac{\sin[2\pi Wt + n\pi]}{2\pi Wt + n\pi},$$

which is the set of cardinal functions used in the sampling representation of bandlimited functions.

Example 2. Let $(a, b) = (-\infty, \infty)$, $g(t) = \sqrt{2}e^{-t}u(t)$. Then

$$E_n(j\omega) = \frac{\sqrt{2}}{1 + j\omega}\, e^{jn2 \arctan \omega},$$

$e_n(t)$ is the Laguerre function

$$e_n(t) = \begin{cases} -\sqrt{2}e^t L_{n-1}(-2t)u(-t) & \text{if } n \geqq 1, \\ \sqrt{2}e^{-t}L_{-n}(2t)u(t) & \text{if } n \leqq 0, \end{cases}$$

and $L_n(t)$ is the Laguerre polynomial of degree n,

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n \geqq 0.$$

Example 3. The Laguerre and cardinal functions are the most common sets that belong to this class. However, one can easily construct other sets.

For example:

$$(a, b) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); \qquad G(j\omega) = 2 \cos \omega.$$

Then

$$e_n(t) = g_n(t + 2n + 1) + g_n(t + 2n - 1),$$

where $g_n(t)$ is related to the Anger function $J_\nu(x)$ [7, p. 35] by

$$g_n(t) = \tfrac{1}{2} J_{t/2}(-n).$$

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