ADAPTIVE SIGNAL RECONSTRUCTION

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ABSTRACT

An adaptive filter which reconstructs a continuous signal from its samples is described. This filter is based on the minimum mean-square-error reconstruction filter, assuming an all-pole model for the sampled spectral density of the input signal. The use of this model leads to two important simplifications. First, simple linear regression can be used to identify the unknown parameters of the signal spectral density. Second, the resulting filter has an impulse response which is of finite duration. These simplifications lead to an adaptive filter which is at the same time both generally applicable and easily implemented on a digital or hybrid computer. Experiments with both deterministic and random inputs are described which show that the adaptive filter yields significant improvement over a linear point connector or other commonly used reconstructors with relatively low order models and with relatively short identification times.

I. INTRODUCTION

In many information processing systems it is necessary to reconstruct continuous signals from equally spaced samples. Inexpensive devices for doing this in real time have been widely used for many years, the simplest of these being low pass filters or zero-order (boxcar) holds. These are adequate in many situations. When redundant data is unavailable or comes at a high cost, however, it may become feasible to increase the complexity of the reconstruction process, either to extract maximum information from a digital signal or to minimize bandwidth requirements. This suggests the use of a reconstruction filter which adapts to the spectral density of the incoming sampled signal.

For the case where the signals are wide sense stationary random processes, where the processing is to approximate a linear operation L on the continuous signal, and where a mean-square-error criterion is used, the solution to the optimum filter problem has been known for several years [1,2,3]. Consider the general solution illustrated in Fig. 1, where the original continuous signal is corrupted by additive noise n(t) before sampling. The optimum reconstruction filter, H(s), without regard to realizability is given by

$$H(s) = \frac{L(s)\Phi_{yf}(s)}{\Phi_{yy}(z)} \qquad (1)$$

where
- $L(s)$ is the desired operation on the input signal $f(t)$,
- $\Phi_{yf}(s)$ is the cross power spectral density between $f(t)$ and $y(t) = f(t) + n(t)$,
- $\Phi_{yy}(z)$ is the sampled power spectral density of $y(t)$, where $z = e^{sT}$, and T is the sampling interval.

In order to construct the above optimum filter it is necessary either to know *a priori* the various self and cross spectral densities, or to use some type of identification technique to learn these from the signals themselves [4]. In this paper it will be assumed that only the sampled signal is available for making the identification as would be the case if the reconstruction were at the receiving end of a pulse code modulation system. A previously described identification method will be used [5] and it will be shown that this method, based on an all-pole model for the sampled spectral density $\Phi_{yy}(z)$, leads to simple and useful adaptive reconstruction filters.

II. THE IDENTIFICATION - ADAPTATION SCHEME

Consider first the case where the original analog signal f(t) is uncorrupted by noise. The sampled spectral density $\Phi_{ff}(z)$, and the continuous spectral density $\Phi_{ff}(s)$ must be identified to synthesize the optimum filter. It will be assumed that the sampled spectral density is all-pole, i.e. of the form

$$\Phi_{ff}(z) = \frac{\beta^2}{D(z)D(z^{-1})} \qquad (2)$$

where

$$D(z) = \sum_{n=o}^{p} d_n z^{-n}, \quad d_0 = 1$$

and β^2 is a positive real constant; or can be approximated by such a form for sufficiently large p. While this assumption introduces an approximation problem, it results in the following two vital simplifications:

1. The simple linear regression technique described previously [5,6] can be used to estimate the parameters d_n and β^2. The identification proceeds as follows: first the p + 1 mean-lagged-products

$$c_j = \overline{f_i f_{i+j}} \qquad j = 0,1,\ldots,p$$

are computed from the available samples of f(t). Then the following estimates are used:

$$\begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_p \end{bmatrix} = - \begin{bmatrix} c_0 & c_1 & c_2 & \cdots c_{p-1} \\ c_1 & c_0 & c_1 & \cdots c_{p-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{p-1} & c_{p-2} & c_{p-3} & \cdots c_0 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_p \end{bmatrix}$$

and

$$\beta^2 = c_0 + d_1 c_1 + \ldots + d_p c_p \quad .$$

Experimental evidence [6] has indicated that this all-pole model is useful even when the spectral density to be identified is all-zero, if p is of the order of 4 to 8. This can be justified heuristically in the following way:
Suppose

$$\Phi_{ff}(z) = \frac{N(z)N(z^{-1})}{D(z)D(z^{-1})}$$

where N(z) and D(z) are finite polynomials in z^{-1} with roots inside the unit circle. Then this can be written

$$\Phi_{ff}(z) = \frac{1}{\dfrac{D(z)D(z^{-1})}{N(z)N(z^{-1})}} = \frac{1}{P(z)P(z^{-1})}$$

where P(z) is an infinite series in z^{-1}. The all-pole model identifies D(z) as a truncated version of P(z), which is valid since the coefficients of P(z) converge to zero in magnitude.

2. The impulse response of the optimum reconstruction filter is of only finite duration and the problem of its unrealizability is eliminated if we allow a delay of p sample periods. That is

$$h(t) \equiv 0 \quad \text{for } |t| \geq pT \ .$$

This has been shown elsewhere by the authors [7] and will not be proven here. Hence for the p-th order all-pole processes only p samples in the past and p in the future are needed for the best possible reconstruction at a given time, in the minimum mean-square-error sense. In other words, only a delay of p sampling periods is necessary for the best possible reconstruction. This delay of pT is easily introduced if the reconstruction is performed on a digital computer. Yaglom [8] has referred to processes of this type as p-th order Markov processes.

For processes with all-pole spectral densities, then, the optimum reconstruction filter is given by substituting (2) into (1) (L(s)=1 for reconstruction) :

$$h(t) = \mathcal{L}^{-1}\Big[\sum_{m=0}^{p} \sum_{n=0}^{p} \frac{d_m d_n}{\beta^2} z^{-m} z^{n} \Phi_{ff}(s) \Big]$$

$$= \frac{1}{\beta^2} \sum_{m=0}^{p} \sum_{n=0}^{p} d_m d_n \, \varphi_{ff}(t-mT+nT) \tag{3}$$

where $\varphi_{ff}(t)$ is the inverse z-transform of

$$\Phi_{ff}(z) = \sum_{n=-\infty}^{\infty} \varphi_{ff}(nT) z^{-n}$$

and can be computed uniquely from $\Phi_{ff}(z)$, the estimated sampled spectral density of f(t), with the assumption that all poles of $\Phi_{ff}(s)$ lie in the strip

$$-\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T} \quad \text{in the s-plane} \ .$$

In practical situations the sampling rate will be fast enough so that this will be a reasonable assumption. The specific method employed is to partial-fraction $\Phi_{ff}(z)$ as

$$\Phi_{ff}(z) = \frac{\beta^2}{D(z)D(z^{-1})}$$

$$= \sum_{i=1}^{p} \frac{A_i}{1-e^{-a_i T} z^{-1}} + \sum_{i=1}^{p} \frac{A_i e^{-a_i T} z}{1-e^{-a_i T} z} \ ,$$

and take

$$\varphi_{ff}(t) = \sum_{i=1}^{p} A_i e^{-a_i |t|} \ .$$

Using this, h(t) can be computed from (3) and the reconstructed signal is given by

$$\hat{f}(t) = \sum_{n=k-p+1}^{k+p} f(nT)h(t-nT) \tag{4}$$

where $kT \leq t \leq (k+1)T$. When this operation is being performed on a digital computer, the impulse response (3) is calculated once at sufficiently small increments of time and stored in a table, and the filter (4) implemented as a moving average, filling in points between sampling instants. Fig. 2 shows a block diagram of the identification - reconstruction process. For the experimental results reported in the next section, the adaptive filter was implemented entirely by a digital computer.

If the continuous analog signal is corrupted by an additive noise component, n(t), the situation is slightly more complicated. Assuming that only the samples of y(t) can be used for the identification procedure, the optimum filter cannot be determined since it would not be possible to estimate $\Phi_{yf}(s)$. However, it is normally reasonable to assume that the signal and noise are independent and also that there are periods when the signal is not being transmitted when the noise alone can be identified. With these assumptions the optimum reconstruction filter is given by:

$$H(s) = \frac{\Phi_{ff}(s)}{\Phi_{yy}(z)} = \frac{\Phi_{yy}(s) - \Phi_{nn}(s)}{\Phi_{yy}(z)}$$

Thus the optimum filter when uncorrelated noise is present is made up of two components: the first,

$$\frac{\Phi_{yy}(s)}{\Phi_{yy}(z)} \quad ,$$

can be shown to connect the sample points [7]; the second,

$$- \frac{\Phi_{nn}(s)}{\Phi_{yy}(z)} \quad ,$$

attempts to average out the noise component.

III. EXPERIMENTAL RESULTS

In order to investigate the performance of the adaptive reconstruction filter, the problem was simulated on an IBM 7094 computer and the mean-square reconstruction error was compared with that of a zero-order hold, a first-order hold, and a linear point connector. A delay of p sampling instants is of course inherent in the realization of the filter. In most communication systems a small delay is allowable so that for p-th order all-pole processes one would probably allow a delay of p sampling intervals in any case. The adaptive filter was tested on a variety of deterministic and random signals. Some typical results are presented below.

Experiment 1. Deterministic Sine Wave.

An uncorrupted sine wave

$$f(t) = \sin t$$

was sampled with sampling interval T=1. 230 points were used for identification and a second-order spectral density model was used (p = 2). Fig. 3 shows the resulting impulse response of the adaptive filter. Since no noise was present and since the sine wave has a second-order all-pole z-transform, the identification was near perfect, as was the reconstruction. The following tabulation compares the adaptive filter with some common reconstruction filters:

	Normalized Mean-Square-Error
adaptive filter	0.0000
linear point connector	0.0039
first-order hold	0.0900
zero-order hold	0.1407

Experiment 2. Deterministic Triangle Wave.

The case of a deterministic triangle wave, where the all-pole model is not exact, illustrates the effect of model order on the accuracy of the identification and reconstruction. One cycle of the triangle wave was

$$f(t) = \begin{cases} 4(t-1) & 0 \le t \le 2 \\ 4(3-t) & 2 \le t \le 4 \end{cases}$$

and this periodic wave was sampled every 0.6 seconds so that the sampling instants would not usually coincide with the corners of the triangle wave. Again 230 points were used in the identification. First, Second, Fourth, and Sixth order models were used for the sampled spectral density, with the following results:

p	MSE For Adaptive Filter	MSE For Linear Point Connector	MSE For First-Order Hold	MSE For Zero-Order Hold
1	0.0976	0.0752	1.029	1.429
2	0.0474	"	"	"
4	0.0347	"	"	"
6	0.0131	"	"	"

It is seen that at least a second-order model is required to out-perform the linear point connector. For higher order models the performance of the adaptive reconstruction filter improves steadily. Fig. 4 shows the corresponding impulse responses. Fig. 5 shows the typical behavior of the first-order adaptive filter which is similar to the linear point connector. Fig. 6 shows the behavior of the sixth-order adaptive filter. As in the case of the sine wave, the adaptive structure, when the model order is appropriate, is able to "recognize" the character of the wave and produce a linear filter which will perform much better than any conventional reconstruction device.

Experiment 3. Random Signal.

In a more realistic situation, the signals are random rather than deterministic. To simulate this, the output of a normal random number generator was filtered by the low pass digital filter

$$\frac{10(1-a)^3}{(1-az^{-1})^3}$$

and the resulting signal was interpreted to be the original and continuous input signal f(t). Every tenth sample of this signal was taken as input for the adaptive reconstruction process. Curves of the normalized mean-square-error vs. a (which determines the bandwidth of the original signal f(t)) are plotted in Fig. 7 for the third-order adaptive reconstruction filter, the linear point connector, and the zero and first order holds. As expected the error decreases with the signal bandwidth for all these filters. With only 260 points used for identification, the adaptive filter performed significantly better than the other conventional filters.

Experiment 4. Sine Wave Plus Random Noise.

To illustrate the performance of the adaptive scheme when noise is present, white gaussian noise n(t) with zero mean and unit variance was added to a deterministic sine wave so that

$$y(t) = 4.5 \sin t + \sigma n(t)$$

The samples of n(t) were uncorrelated and the continuous spectral density of n(t) was flat and bandlimited to half the sampling frequency. Fig. 8 shows the resulting normalized mean-square reconstruction error vs. the standard deviation of the noise, σ, for the second order adaptive filter with 290 points for identification, for the linear point connector, and for the zero and first-order holds. With small noise the adaptive filter is much better than the others, because the identification is effective and the situation is like that in Experiment 1. As the noise level increases the advantage gained by identification decreases and the behavior of all the filters tends to become similar.

IV. SUMMARY

An adaptive signal reconstruction filter has been described which is based on the minimum mean-square-error reconstruction filter with an all-pole model for the sampled spectral density of the input signal. The use of this model leads to a simple identification scheme and a filter with an impulse response which is of finite duration. As we have seen these simplifications lead to an adaptive filter which is easily implemented on a digital computer.

Experiments performed with both deterministic and random inputs have shown that the filter yields significant improvement over a linear point connector and zero and first order holds with relatively low order models and with relatively short identification times. When the additional cost and time for reconstruction can be justified this adaptive filter performs significantly better than conventional reconstruction devices.

The way the adaptive filter would be used depends on whether or not an analog signal were desired. The filter as implemented on a digital computer can only fill in data points and so effectively increase the sampling rate. If an analog signal were desired a simple hold circuit might follow the adaptive reconstruction at a sampling rate many times the original.

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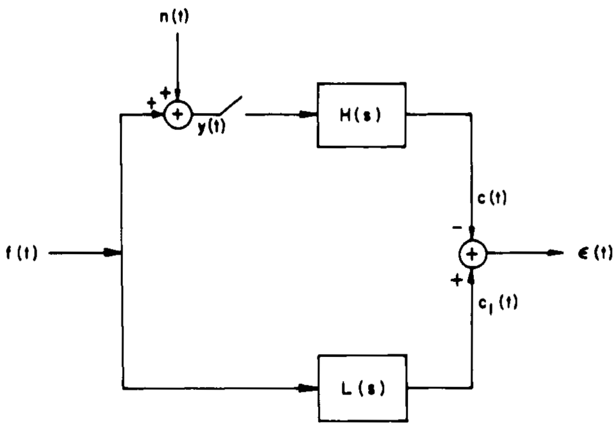

Fig. 1 Block Diagram of Minimum Mean-Square-Error Reconstruction Problem

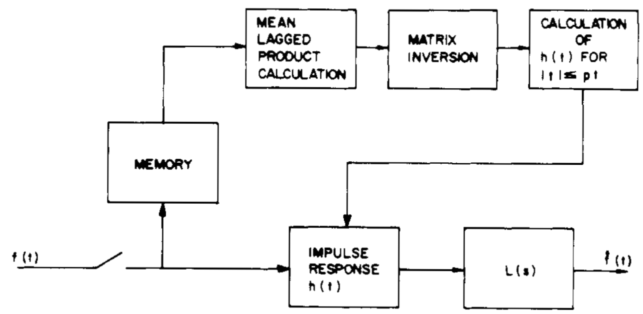

Fig. 2 Block Diagram of Adaptive Filter

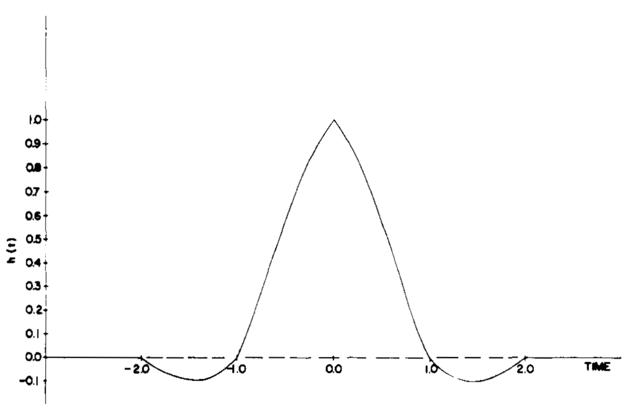

Fig. 3 Impulse Response of Second Order Adaptive Reconstruction Filter for f(t) = sin t .

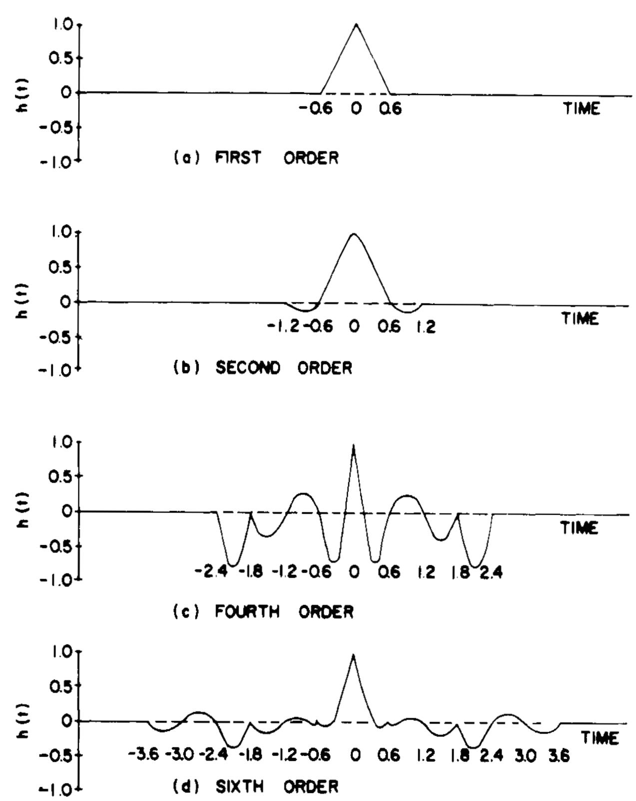

Fig. 4 Impulse Responses of Adaptive Reconstruction Filters for Triangular Wave

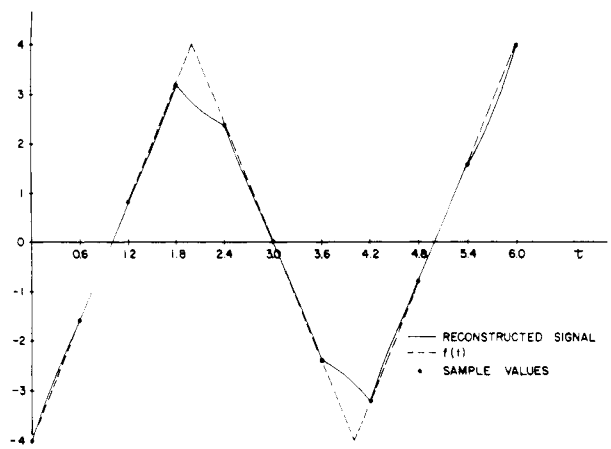

Fig. 5 Reconstruction of Triangular Wave with 1st Order Adaptive Filter.

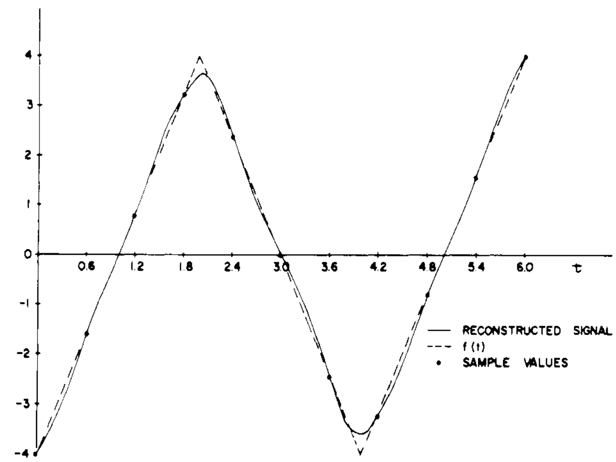

Fig. 6 Reconstruction of Triangular Wave with 6th Order Adaptive Filter

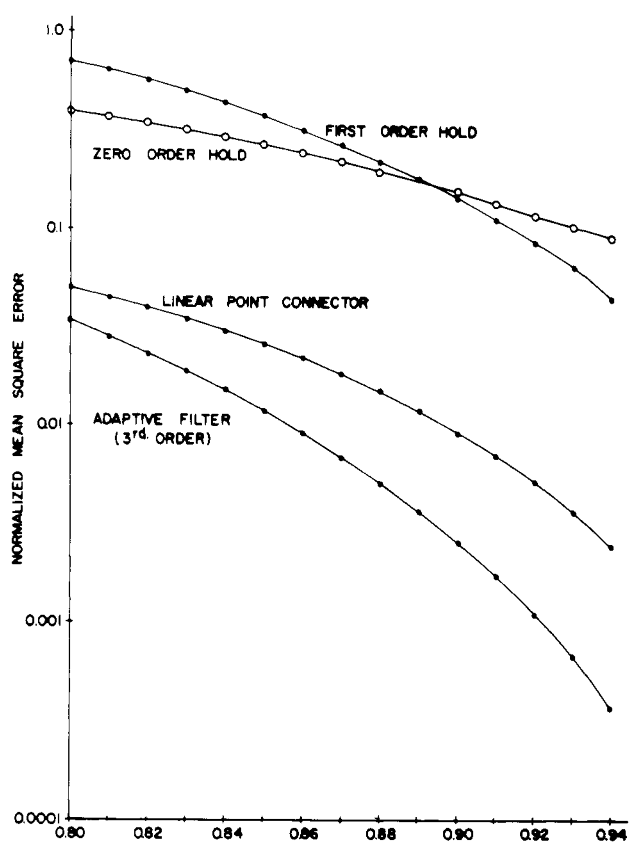

Fig. 7 Normalized Error vs. a . Experiment 3

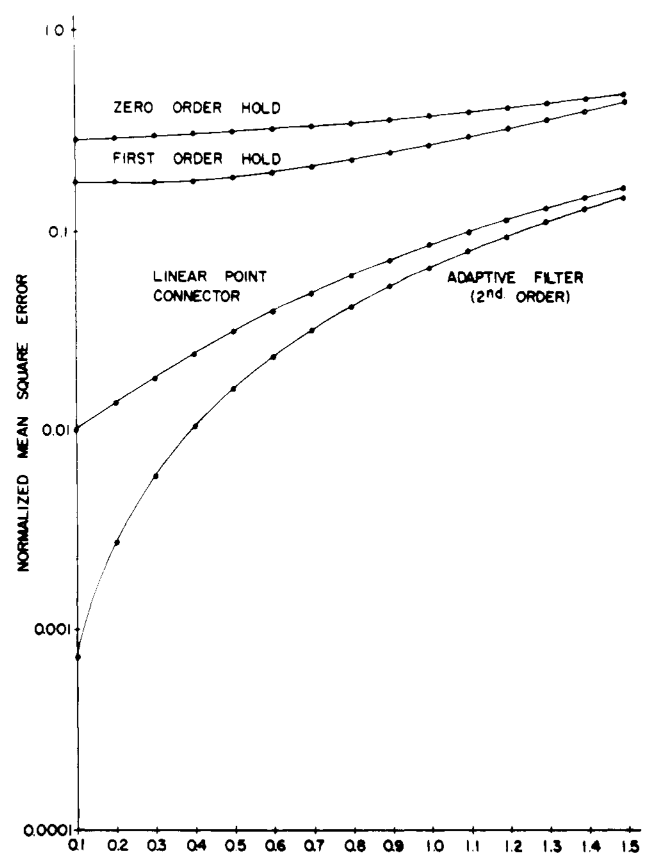

Fig. 8 Normalized Error vs. σ for $y(t) = 4.5 \sin t + \sigma n(t)$. Experiment 4

Note: $n(t)$ generated by a normal random number generator with zero mean and unit variance.