An Article Submitted to

The B.E. Journal of Economic Analysis & Policy

Manuscript 2183

Separating Equilibria in Public Auctions

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Abstract

We consider two private-value auctions where the prize in one is higher than the prize in the other. We show that a separating equilibrium exists in which bidders with a high valuation attend the auction with the higher prize while weak bidders attend the auction with the lower prize. In addition, we prove that a weak separating equilibrium exists where the strong bidders attend the high prize auction while the weak bidders randomize and may attend either auction, although with a higher probability of attending the low prize auction.

KEYWORDS: auctions, separating equilibrium, selling mechanisms, private information.

1 Introduction

There are numerous practical situations where authorities offer the public many objects through auctions. Some examples are sport competitions (with different leagues), cellular frequency licenses, broadcasting rights, mineral rights, research grants, construction contracts (for power plants, parks, dams, public buildings, etc.), among numerous others. These auctions are characterized by nonidentical multiunit prizes. Assuming that the authorities allow for a bidder in the competition to win up to only a single prize, an efficient selling mechanism might be possible in some circumstances, but it is not always welcome. While an efficient outcome is considered as an allocation of the prizes to the bidders with the highest valuation, the authorities may also wish to achieve different social goals. For instance, authorities may seek a mechanism that will give weak competitors the chance to survive and win some of the prizes. This type of mechanism commonly appears in sport competitions where strong teams compete in a high league while weak teams compete in a lower one. In the case of a public auction for mineral rights or construction contracts, the prizes may differ in size or amount and potential participants may attend an auction with a high or low prize. A well known example of auctions with different prizes can be found in frequency (spectrum) auctions for third generation mobile telephones. In the UK, authorities offered $15-15MHz$ (large)[1] licences and $10-10MHz$ (small) licences (for details see Börgers and Dustmann 2003, 2005). In US auctions for nationwide narrowband frequencies, the FCC offered three different licenses: $50-50KHz$, $50KHz$, and $50-12.5KHz$ (see Cramton 1997).

We consider a private value model with two simultaneous auctions and different prizes. A bidder is informed about his valuation v and decides whether to participate in the higher prize auction or the lower prize one. While in the low-prize auction the bidder's valuation is equal to his type v, in the high-prize auction his valuation is equal to av, where $a>1$.

[1] $15-15MHz$ represents a bandwidth of $15MHz$ from the network to the users and $15MHz$ from the users to the network. A $15MHz$ licence means that the bandwidth is only from the network to the user.

Börgers and Dustmann (2003, 2005) show that in UK frequency auctions, the bidders evaluate the high prize as equal to 1.5 of the low prize. In this case we can assume that $a = 1.5$. Similarly, Cramton (1997) shows that in the US, for FCC auctions for nationwide narrowband frequencies, the final winning bids for all licenses in terms of \$/MHz-pop is almost identical although some licences have a different bandwidth.[2] We can find in Cramton (1997) that bidders evaluate a licence with $50KHz$ bandwidth as 50% of the licence with $100KHz$ bandwidth (namely $a = 2$).[3] If we limit the bidders to apply only to one auction,[4] then the intuition about the bidder's behavior will be ambiguous. A high valuation bidder may believe he should participate in the lower prize auction since as a strong bidder, the probability of winning is high. On the other hand, a low valuation bidder may take a chance and participate in the high prize auction hoping to be the sole participant and win the higher prize. We show that a separating equilibrium exists where a bidder of a type higher than a cutoff c will participate in the higher prize auction while a bidder of a type below the cutoff c will participate in the lower prize auction. Moreover, by properly setting the minimum bid (reservation price) in the auction with the higher prize, the seller can control the value of the cutoff type c. In addition, we show that a weak-separating equilibrium exists in the sense that for a certain cutoff $\underline{c}$, bidders with a type above this cutoff will attend the higher prize auction while bidders with a type below this cutoff will randomize between the two auctions, with a higher probability to join the low prize auction.

[2] The \$/MHz-pop is a common measure used in frequency auctions. It represent the ratio between total payment for the license and the multiplication of frequency bandwidth in terms of MHz and population size.

[3] According to Table 1 in Cramton (1997), the maximum difference between the winning bids in terms of \$/MHz-pop is 8%.

[4] In many public goods auctions, the authorities limit the bidder's participation by quantity restriction or by blocking incumbent bidders such as in US, FCC auctions (Cramton 1997) and UK third generation cellular auctions (Börgers and Dustmann 2003). However, in the UK third generation cellular auctions the bidders can switch from one auction to the another. Recently, in the 2008 Canadian spectrum auction, the authorities set aside some of the licences exclusively for new players (Industry Canada, Specturm Auctions at http://www.ic.gc.ca/epic/site/smt-gst.nsf/en/h_sf01714e.html).

Although the model we study deals with auction selection in the sense that a bidder has to choose between auctions, we deviate from the literature since the objects in the auctions are not identical and the emerging equilibrium splits the bidders into strong and weak subgroups. Moreover, in our setting there is only one auctioneer who sells his two objects in two different auctions and thus, there is no competition between auctioneers. This setting differs from other studies that consider auction selection, for example, in McAfee (1993), Peters (1997) and Peters and Severinov (1997) the model of auction selection has a stock of sellers, each with an identical single object for sale and a stock of buyers. The sellers independently announce the auction rules and the buyers independently choose in which auction they are going to participate. In this type of model, there are two levels of competition. The first is the competition between the sellers trying to attract the buyers and in the second, the bidders, after they choose an auction, compete over the prize. In the limit case (where the number of auctioneers increases to infinity), all sellers announce the same auction mechanism which is equivalent to a second-price auction with a zero reserve price, and the buyers randomize between the sellers. Hernando-Veciana (2005) shows the same result for a finite number of auctioneers in a similar model with a finite set of possible minimum bids.

Burguet and Sákovics (1999) study a model of two sellers with identical objects (the case when $a = 1$ in the present paper) who compete in a one-shot game to attract a given set of bidders. In their model, the sellers' strategy is to choose the reserve price. They find a symmetric equilibrium when the sellers set the same positive reserve price and the bidders randomize between the auctions independently of their valuation. This result indicates that the case of a large market with many sellers is different from the case of a small number of sellers. A small number of sellers leads to an inefficient equilibrium in the sense that the reserve price is positive and the object may not be sold. In contrast to Burguet and Sákovics (1999), we look for asymmetric behavior when bidders separate themselves by applying to an auction according to their valuation such that high-value bidders will prefer to compete in the high-prize auction and the low-value bidders will prefer the low-prize auction.

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we show the existence of a strong separating equilibrium and its characteristics. In Section 4 we introduce the weak separating equilibrium. In Section 5 we generalize the results for any selling mechanism and general high-prize transformations and make concluding remarks.

2 The Model

Consider two similar (but not identical) nondivisible objects H and L offered for sale in two simultaneous and independent second-price auctions, henceforth referred to as 'auction H' and 'auction L' (later on we will generalize to any selling mechanism). There are n potential buyers with a private valuation for the object $v_i \in [0,1]$, $i = 1, 2, \ldots, n$. v_i is private information of buyer i and is drawn according to a continuous distribution function $F(v)$, $v \in [0,1]$. If a buyer wins the object in auction L, his gain is v_i while if he wins the object in auction H, his valuation is av_i, where $a > 1$. We assume that each buyer can participate only in one of the auctions. Once buyer i is informed about his privately-known valuation for the object, v_i, he decides in which of the two auctions he will participate and then submits a bid $b \geq 0$. We assume that when a buyer decides in which of the auctions he will participate, he is not informed about the number of buyers who have decided to participate in the same auction. However, this assumption does not restrict our analysis and the results hold also for mechanisms where the auctioneer announces the number of bidders in each auction after the bidders choose their auction. Under this setting, a bidders' strategy is $s_i = (j, b), j = H, L, b \geq 0$, for $i = 1, 2, \ldots, n$. We denote by $r_j \geq 0, j = H, L$ the minimum price in each auction announced by the auctioneer (the reservation price). A bidder participating in auction j wins the object if his bid is the highest in that auction and he pays the second highest bid (or the reservation price r_j if there is no second highest bid in auction j).

3 Strong-Separating Equilibrium

We start by proving the existence of a strong-separating equilibrium when a strong (high type) bidder participates in auction H, and a weak (low type) bidder participates in auction L. Then, we show the conditions for which a separating equilibrium can exist. We demonstrate that these conditions are mild and that this type of equilibrium is likely to exist.

Proposition 1 *Let $c \in (0,1)$ satisfy the condition*

$$\int_{r_L}^{c} (1 - F(c) + F(s))^{n-1} ds = F^{n-1}(c)\,(ac - r_H) \tag{1}$$

and the constraint

$$1 \leq aF^{n-1}(c). \tag{2}$$

Then, if $ar_L < r_H$, the strategy

$$s(v) = \begin{cases} j = L, \quad b(v) = v, \quad \text{if } r_L \leq v \leq c, \\ j = H, \quad b(v) = av, \quad \text{if } c < v \leq 1, \end{cases} \tag{3}$$

forms an equilibrium where a bidder of a type below r_L does not participate in any auction.

Proof: See Appendix A. $\boxdot$

The pivotal type c is the type of bidder who is indifferent between participating in auction L and winning with certainty, or participating in auction H and winning only if he is the sole bidder attending auction H. Equation (1) is the condition that constrains the pivotal bidder's utility to be identical in both auctions. Observe that the equilibrium bid function (3) is the same as in a regular second-price auction. From the proof for Proposition 1 we can find that condition (2) is required for an equilibrium to exist. To gain more insight into this condition, we can present (2) as $c \leq acF^{n-1}(c)$ which indicates that the expected income for a bidder with a cutoff value when he participates in auction L is less than his income when he participates in auction H. According to Myerson (1981), the expected payoff for a bidder in an auction is $\int_r^v P(t)dt + constant$ where $P(t)$ is the

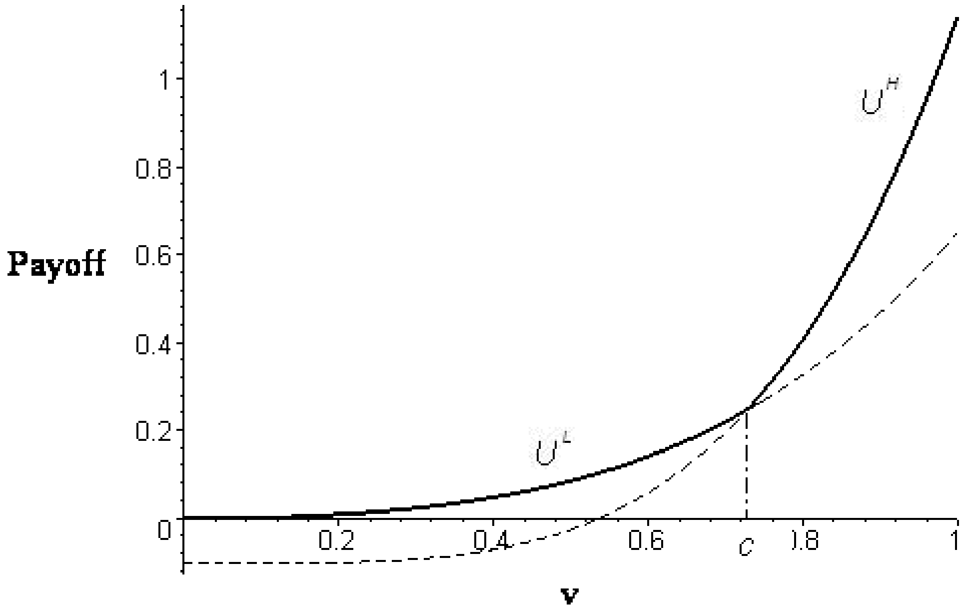

Figure 1: Bidder's expected utility as a function of his type v (bold curved line). The dashed lines represent the extension of the low and high utilities.

probability that type t will win the object (the constant appears in case of a participation fee or a reservation price). In our model, the payoff for type v in auctions L and H are given by $\int_{r_L}^{v} P^L(t)dt$ and $a\int_{c}^{v} P^H(t)dt$, respectively, where the constants are omitted. It is simple to verify that the marginal expected payoff for the pivotal type is 1 in auction L and $aF^{n-1}(c)$ in auction H. Thus, condition (2) implies that the cutoff type marginal revenue is higher in auction H. From Figure 1 we can see that the expected payoff for a bidder as a function of his type is continuous at $v = c$ (but not smooth) since the pivotal bidder is indifferent between participating in auction L or H. Furthermore, the slope of the payoff is sharper when $v > c$ than when $v < c$, which ensures that high type bidders will not try to play as a weak type by participating in auction L. If (2) is binding then the expected utility is smooth at the pivotal type since the slope is equal from both sides.

Condition (2) is misleading since it gives the impression that it can be satisfied merely by increasing the value of a. However, the value of the pivotal type c depends on a through (1); increasing a affects the value of c and thus, the condition may not hold. If we increase a, we increase the value for bidders who participate in auction H but, at the same time, we have to increase the minimum price such that a bidder will not defect from auction L to auction H. Note that a, c and r_H, which satisfy the conditions in the proposition, are not unique and there is a range of possible parameters that generate a separating equilibrium. Thus, a separating equilibrium is more likely to exist than expected.

The condition $ar_L < r_H$ introduced in Proposition 1 guarantees that a weak bidder with a valuation below r_L (the minimum bid for participating in the auction L) will not try to participate in auction H. In other words, the reservation price in auction H is high enough or the reservation price in auction L is sufficiently low to completely block bidders with a value below r_L from participating in any auction.

While in Proposition 1 we show that given a cutoff that satisfies conditions (1) and (2) a separating equilibrium exists, there is still a question of whether or not a separating equilibrium exists for a given $F(v), a, n$. We can expect that the auctioneer has the freedom to set the minimum prices $r_A, A = L, H$, but the value of the cutoff c is dictated only by the bidders. We show that for any set of parameters a and n and distribution $F(v)$, choosing an appropriate minimum bid in both auctions can support a separating equilibrium. Moreover, if we let $\underline{c}$ be the solution to $aF^{n-1}(\underline{c}) = 1$, then for every $c \geq \underline{c}$ we can find minimum bids $r_A, A = L, H$ such that a separating equilibrium with pivotal type c exists.

Proposition 2 *For every a, n, F and $c \geq \underline{c}$ that satisfy (2), there exists r_L and r_H such that a separating equilibrium exists.*

Proof: Let $r_L < \underline{c}$ be any minimum bid and let $y(r_H) = \int_{r_L}^{c} (1 - F(c) + F(s))^{n-1} ds - F^{n-1}(c)\,(ac - r_H)$. Observe that $y(ar_L) < 0$ since $y(ar_L) = \int_{r_L}^{c} (1 - F(c) + F(s))^{n-1} ds - aF^{n-1}(c)\,(c - r_L) < c - r_L - aF^{n-1}(c)\,(c - r_L) = (1 - aF^{n-1}(c))\,(c - r_L) \leq 0$. In addition, $y'(r_H) = F^{n-1}(c) > 0$ and $y(ac) = \int_{r_L}^{c} (1 - F(c) + F(s))^{n-1} ds > 0$, which proves that there

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exists $r_H^0 = r_H^0(c)$ which solves (1) (i.e., $y(r_H^0) = 0$) and satisfies condition (2). Finally, since $y(ar_L) < 0$, $y(r_H)$ is increasing and $y(r_H^0) = 0$, it follows that $ar_L < r_H^0$, which completes the proof. □

Since we expect the auctioneer to set the minimum prices and the bidders to dictate the cutoff type c through equilibrium, we should find out how the minimum prices would affect the cutoff.

Corollary 1 *The pivotal type c decreases with r_L and increases with r_H.*

Proof: Differentiating (1) with respect to r_L and r_H shows that $\partial c/\partial r_L < 0$ and $\partial c/\partial r_H > 0$. ⊡

It is surprising to find that by increasing r_L the cutoff decreases. The reason behind this finding is that since by increasing the minimum price in auction L we reduce the expected payment for the pivotal bidder making him better off by participating in auction H . Thus, the pivotal type decreases.

Corollary 2 *The pivotal type c is decreasing with the value of auction H's prize factor a.*

Proof: By fully differentiating (1) with respect to a, rearranging and using (2) we find that $\partial c/\partial a < 0$. ⊡

Corollary 2 shows that increasing a while holding the other parameters fixed will decrease c since more bidders closer to the pivotal type c will be better off by participating in auction H and thus, c decreases. While the monotonicity with respect to a is clear, the behavior of the pivotal type with the number of bidders is ambiguous. It is simple to show an example where c is not monotonic with the number of bidders n.

4 Weak-Separating Equilibrium

In the previous section we demonstrated the existence of what we termed a *strong separating equilibrium* where bidders signal their type by choosing the auction they prefer.

Now we show that there also exists an equilibrium where the separation is partial. In this case, bidders with a type above a given cutoff will participate in auction H with certainty, while bidders with types below this cutoff will randomize between the two auctions. For simplicity, we assume $r_L = r_H = 0$ (it is straightforward to show that the result in this section holds also for $ar_L = r_H > 0$), which excludes the possibility of a strong separating equilibrium. Note that the condition $ar_L < r_H$ from the previous section is not needed here.

Proposition 3 *Let and $\underline{c}$ be the solution of $aF^{n-1}(\underline{c}) = 1$. Then there exists an equilibrium where bidders with value $v > \underline{c}$ participate in auction H and bidders with value $v \leq \underline{c}$ randomize and participate in auction L with probability $\alpha = \frac{1}{1+F(\underline{c})}$ and in auction H with probability $1 - \alpha = \frac{F(\underline{c})}{1+F(\underline{c})}$. Once a bidder participates in an auction he bids his value v in auction L and av in auction H.*

Proof: See Appendix B. □

In Proposition 3 we found that the probability that a bidder of a type below $\underline{c}$ will participate in auction L is $\alpha = \frac{1}{1+F(\underline{c})} > \frac{1}{2}$, which indicates that a bidder with a low valuation is more likely to participate in auction L. In the extreme case, when $a = 1$, the cutoff is $\underline{c} = 1$, and the probability of participating in each auction is 0.5. This case appears when the objects are identical as shown by Burguet and Sakovics (1999). We can observe that, according to the previous section, there exists also a strong separating equilibrium with the same cutoff $\underline{c}$. However, in this case r_H has to be positive.

5 Generalization of the Model and Concluding Comments

In this section we consider two possible generalizations for our model. First, we question the robustness of the results when the valuation for the prize in auction H is $g(v)$ and not just linear. Second, we consider other selling mechanisms and not just the second-price auction.

5.1 General High-prize Transformation

Assume that the relation between a bidder's type v and that his high-prize valuation is given by $g(v)$ where $g(v) > 1$ is monotonically increasing and concave. Assume also that c satisfies the condition $\int_{r_L}^{c}(1-F(c)+F(s))^{n-1}ds = F^{n-1}(c)\,(g(c)-r_H)$ and in addition that $g'(c)F^{n-1}(c) > 1$ and $g(r_L) < r_H$. Then, Proposition 1 holds. The arguments in this case are similar to the discussion in Section 3. It is simple to verify that the equilibrium expected payoff for a bidder in auction H is given by $U^H(v|v) = F^{n-1}(c)\,(g(c)-r_H) + \int_c^v g'(s)F^{n-1}(s)ds$ and thus the marginal payoff for the pivotal bidder when he participates in auction H is $g'(c)F^{n-1}(c)$. The marginal payoff for this bidder when participating in auction L is 1. Similarly to (2), the equilibrium condition $g'(c)F^{n-1}(c) > 1$ requires that for a high type, participating in auction H is better than participating in L. As in Proposition 1, the pivotal bidder is indifferent between the H and L auctions.

5.2 General Selling Mechanisms

Until now we considered only second-price auctions. However, we can show that the results of this paper are more general and are not limited to any specific type of auction by considering different selling mechanisms for the different auctions. The generalization follows from the Revenue Equivalence Theorem and its generalization for a random number of bidders (see Myerson (1981) and Riley and Samuelson (1981)).

Let $b^L(v)$ be a bidder's equilibrium in a given selling mechanism L with a minimum bid r_L, a distribution of types $F(v)/F(c), v \in [0, c]$, and a random number of bidders distributed according to a binomial distribution with parameters n (number of trails) and $F(c)$ (probability of success). Similarly, let $b^H(v)$ be a bidder's equilibrium in a given selling mechanism H with a minimum bid r_H, a distribution of types $(F(v)-F(c))/(1-F(c))$, and a random number of bidders distributed according to a binomial distribution with parameters n and $1 - F(c)$. Finally, let $\hat{r}_L$ be the lowest type that will participate in auction L, and $E(c, r_H)$ be the expected payment made by a bidder of type c in auction H.[5]

Proposition 4 *Let $c \in (0,1)$ satisfy the pivotal condition*

$$\int_{\hat{r}_L}^{c} (1 - F(c) + F(s))^{n-1} ds = acF^{n-1}(c) - E(c, r_H) \tag{4}$$

and the equilibrium constraint

$$1 \leq aF^{n-1}(c). \tag{5}$$

Then if $a\hat{r}_L < E(c, r_H)$ the strategies

$$s(v) = \begin{cases} j = L, \quad b(v) = b^L(v) & \text{if } \hat{r}_L \leq v \leq c, \\ j = H \quad\quad b = b^H(v) & \text{if } c < v \leq 1, \end{cases} \tag{6}$$

form an equilibrium where a bidder of type below r_L does not participate in any auction.

Proof: See Appendix C. ⊡

5.3 Concluding Comments

We introduced a model where two nonidentical objects are being sold in a simultaneous independent auction where the bidders are allowed to participate in only one of those auctions. We found that there exists a separating equilibrium where bidders signal their type by participating in one of the auctions. Moreover, we found that there exists an equilibrium where bidders imperfectly signal their type by randomizing between auctions. The model we introduced has implications for authorities who wish to sell many public properties simultaneously and would like, in addition to increasing efficiency, to limit monopolistic behavior by giving weak players a chance to enter the market thereby, increasing future competition.

[5] In first- and second-price auctions, $E(c, r_H) = F^{n-1}(c)r_H$.

A Proof of Proposition 1

Proof of Proposition 1

Let $U^A(\hat{v}|v)$ be a bidder's expected utility given that his type is v, he acts as if his type is $\hat{v}$ and he participates in auction $A = L, H$, while the other $n-1$ bidders play according to (3).[6] Since the mechanism used in auction A is a second-price auction, a bidder's dominant strategy is to submit a bid equal to his type. Namely, in auction L the bidder submits $b = v$, and in auction H he submits $b = av$. Thus, $U^A(v|v)$ is the equilibrium expected revenue in auction A. It remains to show that no bidder will defect from one auction to another one. In terms of incentive compatibility conditions, we require that for every v and $\hat{v}$,

$$U^L(v|v \leq c) \geq U^H(\hat{v}|v \leq c), \tag{7}$$

$$U^L(\hat{v}|v \geq c) \leq U^H(v|v \geq c). \tag{8}$$

Obviously, if $v < r_H/a$, a bidder will not participate in auction H, and if $v < r_L$ he will not participate in any auction. Let $P^L(v) = (1 - F(c) + F(v))^{n-1}$ be the equilibrium probability that a bidder of type v, $r_L \leq v \leq c$ will win in auction L (if $v < r_L$ then $P^L(v) = 0$). By a standard argument

$$U^L(v|v \leq c) = \int_0^v P^L(s)ds = \int_{r_L}^v (1 - F(c) + F(s))^{n-1}ds.$$

Similarly, since $P^H(v) = F^{n-1}(v)$ is the probability that a bidder of type $v \geq c$ will win in auction H,

$$U^H(v|v \geq c) = const + a\int_c^v P^H(s)ds = (ac - r_H)F^{n-1}(c) + a\int_c^v F^{n-1}(s)ds.$$

The constant $(ac - r_H)F^{n-1}(c)$ appears since a bidder's expected utility in auction H is positive even if his type is c. The probability $P^H(v)$ is multiplied by a since the gain is multiplied by a (i.e., av). The constant $(ac - r_H)F^{n-1}(c)$ is the expected utility if the bidder's type is $v = c$. The bidder's expected utility is given by

[6] We use the notation $U^L(\hat{v}|v \leq c)$ for a bidder's revenue when his type is v, $v \leq c$.

$$\begin{aligned} U^L(\hat{v} &\leq c|v \geq c) = U^L(\hat{v} \leq c|\hat{v} \leq c) + (v - \hat{v})\Pr(\text{wins with bid } v) = \\ &= \int_{r_L}^{\hat{v}} (1 - F(c) + F(s))^{n-1} ds + (v - \hat{v})(1 - F(c) + F(\hat{v}))^{n-1}. \end{aligned}$$

The last expression gets his maximum at $\hat{v} = c$ and thus, it follows that

$$\begin{aligned} U^L(\hat{v} &\leq c|v \geq c) \leq \int_{r_L}^{c} (1 - F(c) + F(s))^{n-1} ds + v - c = \\ &= (ac - r_H)F^{n-1}(c) + v - c \leq (ac - r_H)F^{n-1}(c) + (v - c)aF^{n-1}(c) \leq \\ &\leq (ac - r_H)F^{n-1}(c) + a\int_{c}^{v} F^{n-1}(s) ds = U^H(v|v \geq c) \end{aligned}$$

and thus (8) is satisfied. The second equality and the subsequent inequality is derived from (1) and (2). Since H is a second-price auction, it follows that

$$U^H(\hat{v} > c|v \leq c) \leq U^H(\hat{v} = c|v \leq c) = (av - r_H)F^{n-1}(c).$$

To complete the proof, we show that $U^L(v|v \leq c) = \int_{r_L}^{v} (1 - F(c) + F(s))^{n-1} ds \geq (av - r_H)F^{n-1}(c)$. Define $h(v) = \int_{r_L}^{v} (1 - F(c) + F(s))^{n-1} ds - (av - r_H)F^{n-1}(c)$ and observe that by the assumption $ar_L < r_H$ we have $h(r_L) = -(ar_L - r_H)F^{n-1}(c) > 0$. By the definition of c in equation (1) we have $h(c) = 0$. Thus, if $h(v)$ is non-increasing, the condition $h(v) \geq 0$ is satisfied. Since $h'(v) = (1 - F(c) + F(v))^{n-1} - aF^{n-1}(c) \leq 1 - aF^{n-1}(c)$ it follows by (2) that $h'(v) \leq 0$. Thus,

$$U^L(v|v \leq c) = \int_{r_L}^{v} (1 - F(c) + F(s))^{n-1} ds \geq (av - r_H)F^{n-1}(c) \geq U^H(\hat{v} > c|v \leq c)$$

and (7) is satisfied. $\boxdot$

B Proof of Proposition 3

Assume that bidders with value $v \leq \underline{c}$ participate in auction L with probability α, and in auction H with probability $1 - \alpha$. Since the auction is a second-price auction, once a bidder with value $v \leq \underline{c}$ approaches one of the auctions, he will bid his value v in auction L and av in auction H. A bidder's expected payoff is given by $\int_0^v \Pr^L(t)dt$ and

$a\int_0^v \Pr^H(t)dt$, where $\Pr^i(t)$ $i = L, H$ is the probability that the bidder will win in auction i (see, for example, Myerson (1981)). In a mixed-strategy equilibrium, a bidder of type $v \le c$ should be indifferent between the two auctions. Thus, $\int_0^v \Pr^L(t)dt = a\int_0^v \Pr^H(t)dt$ for every $v \le \underline{c}$. Since for $v = 0$ both expected payoffs equal zero, it remains to show that the derivative is equal for every $v \le \underline{c}$, i.e., $\Pr^L(v) = a\Pr^L(v)$. The probability of a bidder with value $v \le \underline{c}$ to win is $\Pr^L(v) = [1 - \alpha F(\underline{c}) + \alpha F(v)]^{n-1}$ in auction L, and $\Pr^H(v) = [\alpha F(\underline{c}) + (1-\alpha)F(v)]^{n-1}$in auction H. After substituting $\alpha = \frac{1}{1+F(\underline{c})}$, we have $\Pr^L(v) = \left[\frac{1+F(v)}{1+F(\underline{c})}\right]^{n-1}$ and $a\Pr^H(v) = \frac{1}{F^{n-1}(\underline{c})}\left[\frac{1}{1+F(\underline{c})}F(\underline{c}) + \frac{F(\underline{c})}{1+F(\underline{c})}F(v)\right]^{n-1} = \left[\frac{1+F(v)}{1+F(\underline{c})}\right]^{n-1}$ and thus, $\Pr^L(v) = a\Pr^L(v)$.

To complete the proof it remains to show that a bidder with value $v > \underline{c}$ will not deviate and participate in auction L. Observe that $\Pr^H(v) = F(v)^{n-1}$ if $v > \underline{c}$ and $\Pr^H(v) = \left[\frac{F(\underline{c})}{1+F(\underline{c})}(1+F(v))\right]^{n-1}$ if $v \le \underline{c}$. Thus,

$$U^H(v|v \ge \underline{c}) = a\int_0^v \overset{H}{\Pr}(t)dt = \left[\frac{1}{1+F(\underline{c})}\right]^{n-1}\int_0^{\underline{c}}(1+F(s))^{n-1}ds + a\int_{\underline{c}}^v F(s)^{n-1}ds,$$

and if a bidder with valuation $v \ge \underline{c}$ applies to auction L (in that case he will bid as he has the highest type $v = \underline{c}$) his expected payoff is

$$\begin{aligned} U^L(v|v \;\; &\ge \;\; \underline{c}) = U^L(\underline{c}|v = \underline{c}) + (v - \underline{c}) = \int_0^{\underline{c}} \overset{L}{\Pr}(t)dt + (v - \underline{c}) \\ &= \;\; \left[\frac{1}{1+F(\underline{c})}\right]^{n-1}\int_0^{\underline{c}}(1+F(s))^{n-1}ds + (v - \underline{c}). \end{aligned}$$

To prove that $U^H(v|v \ge \underline{c}) \ge U^L(v|v \ge \underline{c})$ it remains to show that $a\int_{\underline{c}}^v F(s)^{n-1}ds \ge (v-\underline{c})$. Since $aF(\underline{c})^{n-1} = 1$ we have that $a\int_{\underline{c}}^v F(s)^{n-1}ds \ge a(v-\underline{c})F(\underline{c})^{n-1} = (v-\underline{c})$ and the proof is completed. □

C Proof of Proposition 4

Calculating the expected payoff for a bidder in auction L, given that his type is v and there are a random number of competing bidders distributed according to binomial distribution with $n-1$ trails, the probability of success (approaching auction L) $F(c)$ and distribution over types $F(v)/F(c)$, $v \in [0, c]$ will give exactly the same results as in the case of the second price auction shown in the previous section. This follows from the revenue equivalence theorem that is valid also for a random number of bidders (see Milgrom 2004).[7] Similar arguments hold for the expected payoff in auction H, however, here we need to be a bit more careful since revenue equivalence assumes that the bidder with the lowest valuation will obtain zero expected payoff. In auction H, the expected payoff for the bidder with the lowest valuation c (or actually ac) is equal to the expected payoff of the highest valuation in auction L and is strictly positive. The proof continues similarly to the proof of Proposition 1. $\boxdot$

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