Information transfer between solitary waves in the saturable Schrödinger equation

Mariusz H. Jakubowski and Ken Steiglitz
Computer Science Department, Princeton University, Princeton, New Jersey 08544

Richard Squier
Computer Science Department, Georgetown University, Washington, D.C. 20057
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In this paper we study the transfer of information between colliding solitary waves. By this we mean the following: The state of a solitary wave is a set of parameters, such as amplitude, width, velocity, or phase, that can change during collisions. We say information is transferred during a collision of solitary waves A and B if the state of B after the collision depends on the state of A before the collision. This is not the case in the cubic nonlinear Schrödinger, Korteweg–de Vries, and many other integrable systems. We show by numerical simulation that information can be transferred during collisions in the (nonintegrable) saturable nonlinear Schrödinger equation. A seemingly complementary feature of collisions in this and similar systems is radiation of energy. We give results that show that significant information can be transferred with radiation no greater than a few percent. We also discuss physical realization using recently described spatial solitary light waves in a saturable glass medium. [S1063-651X(97)05812-1]

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I. INTRODUCTION

A solitary wave can carry information in its envelope amplitude, width, and position; its group and phase velocities; and its carrier phase; and this information can be exchanged in collisions with other solitary waves. This paper is devoted to the question of whether such information transfer can occur in a way that is useful as a basis for computation while still preserving particle identities. If this is possible, it suggests that general computation can be performed via interacting waves in a uniform medium, such as a nonlinear optical material.

In the usual conception of optical computing, one builds discrete gates based on the propagation of light and then essentially mimics the construction of a conventional computer. We describe here an alternative approach to building an all-optical computer, using only solitary waves in a homogeneous nonlinear optical medium. In our approach, programs and data are encoded as streams of solitary waves, which are injected into the medium at a boundary and compute via the information transfer effected by solitary-wave collisions.

A general Turing-equivalent model for such ''gateless'' computation, the *particle machine*, was introduced in [1,2]. By exploiting the fine-grain parallelism of particle systems, this model supports fast and efficient execution of many operations, including arithmetic and convolution. Briefly, particle machines treat solitary waves as particles whose collisions can change particle states, thus performing computation. Such computation requires that if solitary waves A and B collide, then (i) some part of the resulting state of A depends on the state of B and (ii) (this is essential) the state of B is changed by collisions. In a word, information should be transferred from one wave to the other in ''interesting'' ways.

There is much already known about the phenomenology of collisions in nonintegrable versions of nonlinear Schrödinger equations (NLSEs) [3–8]. Generally, solitons in these systems, including the saturable NLSE (SNLSE), can change amplitude and velocity after collisions. We emphasize that this is not in itself sufficient to meet our criterion of computationally ''interesting.''

The properties that are useful for our computational purposes are the opposite of those usually considered useful in communication optics: At least some of the collision products must effect a nontrivial transformation of information in the colliding waves. The reason for this is that general computation requires a transformation of information in basic logic operations. Unfortunately, many commonly studied systems that support waves do not have this behavior. For instance, because of linear superposition, colliding plane waves in a linear medium do not interact, i.e., do not undergo any state changes, and therefore cannot have information interaction among colliding waves.

An example of a system in which collisions cause a change of state but nevertheless cannot transform information in a nontrivial manner is the cubic nonlinear Schrödinger equation (3NLSE). In order to perform a computation, solitary waves must carry information from one collision to the next; such information must be coded in parameters that are not constant. However, in the 3NLSE system, the state parameters that cause the information transfer are themselves invariant: The only change of state occurs in the spatial (or temporal) position and carrier phase, and this change depends only on the amplitudes and velocities of the envelopes of the incoming solitons. We conjecture in [9,10] that all solitary-wave collisions in integrable systems have this property and we show that particle machines based on such systems are very limited in computational power. In particular, these particle machines are not Turing equivalent. We must therefore look to solitary waves in nonintegrable systems for computationally useful collisions.

For solitary waves to carry information, they must also preserve their integrity after a sequence of collisions and lose negligible energy through radiation. These requirements are

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apparently antagonistic to the information-transform capability necessary for computation, but our goal is to find systems that meet all these requirements. The results shown here suggest that the SNLSE describes such a system.

II. INFORMATION TRANSFER

To be more precise about the definition of *information transfer*, suppose that a medium supports a set of solitary waves. Then a selected set of properties that can change during a collision define a *state* $S(A)$ of a wave A, whereas a set of constant wave properties that are unaffected by collisions define an *identity* $I(A)$ of A. Note that we may define different types of states S and identities I for the same type of wave. Denote by A' the solitary wave A after a collision with wave B. Then a collision of A with B supports the transfer of information if $S(A')$ depends on $S(B)$ for some $S(A)$ and $S(B)$; otherwise, the collision transfers only trivial information [if $S(A')$ depends on only $I(B)$] or no information [if $S(A')=S(A)$].

We illustrate the above definition using the cubic NLSE and the saturable NLSE systems. The cubic NLS equation supports solitons whose variable states are phases and whose constant identities are amplitudes and velocities. Collisions of such solitons transfer only trivial information since the phase shifts due to soliton collisions are a function of only the amplitudes and velocities, i.e., the identities, of the colliding solitons. On the other hand, the saturable NLSE system gives rise to solitary waves whose variable state includes phases, amplitudes, and velocities. This system supports collisions that transfer nontrivial information since the state changes due to collisions are a function of the states of the colliding waves.

III. COMPUTATIONAL POWER

To examine how information transfer relates to computational power, we briefly review the notion of Turing equivalence, or computational universality. Informally, a Turing machine is a computational model in which programs and data are stored on an infinite tape of discrete cells. A read-write head processes information by reading cell contents, writing new cell contents, and moving back and forth along the tape, all according to a transition function that considers both the state of the head and the symbol read from underneath the head. The machine can enter a special ''halt'' state, which signals the end of computation and the presence of the machine's final output on the tape.

It is generally accepted (by virtue of Church's thesis [11]) that given enough time and space, a Turing machine can implement any algorithm; that is, in terms of the results that can be computed, a Turing machine is as powerful as any computer. A computational system is Turing equivalent, or computation universal, if it can simulate a Turing machine. While this property is not absolutely necessary for a system to perform useful computation, universality nevertheless serves as a good measure of a system's computational potential.

Intuitively, in order for a computation to take place in a solitary-wave system, colliding waves should interact and transfer information that is necessary to execute steps of an algorithm. In [10] we show that only at most cubic-time computation can be done on a particle machine that models a system in which collisions transfer at most trivial information. This upper bound on such a system's computation time proves that this system cannot be Turing equivalent since universal computation can take an arbitrarily long time. Moreover, solitary-wave interactions in this system are computationally very limited, and designing algorithms based on these interactions appears tedious and impractical. It is unclear whether or not collisions supporting only trivial information transfer can encode any useful computation at all.

Solitary-wave systems in which collisions transfer nontrivial information are more readily applicable to encoding computation. We have shown in [10] that such a system can be Turing equivalent provided the solitary-wave state changes are sufficiently complex.

IV. THE NLS EQUATION AND ITS SOLUTIONS

To study the information-transfer capabilities of NLSE solitary waves, we first review the one-dimensional NLS equation and its solutions. We consider the following form of the NLS equation [4,12]:

$$-i\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+N(|u|)u. \tag{1}$$

Here x is space, t is time, u is the complex amplitude of waves described by the equation, a is real, and $N(|u|)$ is a nonlinear function of $|u|$. For the 3NLSE, $N(|u|)=k|u|^2$; for the SNLSE, $N(|u|)=m+k|u|^2/(1+|u|^2)$, where k and m are real constants.

The nonlinearity $N(|u|)$ determines the integrability of Eq. (1) and the existence of closed-form solitary-wave solutions. To find solitary waves, either analytically or numerically, we assume that each such wave consists of an envelope modulating a sinusoidal carrier wave. Following [13], we make the ansatz

$$u(x,t)=\Phi(x-u_e t)e^{i\theta(x-u_c t)}, \tag{2}$$

where $\Phi(x-u_e t)$ is the envelope, $e^{i\theta(x-u_c t)}$ is the carrier, and u_e and u_c are the envelope and carrier velocities, respectively. We find that the carrier function θ is given by

$$\theta(x-u_c t)=\frac{u_e}{2a}(x-u_c t)+\phi_0, \tag{3}$$

where ϕ_0 is an arbitrary constant. The envelope function Φ can be found from

$$x-u_e t=\int_{\Phi(0)}^{\Phi(x-u_e t)}\frac{d\Phi}{\sqrt{\alpha\Phi^2-2\int N(|\Phi|)d\Phi}}, \tag{4}$$

where $\alpha=(u_e^2-2u_e u_c)/(4a)$.

If the integral in Eq. (4) can be evaluated analytically and used to solve Eq. (4) for the envelope function $\Phi(x-u_e t)$, then Eq. (2) gives an exact expression for a solitary wave, as is the case with the 3NLSE. Otherwise, the integral and $\Phi(x-u_e t)$ can be computed numerically, using boundary

conditions chosen to yield solitary waves. We explain how to do this to obtain the SNLSE solitary waves used in our numerical simulations.

We consider the following form of the SNLSE:

$$-i\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\left(m+\frac{k|u|^2}{1+|u|^2}\right)u. \tag{5}$$

Here m and k are real constants. Solitary-wave solutions are of the form of Eq. (2), where the carrier θ is given by Eq. (3). The envelope Φ can be found from Eq. (4), which simplifies to

$$x-u_e t=\int_{\Phi(0)}^{\Phi(x-u_e t)}\frac{d\Phi}{\sqrt{c\Phi^2+k\,\ln(1+\Phi^2)}}, \tag{6}$$

where $c=\alpha-m-k$. We evaluate the above integral numerically using the boundary conditions $\Phi(0)=A$ and $\Phi(\pm\infty)=0$, where A is the maximum amplitude of the envelope and is determined by u_c and u_e. The integration yields $x-u_e t$ as a function of the envelope $\Phi(x-u_e t)$. We invert the result of the integration to compute the envelope $\Phi(x-u_e t)$ as a function of $x-u_e t$. We multiply this numerical envelope by the exact carrier [Eq. (3)] to plot solitary waves on a discrete one-dimensional grid and use numerical methods to study the behavior of propagating and colliding waves.

Note that SNLSE solitary waves are characterized by four parameters: amplitude (A), envelope velocity (u_e), carrier velocity (u_c), and phase (ϕ_0). Using Eq. (6), it can be shown that any two of A, u_e, and u_c determine the third. We may choose ϕ_0 freely, so that there are three degrees of freedom in choosing the initial state of a SNLSE solitary wave.

V. INFORMATION TRANSFER IN COLLISIONS OF NLSE SOLITARY WAVES

In the integrable 3NLSE system, solitary waves are true solitons whose collisions can change only their envelope position and carrier phase; envelope amplitude and velocities are conserved. In addition, the spatial and phase shifts of colliding solitons depend only on their constant amplitudes and velocities. Thus such collisions transmit only trivial information and are computationally very limited, as we demonstrate in [10]. (See Fig. 1.)

The nonintegrable SNLSE gives rise to solitary waves whose collisions support nontrivial information transfer. (See Fig. 2.) In particular, phases, amplitudes, and velocities can all change as a function of the parameters of the colliding waves. We have observed that the most computationally useful collisions occur when the solitary waves have a low relative speed (approximately 4.0 and below). The magnitude of information transfer decreases gradually as the relative speed of the waves increases. To estimate this magnitude, we measured the amplitude and velocity changes following collisions of low-velocity waves at various initial phases. In Fig. 3, the normalized amplitude change is plotted as a function of the relative phase of two colliding solitary waves.

In practice, it is reasonable to expect that the amplitudes of two colliding solitons cannot be made precisely equal. To test the robustness of the results in Fig. 3 we ran experiments with unequal amplitudes (amplitude ratios of 1.1, 1.3, and 2.0) and found the results to be quite similar, except that the magnitude of the effect was even greater.

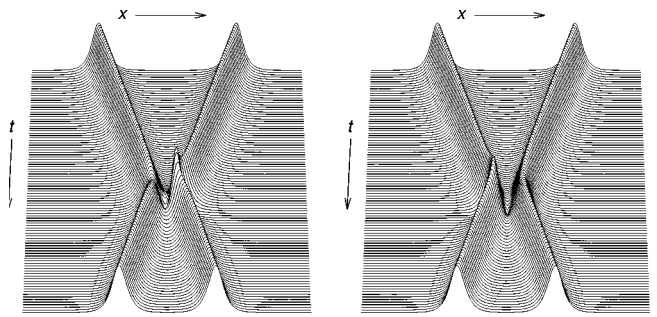

FIG. 1. Trivial information transfer in collisions of 3NLSE solitons. The initial relative phases of the solitons in the left and right graphs are 0.25π and -0.45π, respectively; velocities are ± 0.2. Phase and spatial shifts, though not apparent from these graphs, are a function of only the constant amplitudes and velocities.

It might appear that in the perfectly symmetric case, when the relative phase is zero, there should be no amplitude change. That is, a nonzero value of the amplitude change at zero relative phase shift would imply that energy is transferred from one wave to another, thereby spontaneously breaking the symmetry. To explain this apparent problem, we first note that what is plotted is a change in amplitude, not energy. A nonzero value of the amplitude change at zero relative phase (which is indeed a symmetric situation) then means that the amplitudes of both solitons change. There are three ways that this can happen: (i) Radiation can decrease the energy of both solitons, (ii) the amplitudes can change, but a change in width can compensate to preserve energy, and (iii) the collision products can breathe, which in fact makes the amplitude poorly defined.

VI. RADIATION AND REUSABILITY

In general, computation encoded in a NLSE system must reuse solitons after they have been involved in multiple collisions. To behave like the particles of a particle machine, these solitary waves should be stable; more specifically, collisions should preserve the identities of solitary waves and generate negligible radiation.

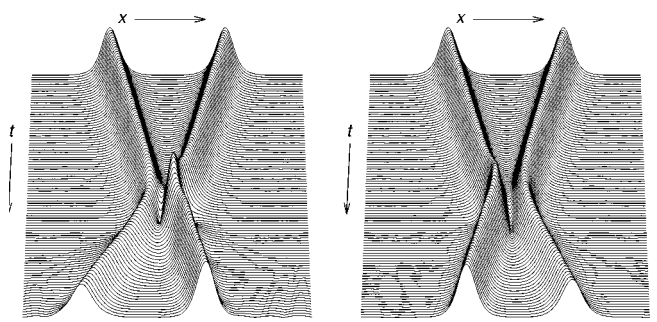

FIG. 2. Nontrivial information transfer in collisions of SNLSE solitary waves. The initial relative phases of the waves in the left and right graphs are 0.25π and -0.45π, respectively; velocities are ± 0.2.

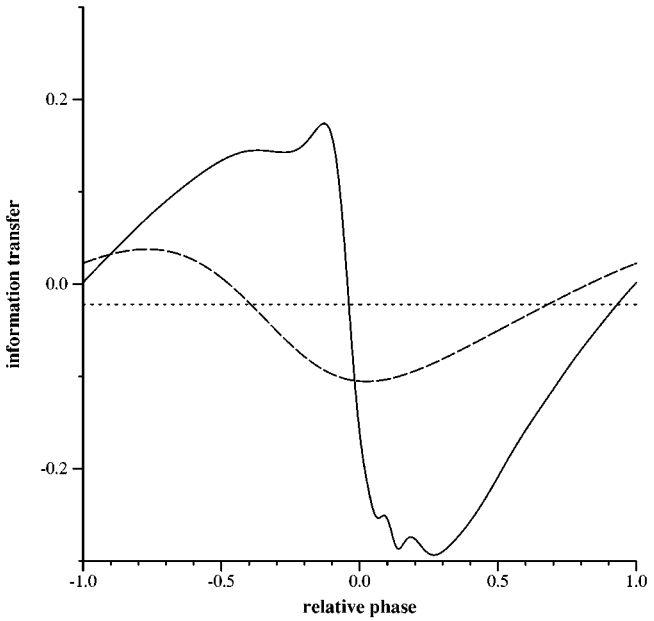

FIG. 3. Information transfer for collisions of two SNLSE solitary waves. Here information transfer is defined as the fractional change in the amplitude of one solitary wave; that is, the transfer is equal to $\Delta A_1/A_1$, where A_1 is the initial amplitude of the right-moving wave and ΔA_1 is the amplitude change due to collisions. The solid, dashed, and dotted curves show information transfer for collisions of two waves with amplitudes 1.0 and velocities ± 0.5, ± 1.5, and ± 10.0, respectively. The relative phase is in multiples of π. Note that in the low-velocity case (solid line) near zero relative phase there is significant radiation and breathing in the collision products, making the amplitude poorly defined. What is shown is the result of measuring the amplitude peak at a fixed time.

Numerical results reveal that information transfer and radiation often go hand in hand. Soliton collisions in the 3NLSE system are perfectly elastic and generate no radiation, but such collisions support only trivial information transfer, as we have seen. In the SNLSE system, large amounts of radiation tend to accompany large magnitudes of information transfer. However, much like other known nonintegrable NLSE systems [4], the SNLSE system supports collisions that transfer information and yet generate only small amounts of radiation. More specifically, our numerical studies have revealed the following.

(i) When at least one of the solitary waves is moving at a high speed (approximately 4.0 and above), their collision generates negligible radiation and supports no measurable information transfer. (This phenomenon in generalized NLSE systems was mentioned by Snyder and Sheppard [5].)

(ii) When the relative phase $\phi_0 = \pi$, the collision is the same as in the above case, regardless of the value of the relative speed v_0.

(iii) When both waves have low speeds (below 4.0) and $0 \leqslant \phi_0 < \pi$, the collision is accompanied by larger amounts of radiation and information transfer. However, as ϕ_0 tends towards π, both radiation and the magnitude of information transfer decrease. For $\phi_0 > \pi/2$, very little or no measurable radiation is generated.

The solitary waves that emerge from collisions in the SNLSE system may or may not be of the form given by Eq. (2), depending on the initial wave parameters. As observed

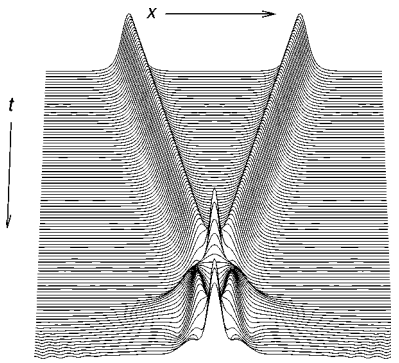

FIG. 4. Fusion of two solitons after collision. In the case shown, the two solitons approach each other with velocities ± 0.2 and their amplitudes are both 1.0.

early in a variety of nonintegrable systems [4,3], and predicted theoretically for a wide range of non-Kerr materials by Snyder and Sheppard [5], certain regimes of operation can lead to breathers and more dramatically to the fusion of colliding waves and the birth of new waves. We show examples of fusion in the saturable NLSE in Fig. 4.

Breathers, fusion, and the birth of new particles may be useful for computation in our context, but are more difficult to study and characterize than collisions that conserve the shape and number of particles, especially because they often seem to be accompanied by more radiation. In fact, our definition of information transfer is not applicable to these situations. However, the idea of information transfer may be generalizable to all interactions in a wide class of nonintegrable systems.

When wave velocities are very low (<1.0) and relative phases are approximately in the range 0.0–0.3, collisions produce breathers, or waves whose amplitude pulsates regularly, that cannot arise from Eq. (2). However, we observed that other collisions result in waves that can be specified by Eq. (2).

To test the hypothesis that collision products are of the same form as the original waves, we measured the amplitudes, envelope velocities, and phases at the peaks of waves after collisions; we then used these parameters to plot ''fresh'' waves and to compare their characteristics with those of the postcollision waves. In particular, we compared the carrier velocities of the fresh and postcollision waves and observed what happens in collisions between two fresh waves and between two postcollision waves. The results do suggest that the postcollision waves have the form of Eq. (2).

We estimated radiation for the collisions of Fig. 3 by finding the fixed-size section of the numerical-solution grid with the lowest root-mean-squared (rms) norm of the grid points. (We use circular boundary conditions in our numerical simulations, so that any radiation generated by collisions remains in the system.) Ideally, this rms norm should be very close to zero for solitary waves. Numerical error caused by the discrete nature of time and space in the grid contributes some noise, which we measured for the analytically solvable case of the 3NLSE by comparing numerical results with exact solutions. Based on these investigations, it appears that our simple measure of radiation gives a good general idea of the usefulness of various collisions for computation. In Fig.

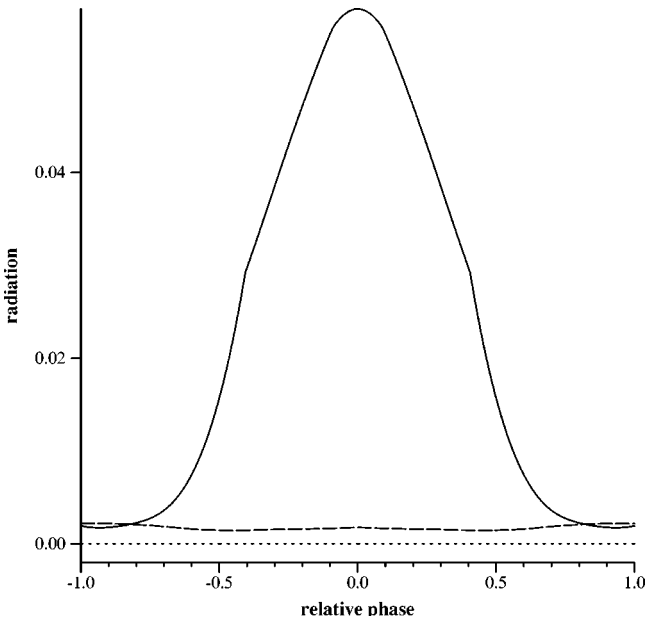

FIG. 5. Radiation due to collisions in the SNLSE system. Radiation is computed by finding the section of the numerical-solution grid with the lowest rms norm of grid points, using sections of size $N/10$, where N is the size of the entire grid; radiation is taken to be this lowest rms norm. The solid, dashed, and dotted curves show radiation for collisions of two solitary waves with velocities ± 0.5, ± 1.5, and ± 10.0, respectively. The relative phase is in multiples of π.

5, radiation is plotted as a function of the relative phase of two colliding waves.

The SNLSE solitary waves that appear to hold promise for encoding computation have relative speeds from about 0.2 to 4.0 and relative phases whose absolute values range from about 0.2π to 0.8π. Frauenkron *et al.* report [7] numerical studies of a quintic perturbation of the cubic NLSE and show that radiation in that system is $O(\epsilon^2)$, while energy exchange is first order, a general indication that the phenomena involved in information transfer can dominate radiation in nonintegrable variations of the NLSE.

VII. PHYSICAL REALIZATION

In this section we mention some physical systems that we might expect to be described by the saturable Schrödinger equation. We are continuing research in this direction, with the hope that computationally useful information-transferring collisions will be observed experimentally.

The integrable 3NLSE describes soliton propagation in so-called *Kerr materials*: materials in which the operative nonlinearity is due to the Kerr effect. In such materials $\chi^{(2)}=0$ and the dominant nonlinearity in the dependence of refractive index on electric field intensity is due to $\chi^{(3)}\neq 0$. This is the case for centrosymmetric and isotropic materials [14] and includes optical fibers in which soliton transmission has been demonstrated over long distances [15,16].

The nonintegrable SNLSE is applicable to simulating various physical phenomena, including the nonlinear effects of laser beam propagation in various media [4]. The SNLSE also describes the recently discovered $(1+1)$-dimensional photorefractive optical spatial solitons in steady state [12,17] and the optical spatial solitons in atomic media in the proximity of an electronic resonance [18].

Both the 3NLSE and the SNLSE describe temporal solitons; with the transformation $t\rightarrow z$, both equations also describe spatial solitons, with x and z being the transverse and longitudinal directions [19,20]. In practice, spatial solitons appear better suited for computation because temporal solitons require long distances to propagate. In addition, spatial solitons also exist in $2+1$ dimensions [17,18], offering an additional degree of freedom and suggesting the possibility of implementing two-dimensional universal systems such as the billiard-ball model of computation [21].

VIII. CONCLUSIONS

The analytically solvable 3NLSE supports only soliton interactions that transfer only trivial information and is thus unlikely to support a useful computational system such as the particle machine [1,2]. The nonintegrable SNLSE supports solitary-wave collisions that transfer nontrivial information and generate acceptable radiation and offers promise for encoding general computation through the particle-machine model.

The next step in this line of work will likely involve searching for configurations of collisions that can be used for simple computations, such as ripple-carry addition. Such an algorithm was implemented using the solitons of a filter automaton [22] and we believe that spatial SNLSE solitary waves support sufficiently general interactions to implement this algorithm in systems of $1+1$ dimensions. Spatial solitary-wave systems of $2+1$ dimensions also offer possibilities for encoding computation.

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