# Notes on the capital acquisition decision

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## 1 Optimal balance between capital and labor for fixed budget

We assume the Cobb-Douglas production function

$$V = AL^{\beta} K^{1-\beta} \ , \tag{1}$$

where $V$ is the output capacity of a firm, $L$ is labor, $K$ is capital, and $A$ is a technology constant. Suppose we have a fixed budget $T$ to spend in the present step, the price of capital is $c$, and the wage is $w$. We first ask what the optimal split is between capital and labor for this fixed budget $T$. If we produce capital $K$ at cost $cK$ we can spend the remainder to buy $(T - cK)/w$ units of labor. Substituting in eq. 1 give us

$$V = A \left( \frac{T - cK}{w} \right)^{\beta} K^{1-\beta} \ . \tag{2}$$

Maximizing with respect to $K$ gives us the optimal choice

$$K^* = (1 - \beta)T/c \ , \tag{3}$$

and

$$L^* = \beta T/w \ . \tag{4}$$

Notice that $T/c$ is the capital we would buy is we spent the entire budget on capital; so this says that to maximize output we buy a fraction $(1 - \beta)$ of that, and, also, a fraction $\beta$ of the total amount of labor that it is possible to buy with the budget. The total $T = cK^* + wL^*$, as it must be.

The optimal ratio of capital to labor is

$$\frac{K^*}{L^*} = \frac{w}{c} \frac{1-\beta}{\beta} \ . \tag{5}$$

We can write this also as

$$\frac{\text{cost of capital}}{\text{cost of labor}} = \frac{1-\beta}{\beta} \ . \tag{6}$$

## 2 Optimal budget given volume

Now think of the output volume $V$ as given, instead of the budget $T$. Substituting the optimal capital and labor split for fixed budget in eq. 1 yields

$$V = A \left( (1 - \beta)T/c \right)^{1-\beta} \left( \beta T/w \right)^{\beta} \ . \tag{7}$$

This can then be solved for the optimal budget in terms of the desired output volume, given that the capital/labor split is optimal:

$$T^* = \frac{V}{A} \frac{c^{1-\beta} w^{\beta}}{(1-\beta)^{1-\beta} \beta^{\beta}} \ . \tag{8}$$

The optimally balanced capital and labor in terms of the fixed volume are

$$K^* = \frac{V}{A} \left( \frac{1-\beta}{\beta} \right)^{\beta} \left( \frac{w}{c} \right)^{\beta} \tag{9}$$

and

$$L^* = \frac{V}{A} \left( \frac{\beta}{1-\beta} \right)^{1-\beta} \left( \frac{c}{w} \right)^{1-\beta} \ . \tag{10}$$

## 3 Scaling when capital production is linear in wage

Assume for this section that the cost of producing capital is proportional to the wage, so $c = wD$, where $D$ is the number of units of labor it takes to produce a unit of capital. Equation 9 then becomes

$$K^* = \frac{V}{A} \left( \frac{1-\beta}{\beta} \right)^{\beta} D^{-(1-\beta)} \propto V \ . \tag{11}$$

In the simple case $\beta = 1/2$

$$K^* = \frac{(V/A)}{\sqrt{D}} \propto V \ . \tag{12}$$

That is, the best choice of capital is proportional to a target volume, with a known and simple constant of proportionality. The same sort of result holds for the choice of labor.

**Result:** *Both capital and labor for efficient production in the Cobb-Douglas model scales scales linearly with output, even though the Cobb-Douglas function is nonlinear.*

**Question:** *For what other production functions does this result hold?*

## 4 Optimal budget when capital is financed

Observe that in section 1 the cost $c$ of capital needs to include any financing costs; in particular, interest on loans. Distinguish this from capital's market price by denoting its market price by $\hat{c} \le c$.

If we aim to achieve the optimal capital/labor balance in eq. 6, we want to buy an amount of capital each step of value

$$\hat{c}K = (\hat{c}/c)\,(1-\beta)T \tag{13}$$

where $T$ is the total budget of the firm per step for both capital and labor. Note that this is the amount spent in the capital market, and does not include financing costs. In other words, the firm must take financing costs into account when planning capital purchases. If in a simple case $\hat{c}/c = 1/(1+r)$, where $r$ is the per-step interest rate, then the firm should aim at spending

$$\hat{c}K = \frac{1}{(1+r)}\,(1-\beta)T \tag{14}$$

on capital in the capital market.

If capital is financed over a period of $n$ steps with equal payments per step, and each loan payment is $P$, the ratio $(\hat{c}/c) = \hat{c}/(nP)$, and the firm's target is to spend

$$\left(\frac{\hat{c}}{nP}\right)(1-\beta)T \tag{15}$$

per step on capital in the capital market. The per-step payment $P$ to finance a given amount of capital can be found using standard mortgage formulas (for example, [1]).

In the realistic situation, then, when the firm is financing capital with loans, it needs to make decisions each step so that its expenditures on capital and labor approach the optimal ratio, based on its information about the current interest rate $r$. It knows how much capital it owns at any given time, and how much total cost that incurs, taking financing costs into account. If that amount is below eq. 15 it should buy capital. If it has too much capital, it can choose either to use up the excess over time, or possibly sell partially depleted capital. The latter option may not be necessary, and may not even be profitable if capital depreciates quickly. Of course the ratio $\hat{c}/(nP)$ should be estimated using (possibly forecast) estimates of $r$ that are as fresh as possible.

Continuing with a loan structured with equal-payments $P$ over a fixed term of $n$ steps, the standard mortgage formulas give

$$\frac{\hat{c}}{nP} = \frac{1}{nr}\left[1 - \frac{1}{(1+r)^n}\right] , \tag{16}$$

where $r$ is the per-step interest rate. When $n = 1$ this reduces to $1/(1+r)$, as expected. This factor is a measure of just what fraction of total capital costs goes to the producer of the capital. The rest, of course, goes to the bank as interest.

For small $rn$,

$$\frac{\hat{c}}{nP} \approx \frac{1}{(1+r)^n} \approx \frac{1}{1+nr} , \tag{17}$$

but this is not much less expensive to compute than the exact form in eq. 16.

# References


[1] J.M. Guttentag. The mortgage professor's web site: Mortgage formulas. `http://www.mtgprofessor.com/formulas.htm`, 2008.