Macroeconomic Effects of Credit Shocks: Loan Rigidities and System Memory

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Abstract

We explicitly model credit generation and flows of physical cash and credit across different sectors of the economy, paying particular attention to the role of long-term capital acquisition and the associated credit required. The financing of capital is tracked in an aggregate loan portfolio, which, when interacting with the various sectors of our model economy, introduces system memory—the current state depends on the status of all outstanding loans issued over past time periods. This system memory has important effects on dynamic behavior and equilibria. It limits the speed of response to policy shifts or other shocks, even in a frictionless setting with no incomplete information. It also results in asymmetric dynamic effects of equal-sized credit expansions and contractions. While expansions generally exhibit smooth transitions, contractions cause more abrupt shifts in state variables, even when model parameters are identical. When combined with downward wage frictions, credit contractions cause widespread unemployment and a loss in production. The memory in the loan portfolio also leads to non-neutrality of monetary policy in steady-state. It does this because inflation discounts the value of past loans, allowing banks to expand credit, and this, in turn, lowers the real interest rate. Additional experiments compare the effects of an increase or decrease in liquidity injections by a central bank with a respective credit expansion or contraction introduced by the financial sector. Both methods exhibit similar asymmetries, although liquidity injections can cause larger shifts in interest rate.

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1 Introduction

Long-term financing of capital fuels economic growth and consequently has a major influence on macro-economic dynamics. A challenge to understanding these dynamics is the nature of capital itself: capital goods are heterogeneous in kind and durability, and are employed in interdependent and continually evolving chains of production. Most important, the element of *time* must play a central role in any theory of production, a fact recognized more than a century ago in Böhm-Bawerk's theory of interest and his emphasis on "roundabout" methods of production.[1] The complexity and persistent nature of long-term capital investment makes this activity difficult to aggregate and incorporate in the usual macroeconomic models.

In this paper we abstract away from the details of capital structure, and as an alternative, capture the long-term credit necessary for its financing in a portfolio of contractual bank loans. This requires several important departures from current macroeconomic models, including the careful modeling of the banking sector, the use of monetary flow to enforce intertemporal budget constraints, the differentiation of credit and cash money, and the treatment of monetary policy as an external control on the supply of cash in a baseline model. As we shall see, taking account of the inertia of the loan portfolio has significant macroeconomic consequences that are not manifested in current alternative models.

A key property of the model we present is thus the introduction of *system memory*: the system state includes information about all outstanding loans made over previous time periods up to the maximum loan term, which could span several years. This type of memory associated with capital investments has been relatively neglected in macro-economic modeling, yet is likely to have a substantial influence on macro-economic dynamics. In particular, the conventional class of Dynamic Stochastic General Equilibrium (DSGE), or New Keynesian models focus on shifts in *current* aggregate variables, such as investment and demand. We argue that this class of models is not well-suited to capture the effects of long-term capital investments. One reason is that introducing the associated memory into the model optimizations would either require a stationarity assumption on the random inputs over implausibly long time periods, or the incorporation of nonstationarities that would be difficult to estimate, and which would greatly complicate those optimizations.

We present a model in which the loan portfolio is embedded in a model economy with different sectors for producing capital and consumer goods. The loan portfolio is held by a financial or banking sector, and comprises loans spread over different durations. This variation in duration leads to rigidities in contractual interest rate agreements which, in turn, produce *credit inertia* associated with the expansion and contraction of total credit as a consequence of shocks or policy interventions. This inertia is distinct from the inertia created by frictions associated with price adjustments and credit contracts that have been previously studied.[2] Indeed, it arises in frictionless, complete-information versions of our model. Whereas price and financial frictions are generally viewed as market imperfections, which prevent firms and banks from rapidly adjusting prices and credit in response to exogenous shocks, the speed with which the loan portfolio can be changed in response to changes in other state variables

[1]See, for example, Böhm-Bawerk (1889) and Hayek (1941); for some historical perspective, see Blaug (1997).

[2]Those include the more conventional price frictions associated with monopolistic competition among firms (see, for example, Gali (2015) and Christiano et al. (2011a)), as well as financial frictions due to balance-sheet constraints, agency problems, and monopolistic competition among banks, which have been studied in recent years (see, for example, the surveys Brunnermeier et al. (2012) and Stiglitz (2015)).

is inherently limited by the system memory. That is, the distribution of loan terms held by banks reflects underlying expectations on returns from investing in projects over different time periods. Changes to that distribution in response to a shock cannot occur instantaneously, but are delayed by the time it takes for the banks to re-populate their portfolios with new loans that fit the target distribution. Roughly speaking, complete adjustment to a new distribution of loan terms will take at least as long as the maximum loan term, possibly several years, even in the absence of financial frictions.

To study the effects of the loan portfolio on dynamics, we model both the generation of loans by banks, as well as the demand for loans by Consumer Goods (CG) firms. Specifically, the supply of loans is determined by the willingness of banks to expand credit, as reflected by a fractional reserve requirement, and accounting for current liabilities (existing demand-deposit accounts). The demand for loans arises from CG firms, which borrow to finance capital purchases. Here we assume that *all* capital purchases needed for production of CG are financed.[3] Although it is straightforward to include household credit in addition to CG loans, here we only consider loans to the CG sector.

In contrast with previous models which also include credit generation and financing of capital purchases,[4] we assume that the capital used to produce CG is itself produced, and requires household labor as an input. Hence the model contains both a CG sector and a Capital (K) sector, and household labor is allocated across the two sectors. That allocation depends on the interest rate, which in equilibrium balances the supply and demand for loans. In this way, we can characterize shifts in labor across the two sectors, along with unemployment in each sector, during credit expansions and contractions.

We first present a baseline model with a *fixed* monetary base, or supply of physical cash, in which the state variables consist of monetary flows among four sectors: aggregate households, consumer goods, capital, and banks.[5] We distinguish between flows of *physical cash* and *credit*, which is generated by creating new demand deposits. In this way, it is possible to model particular types of loan contracts, fractional reserve banking rules, and to study the effects of changing the mix of circulating cash versus credit. In particular, increasing the amount of credit generated for loans, relative to cash transfers, generally increases the total credit generated, and lowers the real interest rate. This effect becomes more pronounced as loan durations become shorter, reducing system memory. The introduction of loan defaults then causes a more substantial shift in equilibrium when loans are issued by credit rather than by cash transfers. The total *cash* supply can be controlled by a Central Bank (CB), not part of the baseline model, that buys or sells loans, modeling open market operations.[6] In this way the supplies of cash and credit are explicitly modeled, and play a central role in determining dynamics.

There are two ways to introduce credit expansions and contractions in this model: (1) lowering (respectively, raising) the fractional reserve requirement, and (2) increasing (respec-

[3]Conversely, "capital" in our model is defined as the aggregate of inputs to the CG production function, which are financed. The CG sector does not take out loans to pay for labor.

[4]See, for example, Gertler and Karadi (2011) and Pesaran and Xu (2016), where the loan interest rate is also determined endogenously. Other papers which focus on the effects of different frictions associated with credit markets include Gerali et al. (2010), Christiano et al. (2011b), and Benes et al. (2014).

[5]Prices are then inferred by the associated flow variables. For example, the consumer price index is the aggregate amount spent on consumer goods (a state variable) divided by total consumer goods produced.

[6]The baseline model corresponds to a gold standard, hence "cash" in that model can be interpreted as gold. In contrast, the "cash" injected by a CB is presumably paper currency.

tively, decreasing) the rate at which a CB injects cash. In both cases credit expansions push the real interest rate down, and contractions push the real interest rate up. The dynamics (transients) associated with expansions and contraction are *asymmetric*: it generally takes longer to build out the loan portfolio during an expansion than to shrink the loan portfolio after a sudden contraction. When combined with downward wage friction, contractions cause widespread unemployment across both the CG and K sectors, with an associated transient loss in production.[7] Moreover, the dynamic persistence (durations of the transients) generally increases with loan durations, and can be significantly longer than the maximum loan duration. This is an inherent property of the model due to the memory embedded in the aggregate loan portfolio, and cannot be eliminated via forecasting on the part of agents.[8]

Results from simulated policy experiments are presented in which the CB continuously injects cash at a fixed percentage of the monetary base per iteration, increasing the monetary base at a given rate. Perhaps surprisingly, increasing the rate of injections lowers the *real* interest rate in steady state and hence shifts some labor from the CG sector to the K sector. (The nominal interest rate increases.) This occurs even though agents' decisions account for the associated inflation. This is again due to the memory in the loan portfolio: inflation discounts the liabilities associated with past loans, creating additional room for the banks to increase the supply of loans relative to demand.[9] Conversely, decreasing the rate of CB injections increases the real interest rate in steady state. Additional experiments illustrate the effects of cash injections by the CB versus credit expansions by the banks; injecting cash through the Bank sector versus giving the cash directly to households; and loan defaults.

The paper is organized as follows. The remainder of this section motivates the main modeling assumptions and relates those to prior work. We then summarize our main results in Section 2. Section 3 describes the baseline model without the CB, and specifies the associated set of recursions (periodic updates). Section 4 outlines the derivation of the equilibrium for the baseline model, with the details in an appendix. Section 5 then presents comparative statics, illustrating the equilibrium effects of varying fractional reserve rate, fraction of transactions in cash, and maximum loan term. Section 6 extends the baseline model to include a CB along with loan defaults, and Section 7 presents results from policy experiments, followed by some additional discussion in Section 8. Possibilities for future work, building on the model presented here, are discussed in Section 9.

1.1 Relation to Prior Work

In recent years an extended literature has been developed to explain how a financial sector can extend and amplify downturns associated with economic contractions. Those studies have focused on different types of frictions that interrupt the flow of credit to firms and households (Brunnermeier et al. (2012) and Stiglitz (2015)). The models generally build on

[7]Empirical support for such asymmetries has been presented in Dell'Ariccia and Garibaldi (2005); Kandil (2001). Related asymmetries associated with both monetary and fiscal policy have been extensively studied since the Great Depression. For a comprehensive review, see Sensier et al. (2002).

[8]In practice, transients will be influenced by traders and a financial sector, not modeled here. While that will strongly influence dynamics, and may serve to shorten transients in situations where traders are governed by fundamentals, it must always take significant time to re-populate the loan portfolio after a shock.

[9]This assumes that the fractional reserve requirement is fixed and binding, which may not be the case in practice (e.g., see Benes et al. (2014)). However, the real effects of cash injections are expected to persist even allowing for a change in reserve requirement. See the discussion in Section 7.

the conventional DSGE framework presented in Gali (2015) and Christiano et al. (2005), adding a financial (bank) sector with frictions associated with intermediation, information asymmetries, incomplete information (as in Sims (2010)), and balance sheet constraints. That work is motivated by dual goals: to understand what causes and prolongs deep downturns, and to design monetary policies that can mitigate the severity of such downturns.

In contrast, the system memory associated with the loan portfolio is not a source of friction due to misaligned incentives or lack of information. Rather, it arises from the financial sector's desire and ability to finance projects with returns that span multiple time periods. This leads to an inherent non-neutrality of monetary policy even in the absence of frictions with continuously optimizing agents. Hence our objective is to understand how basic changes in monetary policy affect the state of the economy, as opposed to optimizing a monetary policy that *reacts* to that state. Here the state includes how labor resources are allocated across the CG and K sectors. Those shifts potentially represent a misallocation of resources during a credit expansion, which are not captured in the conventional DSGE (New Keynesian) models (see the related discussion in Rognlie et al. (2015)).

The enforcement of monetary flow constraints, as in our model, forms the basis of the stock-flow accounting models presented in Godley and Lavoie (2007). Their more complicated models also distinguish flows of cash versus credit, as defined here, and include a Central Bank that can inject cash by purchasing particular assets. However, the key structural features of the model we present, as described in Section 1.2, including the system memory embedded in the loan portfolio, are not captured in the models presented in Godley and Lavoie (2007).

The model we present instead has its roots in classical studies of capital production and financing, going back to Böhm-Bawerk (1889), Wicksell (1898) and Hayek (1941). The loan array represents an initial step towards formalizing the role of time in capital production. Here, however, our objective is to provide insight into the macro-economic effects of *financing* capital, as opposed to the effects caused by *producing* heterogeneous capital goods. We therefore do not explicitly model the layers, or chains of production described in those early studies, but rather assume that all capital enters a Cobb-Douglas production function in the same manner.[10] We also do not explicitly model delays in capital production.[11] Introducing such delays would likely exaggerate and complicate the persistent dynamics we observe here, particularly with forward-looking CG firms, but would not change equilibria. Modeling a finer division of the CG and K sectors along with interactions among sub-sectors of production with associated delays is left for future work.

1.1.1 Model Departures

Guided by the preceding objective, we make several major departures from the conventional DSGE modeling framework. These are summarized by the following set of assumptions:

- All state variables are monetary flows except for the loan interest rate.
- Consumer goods and capital firms are perfectly competitive (zero-profit), hence we ignore frictions associated with price-setting.

[10]This might be interpreted as a chain of production with only two stages: production of capital followed by consumer goods.

[11]In this sense our model for capital *production* is memoryless. The system memory is instead due to capital financing.

- Unemployment is caused solely by downward wage friction, or equivalently, a reduction in the flow of total wages into a household labor sector.

- Banks generate an aggregate supply of loanable funds that satisfy an exogenous fractional reserve requirement, assumed to be binding.

- The model is deterministic, i.e., there are no random inputs. Shocks are modeled as step-changes to model parameters.

- Consumer goods firms allocate revenue across labor and capital, and households allocate their wealth across savings and consumption. Those splits account for inflation, but depend only on the current system state, reflecting diffuse priors about the occurrence of future (deterministic) shocks. These behavioral assumptions are discussed in Section 1.1.2.

According to the first bullet, the price levels of consumer goods, capital, and wages are inferred by the corresponding monetary flow state variables. This reverses the causality relation between state variables and money supply within the DSGE framework. Namely, in conventional DSGE models the money supply must implicitly adapt in a frictionless manner to variations in state and control variables such as the inflation and interest rates. In contrast, here the monetary base is either fixed, or directly controlled by the CB through open market operations, and that constrains the money supply and the dynamics of the remaining state variables. The CB injections can, in principle, be used to target the interest rate, as is typical in DSGE models. However, this becomes more challenging with long-term loans since the associated lags and rigidities introduce memory and delay in the trajectories of the supply and demand for loans over time. That in turn complicates the determination of a sequence of cash injections intended to control state variables.

The second bullet is due to our focus on the persistent effects caused by loan memory. We therefore do not need to model the associated monopolistic, or more generally, imperfect competition (Gali (2015)) associated with price frictions. This restriction is easily relaxed by introducing forms of "viscosity", which would restrict the rate at which particular flows can change. That could be applied to CG prices, wages, and credit flows, and would exacerbate the persistent dynamics associated with loan portfolios. In equilibrium all flows in and out of sectors must balance, and the zero-profit condition implies single aggregate unit prices for both consumer goods and for capital. Similarly, laborers are perfectly competitive across both CG and K sectors, so that there is a single wage rate.

While it would be straightforward to drive the model with random inputs or shocks, as in the DSGE framework, this is not essential here, since our main results do not rely on calibrating model parameters with actual data, and we do not seek optimal policies in response to random shocks. Rather we show how deterministic shocks affect both dynamics and steady-state (equilibrium) properties. We will add motivation for the preceding departures, discussing the relation to other models, in subsequent sections.

1.1.2 Household and CG Optimizations

In contrast to conventional DSGE models, where representative agents forecast into the future, given knowledge of the model and the current system state, here we assume those optimizations depend only on the current state, including the loan portfolio, and the inflation rate.[12] Specifically, the savings-consumption split by households depends only on an estimate of the return on savings, and the labor-capital split by CG firms depends on the output elasticities of labor and capital (model parameters), which are fixed.[13] Hence we effectively assume that all (deterministic) shocks are unanticipated, analogous to impulse- or step-response experiments with DSGE models. Here, however, we assume that the agents do not know the model, and therefore cannot exploit that knowledge to forecast the trajectory of state variables following a shock. Rather, actors within each sector determine their actions (most important, the supply and demand for loans) to achieve an equilibrium, assuming the current state persists.

The main justification for this restriction is that it simplifies the model and does not alter our main results. That is, although rational expectations based on complete information of both the current state and the model will influence system dynamics, it does not eliminate the inertia due to the loan portfolio following a step-change in model parameters. Even if all agents were given complete information about the model, it still takes time to repopulate the loan portfolio, reflecting the underlying shift in expected returns on different types of long-term capital projects. Also, in equilibrium (or steady-state with fixed inflation), forward optimization based on full information becomes a myopic decision that satisfies our assumption. Hence this assumption also does not affect the non-neutrality of monetary policy in steady-state. A more detailed study of dynamic behavior will, of course, require more sophisticated models for agent behavior than assumed here, as well as the addition of a financial sector with traders endowed with different forms of side information and investment objectives. (See Section 8.)

To provide additional motivation for our departures from the DSGE framework, we point out that introducing a loan portfolio into the DSGE framework poses significant challenges. Consider the usual core DSGE optimization (see, for example Beccarini and Mutschler (2012)):

$$\max_u E_0 \left[\sum_{t=0}^{\infty} \beta^t U(x_t, u_t)\right] \tag{1}$$

$$\text{subject to} \quad x_{t+1} = F(x_t, u_t, z_t), \tag{2}$$

where u_t is the control vector (to be chosen by the representative agent), E_0 the expectation operator with scope starting at 0, β the discount factor, U the agent utility function, x_t the state vector, F the system update function, and z_t an exogenous random disturbance. The dimension of the state space of this (normally nonlinear) dynamic system is typically on the order of 10, and its solution, except for unrealistic examples, usually involves approximation techniques, even to find the equilibrium.

Consider now what happens when we add a bank loan portfolio. We have in mind here an array of contractual loans that extend backwards in time over the maximum loan term, which in our examples will be on the order of 60 periods, or 5 years if we consider a period to be one month. We can fix the distribution of loans over terms, but we still must carry the history of their implied contractual obligations as part of the system state. This increases the dimensionality of the dynamic system dramatically, from about 10 to about 70, which renders the resulting stochastic equations essentially intractable with present techniques.[14]

[12]The inflation rate is a function of state variables, but is not itself a state variable.

[13]The actual amount spent on capital each iteration is determined so that the target capital expenditures equals expected loan payments, accounting for inflation.

[14]We can gain further insight by considering the continuous-time version of DSGE in Fernández-Villaverde

More important than the increase in computational complexity is the implicit assumption that with loan memory the random processes driving the system should be stationary over the loan periods (several years). Otherwise, the underlying changes in statistics that govern how much credit is generated per period would have to be incorporated in the optimization. That becomes especially problematic at an inflection point where the economy is transitioning from an expansion to a contraction, or vice versa. The stationarity assumption clearly becomes invalid, and measuring and incorporating the nonstationarities into the optimization poses a formidable challenge.

Current DSGE models avoid these problems by effectively approximating potential sources of system memory with random updates. In this way, the system dynamics retain a Markov property in which future states are independent of the past, given current prices and aggregated variables. A standard example consists of random Calvo price updates, which do not lead to a significant augmentation of the state space. Another example is in Teranishi (2008), where staggered loan contracts by banks are introduced according to a similar type of random update rule. (See also the related discussion in Posch and Trimborn (2013), which addresses the memory introduced by modeling rare shocks to the economy.) Approximating loan memory in this way serves to introduce dynamic persistence, but misses important effects due to the association of the underlying loan terms with monetary flows.

1.2 Features of the Baseline Model

Before presenting our main results we highlight important features of the baseline model with a fixed cash supply, followed by the model for open market operations by the CB. The baseline model has four sectors: Household (H), Consumer Goods (CG), Capital (K), and the Bank (B) sector. (See Fig. 1.) Throughout the paper the *sector* (H, CG, K, B) will be capitalized, whereas *constituent* households, consumer goods, capital firms, and banks will be written out in lower case. The CG sector is assumed to produce both private and public goods, so that we do not explicitly model a separate government sector funded by taxation. Each sector should not be viewed as an individual agent, but rather as consisting of a continuum of many heterogeneous entities. The monetary flows between sectors then represent aggregate amounts summed over all transactions within a period among different types of firms, banks, and household agents. The model is *dynamic* in that state variables representing monetary flows, as well as the interest rate, are updated each period. In the baseline model the total cash (monetary base), is fixed, although the Bank sector is able to generate additional credit. The baseline model has the following key features.

The interest rate is determined by a market for loans. The CG sector employs labor from the H sector and finances the purchase of capital from the K sector to produce output (consumer goods). All capital purchased by the CG sector is financed and the financing cost is added to the cost of capital. The fraction of revenue that the CG sector spends on labor versus capital, or *labor-capital split*, then determines the *Demand for Loanable Funds (DLF)*. The *Supply of Loanable Funds (SLF)* is determined by the amount of credit generated by the Bank sector (described next). SLF can also depend on the *savings-consumption split*,

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et al. (2011). The continuous-time limit of a simple DGSE model gives a standard ordinary differential equation (ODE) of low dimension. Adding long delays (with respect to the period, or update increment) instead leads to a *delay-differential equation* (DDE), which exhibits much more complex behavior than a standard ODE (Erneux (2009)).

which is the fraction of household *wealth* (not income) that is saved versus consumed. In equilibrium the Bank sector sets the interest rate so that SLF equals DLF. The two key decisions: the labor-capital split, selected to maximize output, and the savings-consumption split, are coordinated by this equilibrating choice of interest rate.

Since the interest rate is determined endogenously, an interest rate policy is not needed to close the model, as in conventional DSGE models (Gali (2015), Christiano, Trabandt, Walentin (2010)). However, the interest rate can be controlled through CB open market operations. The equilibrium interest rate in our baseline model reflects the personal time-discount rate, which determines the savings-consumption split, and can be interpreted as the "natural" interest rate, corresponding to an equilibrium level of production with *zero* inflation. In contrast to other models, here inflation due to CB injections generally changes the real interest rate. Growth due to technology and other factors is not explicitly included, but is easily added; it is not important for our main purposes. With a fixed monetary base an increase in production then corresponds to falling prices.

The Bank sector generates credit for loans subject to a fractional reserve requirement. The Bank can generate credit in the form of deposit accounts (owned by the CG sector) up to a maximum limit determined by the reserve requirement. That determines SLF. The fractional reserve requirement represents either a binding legal constraint on all banks, or can be viewed as the result of optimizing the balance between risk (probability of default for loans on the margin) versus return.[15] We explicitly distinguish between *cash* flows from one sector to another, and *credit* flows. In particular, the Bank can generate loans by transferring cash to the CG sector (taken from its reserves), and/or by creating credit (demand-deposit accounts) without exchanging cash. The characteristics of a credit expansion and contraction then depend on the fraction of transactions that are cash versus credit.

Banks are assumed to be competitive and zero-profit, so that all interest from loans is passed through to escrow savings accounts held by CG firms (to be explained) and savings accounts held by households. Hence we do not explicitly model financial frictions that have been previously identified as potential causes of prolonged downturns. Those include frictions due to monopolistic setting of loan rates (Gerali et al. (2010)) and financial intermediation (Gertler and Karadi (2011), Pesaran and Xu (2016), Christiano et al. (2010)), monitoring costs for risk premiums (Bernanke et al. (1999), Gerali et al. (2010)), agency problems (Meh and Moran (2010)), and balance sheet effects (as introduced in Kiyotaki and Moore (1997) and Bernanke et al. (1999), and developed further in Gertler and Kiyotaki (2010); Gertler and Karadi (2011); Christiano et al. (2010); Gertler et al. (2016); Benes et al. (2014); Brunnermeier and Sannikov (2014)). We also do not distinguish wholesale from retail banks, which may operate with different fractional reserve rates. (See Gertler et al. (2016).) Rather the fractional reserve rate is set so that the supply of loans represents the aggregate of all credit generated across the financial sector.

We can mimic the overall effects of those frictions by exogenously changing the reserve requirement, which in turn changes the supply of loans, along with the loan default rate. This type of aggregate modeling does not provide insight into how crises may spread within

[15] See also the discussion in Section 3.3 concerning additional constraints on setting the reserve requirement created by the memory in the loan portfolio.

the financial sector,[16] but suffices to illustrate broadly how credit contractions affect the rest of the economy. In particular, an increase in defaults would likely trigger an increase in the reserve ratio. Here this effect is captured exogenously, as opposed to the model presented in Benes et al. (2014), where the reserve ratio is determined endogenously based on a particular model for default risk.

Loans are issued and repaid over a distribution of loan terms. Loans to the CG sector are partitioned into different loan durations (windows) modeling a range of short- to long-term bonds. Here each loan is repaid with interest when it expires, corresponding to a zero-coupon bond. To meet its loan obligations the CG sector makes payments each period into an Escrow demand-deposit account, held by the Bank sector. The interest on all loans is passed along to both the H and CG sectors (for Escrow accounts) as a return on savings. Furthermore, any cash deposited in Escrow adds to Bank reserves, allowing a further expansion of credit and an increase in SLF. The continuing cycle of capital financing introduces memory in the model that influences both the equilibrium and dynamics.

The loan durations might be interpreted as corresponding to the durations of different types of capital goods. However, only the loan amounts and durations are treated as state variables. That is, the total amount of capital is determined by total loans, and is an input to the production function, but does not itself influence the system state. Hence the CG sector's objective is to adjust its capital expenditures to maximize output assuming that the total amount of capital is proportional to total loans. With this simplification the CG sector does not need to forecast capital requirements based on a more elaborate model for capital production and expiration.

Unemployment can occur when total wage flow decreases. Unemployment is relative to an equilibrium baseline level, taken to be zero, at which the labor market clears. It is a direct consequence of downward wage friction caused by a reluctance or inability to lower wages during a monetary contraction. In the absence of such friction, a decrease in total wages into a sector with a fixed number of laborers implies a corresponding decrease in wage rate averaged over the sector. Downward wage rate friction then causes the total wages to be spread over a smaller subset of laborers, with consequent unemployment. This modeling therefore allows unemployment to be be sector-specific (that is, in CG or K), depending on the flow of wages into each sector.

This characterization is motivated by our focus on the effects of credit shocks, as opposed to shocks due to technology or other exogenous forces, which can also affect unemployment (Christiano et al. (2011a)). We also do not explicitly include frictions associated with the movement of labor between the CG and K sectors, although those effects are easy to infer.

Dynamics and stability are controlled by interest rate adaptation and sector forecasting. Shocking a system parameter, such as the fractional reserve requirement, for example, leads to a disequilibrium in which the interest rate and other state variables are mismatched to the new set of system parameters. The system can adapt to the new equilibrium by adjusting the interest rate, using, in the present model, a tâtonnement process. In addition, side-information and forecasts regarding state variables within each sector can

[16] See the review of the literature on this topic by Brunnermeier and Oehmke (2013).

be incorporated in state updates, reflecting assumptions about expectations and any state information that may be available to each sector.

Starting from a disequilibrium state with fixed parameters, the system generally cannot immediately reach a new equilibrium in response to a deterministic shock—*even in the absence of price frictions.* This remains true even when the interest rate can be instantaneously set to clear the loan market and existing loan terms can be reset without penalty. The reason is that in general, reaching a new equilibrium requires a change in both loan amounts and the loan window distribution, which can only be achieved by making new loans spread over time (reflecting the initiation and completion of capital projects). This prevents other state variables from instantaneously moving to their equilibrium values. Rather, the system evolves through disequilibrium states as the interest rate is adjusted and the sectors interact. Those changes are fundamentally driven by changing expectations about the returns expected from long-term projects. Hence the system memory adds inertia, limiting the rate at which the system is able to reach the new equilibrium. This inertia cannot be overcome by forward-looking agents with perfect information about the model parameters; time is still needed to update the loan portfolio with the new equilibrium values. The associated transient increases in duration with the duration of loans, and can be long-lasting.[17]

Because the model is nonlinear, transient effects due to a shock can be complicated and depend critically on how the interest rate is adjusted. For example, rapid adaptation of the interest rate often leads to instability without appropriate filtering or forecasting of other state variables. Hence the interest rate must adapt slowly and in a manner consistent with the dynamics of other state variables. We emphasize that this mechanism is quite different from that of ad hoc frictions.[18] Here our focus is on identifying consistent dynamic trends caused by credit expansions and contractions, as opposed to studying how dynamics change with different assumptions concerning side-information and forecasts. Those trends should apply over a range of system parameters (such as the adaptation step-size) and reasonable assumptions concerning the information available to sectors.

1.3 Additional Features with a Central Bank

To gain insight into the effects of monetary policy, we introduce a CB that can inject liquidity (cash) into the economy represented by the baseline model. (See Fig. 6.) Key additional features of the expanded model follow.

The CB controls the monetary base (cash supply) by purchasing CG debt. The CB can inject liquidity by purchasing CG debt, meaning that part of the loan portfolio owned by the Bank is transferred to the CB in exchange for cash (computed as the total principal plus a premium that reflects accumulated interest).[19] If the CB injects cash so that the

[17]The ability to renegotiate loan terms may help to alleviate some of the inertia when interest rates are falling; however, loan terms are more likely to be rigid when rates rise.

[18]For example, Gerali et al. (2010) assume that banks incur a quadratic cost for adjusting retail rates, which fits within the DGSE modeling framework, but is admittedly a shortcut that "begs the question of its microfoundations". See also Teranishi (2008).

[19]In practice, the CB typically buys government debt. Because both public and private sectors are merged in CG, we do not need to distinguish explicitly between government and private CG debt, although we will assume that CB-owned debt is repaid directly from CG revenues, as opposed to being funded through regular escrow payments.

monetary base increases at a fixed rate, then the system reaches a "steady-state" (assuming stability) in which all monetary flows (and therefore prices) increase at the same rate as the monetary base, reflecting a constant rate of inflation. The CB does not pay interest on reserves, which would simply add to cash injections. We therefore do not explicitly include a Federal Funds Rate, but rather assume that this is embedded in the reserve ratio.

Loan repayments to the CB can produce deflation. The net flow of cash into the economy is the injection rate minus the rate of cash flowing out due to loan repayments to the CB. A direct consequence is that steady-state injections with full repayments of principal plus interest cause *deflation* in steady-state. To maintain a positive steady-state inflation rate, the CB must either continuously buy an increasing amount of debt (relative to the current monetary base) to compensate for the interest repayments, or collect only partial repayments, effectively gifting the CG sector with additional cash, corresponding to a "helicopter drop".

Loan defaults are imposed exogenously. Loan defaults eliminate the corresponding Bank assets (and liabilities), which reduces the return on savings to the Household sector. We will assume that a given fraction of all loans default each period. In practice, the default rate varies according to the level of risk incurred by the Bank sector, which can vary across subsectors and loan categories (e.g., short- versus long-term). Such dependencies, although not included here, could easily be incorporated into the model.

2 Summary of Results

The results described in this section are illustrated in the numerical examples shown in the sequence of figures in Section 7.

Credit expansions and contractions are asymmetric. Credit expansions are driven by an increase in the supply of loans, either by lowering the reserve rate or by cash injections by the CB. The rate at which credit expands is then limited by the *demand* for loans, which rises incrementally as new loans are taken out and propagate through the economy. In contrast, a credit contraction means the supply of loans decreases, so that actual loans must fall accordingly. The transient in that case is due to the additional credit from past loans, which disappears as those loans are repaid. Eliminating that credit takes a maximum loan window. This is illustrated in Figs. 9 and 10.

Inflation due to CB injections has a real effect on interest rate and resource allocation. This pertains to the scenario in which the CB injects cash at a fixed rate known to all agents. In steady-state SLF therefore increases at the inflation rate (same as the net injection rate), and the CG sector anticipates the nominal increase in target capital expenditures due to inflation when determining DLF. Even so, injections cause SLF *normalized by the monetary base* to *increase* relative to the *normalized* DLF, causing the *real* interest rate to fall in steady-state in order to clear the market. (The steady-state *nominal* interest rate increases, but by less than the inflation rate.)

This effect is due to the memory in the system introduced by the distribution over loan windows. The injections have the effect of *discounting* past loans issued by the Bank sector, thereby reducing bank liabilities. Given a fixed reserve requirement, that leaves more room for banks to expand credit by increasing lending, causing an increase in SLF and a drop in the real interest rate. That produces a shift in resource (labor) allocation from CG to the K sector. There is an associated increase in production due to the reduced friction for funding capital expenditures. That increase effectively represents the continual transfer of future consumption to the present, corresponding to a decrease in the time-value of money, as reflected by the drop in real interest rate. This is illustrated in Fig. 7.

The preceding scenario occurs when the CB injections are inflationary, meaning the flow of cash from the CB into the economy exceeds the flow of cash out of the economy from loan repayments to the CB. Conversely, deflationary injections (for example, with full loan repayments) causes the real interest rate to *rise*, in turn causing a shift in resource allocation from the K to the CG sector, and a decline in production. Here the decline represents a continual deferment of consumption to the future, as reflected by the increase in the time-value of money. This is illustrated in Fig. 8.

Credit contractions cause unemployment across the entire economy. A credit contraction in the model can be caused by an increase in reserves held by the Bank sector, or a net decrease in the rate at which the CB injects cash. In either case that causes a decrease in monetary flow into the *entire* labor pool, belonging to both the CG and K sectors. The wage rate (total wages divided by number of laborers) then decreases in *both* sectors, and adding friction results in unemployment and a decrease in total productivity. The severity of the decrease increases with the size of the credit contraction. This is illustrated in Figs. 9 and 10. In contrast, a credit expansion does not generally cause unemployment since wages flowing to both sectors *increase*. Additional numerical examples indicate that the transient effects due to credit contractions can be much more volatile than those associated with credit expansions (for example, as loan windows increase).

Widespread unemployment due to a credit contraction is different from *sector-specific* unemployment that may be caused by other types of shocks. For example, a sudden increase in return on capital (output elasticity) causes an increase in K wages and a decrease in CG wages, so that unemployment is confined to the CG sector. This is illustrated in Fig. 11.

Increasing the ratio of credit to cash transactions lowers the real interest rate. The ability to issue loans by creating new demand deposits increases SLF, relative to cash transactions, and leads to a decrease in interest rate. The decrease in going from all-cash to all-credit transactions, where all cash is held by the Bank as reserves and all flows are credit transactions, depends on the durations of loans in the loan array. The longer the loan windows, the more modest is the decrease in interest rate. This is because longer loan windows increase the amount paid into Escrow accounts, allowing cash to recirculate and generate more loans. This diminishes the difference in total credit generated by the all-cash and all-credit models, normalized by the monetary base. The difference between the all-cash and all-credit versions of the model can be extreme, in particular, when the loan durations are very short (e.g., one period, corresponding to a memoryless system in which all loans are paid back the subsequent period), and the fractional reserve requirement is small. A particular example is shown in Fig. 4.

The effect of loan defaults also depends on the ratio of credit to cash transaction. Defaults reduce the return on savings, keeping the reserve ratio fixed. This decrease is somewhat more severe in the all-credit model due to the increased credit flowing throughout the economy (in particular, as wages). Although loan defaults shift the equilibrium, they do not cause a credit contraction. In fact, to maintain the same reserve ratio, or equivalently, the amount of credit generated prior to the defaults, the Bank would need to *increase* loans to replace those that default. A more likely scenario is that the Bank would react to the defaults by increasing the reserve ratio, thereby causing a credit contraction. This is illustrated in Figs. 12 and 13.

CB injections through the Bank sector cause a larger increase in loans, or equivalently capital purchases, than injections through the Household sector. Injecting through the Household sector refers to the scenario in which the CB transfers cash directly to the Household sector, adding to its nominal wealth, instead of to the Bank sector by purchasing loans. For this comparison, we assume the cash injections are not repaid, corresponding to a "helicopter drop". Both types of injections increase loans in steady-state, although injections through the Bank sector cause a greater shift, which increases with the injection rate. This effect can lead to overinvestment in capital and a consequent decrease in production, because the CG sector acquires more loans to meet its target capital expenditures. In contrast, injections through households cannot cause such an over-allocation of capital, since the cash injections are split by households into Savings and Consumption of CG, and not directly used to purchase capital.

3 Baseline Model

Figure 1 shows the flow diagram for the baseline model. We start by describing the monetary flows between sectors and basic behavioral assumptions associated with each sector. The baseline model is then specified by a set of updates for the state variables.

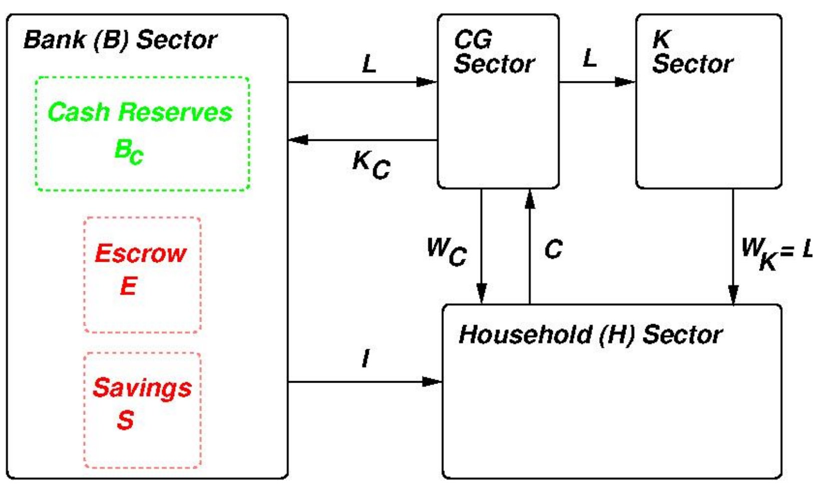

Figure 1: Money transfers in the baseline model.

3.1 Sectors and Assumptions

Referring to the figure, the H sector supplies labor to the CG and K sectors, and the CG sector employs labor and capital to produce its output, consumer goods. The CG sector finances all capital purchases from the K sector via loans through the Bank sector. Because we do not introduce random disturbances, an *equilibrium* associated with a set of input parameters corresponds to a set of state variables that remain constant from period to period. Those include the interest rate, return on savings, flows between sectors, and savings and escrow accounts held at the Bank. The interpretation is that in equilibrium the sectors make the same aggregate decisions each period.[20] Hence in equilibrium the algebraic sum of flows in and out of each sector must be zero; that is, markets clear in aggregate. In addition, we assume that no sector except for the Bank stores any cash or credit. Hence all revenues collected by the CG and K sectors are passed to other sectors, and all household income is either saved or consumed. (Implications are discussed in what follows.) All cash or credit which is not flowing between sectors stays in the Bank sector as reserves or deposit accounts. Next we describe each sector in more detail.

Household: At each period households in the H sector (split across those employed by CG and K firms) receive income consisting of wages W_C and W_K from the CG and K sectors, respectively, and also a return on savings I, which is added to total household wealth $\mathcal{W}$ (including savings). The H sector then decides what fraction of its aggregate wealth to keep in savings S in the Bank sector, and spends the remainder C on consumption. We emphasize that this savings-consumption split is determined from *total wealth*, rather than from income per period. In principle, this decision can take any functional form of past, current, and forecast state variables. Here we make the simplifying assumption that the fraction saved is a nondecreasing function of the return on savings at the current period. While that limits the information used by the H sector to determine savings, which can affect transient (disequilibrium) dynamics, introducing forecasting would not change the equilibrium, since in an equilibrium the state variables remain constant.

Because all income received by the H sector is either consumed or saved, there is no "cash-on-hand", as often assumed in other models. Rather, here it is assumed that savings accounts are liquid in aggregate, so that households can withdraw whatever cash or credit is in the account at any time. With fractional-reserve banking, household cash-on-hand could be motivated by a concern that the Bank sector has insufficient reserves (possibly precipitating a bank run). Here we do not explicitly consider such scenarios, and assume that households trust that their deposits are redeemable.

Although it is relatively straightforward to include household borrowing to finance consumption, which is then combined with the demand for CG loans, we omit this feature in the Baseline model presented here. We therefore do not consider how changes in interest rate and available credit may effect the composition of loans across different subsectors of CG and K such as commercial and industrial, real estate, and discretionary goods. (See the empirical study in Den Haan et al. (2007).)

Consumer Goods: Firms in the aggregated CG sector receive revenue C (equal to consumption) from the sale of the consumer goods they produce, and must decide how much to allocate across capital and labor inputs. We assume that this decision maximizes production for a fixed budget. In the particular implementation described here, the production function is Cobb-Douglas with constant returns to scale, so that the CG sector allocates in equilibrium a fixed fraction of its revenue to capital expenditures and wages. Maximizing output

[20] The uniqueness of the equilibrium will be discussed in Section 4.

implies that the CG sector contains competitive firms, although the firms are interpreted as manufacturing an array of heterogeneous consumer goods. Since all of the CG revenue is allocated across wages W_C and capital expenditures K_C, the CG sector can be interpreted as being zero-profit.[21]

The *average CG price* is defined *implicitly* as the consumption C divided by CG output We omit friction in average CG price adjustments, and assume that in aggregate the CG market clears each period. Hence the average CG price is a function of C, a state variable, but is not a state variable itself. This again is due to the fact that transients in this model are primarily due to the memory associated with the loan portfolio as opposed to price frictions. The flexible CG price then removes the need to model explicitly any market dominance among firms, or monopolistic competition that is typically associated with sticky prices.

The CG sector finances all capital purchases, and in equilibrium takes out loans L from the Bank each period. Those loans are used to purchase capital from the K sector, and are allocated over a distribution of loan windows. We assume that the CG sector repays each loan with a single balloon payment when the loan becomes due. In order to avoid making such balloon payments directly from the CG revenue stream, the CG sector makes an aggregate deposit each period into an aggregate Escrow-savings account held by the Bank sector. That account, along with Household savings, then earns the current return on savings. The demand for loans (DLF), which is L in equilibrium, is computed so that the capital expenditures (escrow deposits) per period, K_C, meets the target amount K_{tar} that maximizes output.

Capital: We assume that capital is produced solely by labor as an input, and that there is a fixed number of labor hours required to make one unit of capital.[22] Hence the amount of capital manufactured per period increases linearly with the fraction of the labor pool allocated to the K sector. The Capital sector receives L per period from the CG sector for capital sales, and passes that to the Household sector as wages for the manufacture of capital. The fraction of households (laborers) that manufacture capital is then $W_K/(W_K+W_C)$, where $W_K = L$ denotes wages for labor in the K sector. This therefore determines the labor-capital split, which determines CG output. (Note that labor is the only resource that is allocated in the economy.)

There are, of course, many ways in which the expiration of capital can be modeled. Here we make the simplifying assumption that the total amount of capital, which enters the production function, is proportional to the amount of labor employed by the K sector. Hence there is no inertia built into the production and consumption of capital. This is in part to focus on the effect the loan portfolio has on dynamics. (In fact, one might argue that inertia in the loan portfolio already captures the inertia in the capital sector, since the loan durations should roughly correspond to the time needed to put the capital in service.)

[21]It is possible to relax this assumption. For example, if firms in the CG sector have the ability to manipulate prices, they may increase revenue by raising their prices, causing households to spend more on consumption and to save less, causing a shift in the equilibrium. Profits in that scenario would be returned to households (shareholders) as part of the return on investment. This change in the model would not affect the main results we present here.

[22]As a refinement, we could assume a Cobb-Douglas model for capital production, in which "second-order" capital is needed to produce the "first-order" capital that appears in the CG production function. This can be iterated, creating longer chains of production with higher-order capital, as introduced in Böhm-Bawerk (1889) and Hayek (1941). Such a refinement may be useful for studying how shocks propagate through subsectors of the economy, especially when delays are taken into account.

In addition, the amount of capital only affects CG production, which is not a state variable, hence the model for capital expiration affects neither transient behavior nor equilibrium values.[23]

Bank: The Bank sector is the sole source of credit, which is issued to the H sector as Savings accounts S, and to the CG sector as Escrow accounts E. We do not explicitly model Bank profits or fees, but assume that any net revenue that is generated by the Bank sector is distributed over all of its accounts. The credit issued is limited by a fractional reserve requirement, and the Bank holds an amount $B_c(t)$ as cash reserves. Specifically, the reserves $B_c(t)$ must be at least a fraction f of total liabilities, which includes all accounts (S and E) held by the Bank sector. SLF, equal to L in equilibrium, is then determined by the reserve requirement along with the amount of credit the Bank sector can generate.

A distinction is made throughout the model between flows of *cash* and *credit*. We assume that loans are a fraction c_k cash, which is deducted from Bank reserves and transferred to CG. The rest is credit, which is issued to CG without changing $B_c(t)$ (e.g., in the form of checks drawn against bank accounts). Consumption is also paid for with a fraction c cash, and the rest credit, which flows between the corresponding demand deposit accounts held by the Bank. (As we shall see, wages to labor in the Capital sector are paid with a fraction c_k cash, and wages to labor in the CG sector with a fraction c cash.) We can therefore distinguish between an *all-cash* model ($c = c_k = 1$) in which all flows in the economy represent cash transactions, and an *all-credit* model ($c = c_k = 0$) in which all cash resides at the Bank sector as reserves, and all flows in the economy represent credit transactions.

In the all-credit model, the total credit issued by the Bank is limited to $B_c(t)/f$, and the money multiplier is therefore $1/f$. In the all-cash model, SLF is limited to $(1-f)\cdot B_c(t)$ each iteration and the usual argument then shows that the total credit is limited to $B_c(t)\cdot(1/f-1)$, so the multiplier is $1/f - 1$.[24] The model allows any $0 \leq c, c_k \leq 1$, and we will see that the particular mix of cash and credit has a real effect on economic activity. Given a fixed monetary base and reserve requirement, the all-credit model generally creates more credit than the all-cash model, but then may potentially experience deeper credit contractions due to a shock.

The fractional reserve requirement f will be exogenously specified, and we will assume that SLF always satisfies this requirement with equality.[25] In practice, banks would attempt to optimize their loan portfolios (both amounts and durations) to maximize their revenue, taking into account the risk of default. We do not model risk here (although we will include defaults), but rather assume that f is set by either the banks themselves or regulators as a proper threshold that balances returns against risk.

The banks competitively adjust the loan interest rate so that SLF matches DLF in equilibrium. Because the interest rate is adjusted via a tâtonnement process, it cannot change

[23]Introducing such inertia in capital production would then require the CG sector to forecast capital inventories when determining DLF. That would influence the dynamics associated with credit shocks, although it would not alter the main dynamic trends illustrated here.

[24]The all-cash model is often used to explain the expansion of credit in fractional-reserve banking (see Mankiw (2013), for example). However, in practice most bank accounts are created by issuing new credit Turner (2013).

[25]In disequilibrium, SLF can be less than DLF, corresponding to a shortage of liquidity. We assume that the available credit is spread uniformly over loan windows and therefore do not model the disaggregated effects of such credit shortages on leveraged borrowers (see Brunnermeier et al. (2012), Stiglitz (2015)).

instantaneously, and in that sense can be thought of as "sticky". This adjustment plays a central role in balancing flows and bringing the economy to an equilibrium. The interest associated with all loan repayments is allocated proportionally across Household Savings and CG Escrow accounts. Interest on Household Savings I flows back to the households and adds to total Household wealth. The principal part of the repayment replenishes the original accounts from which loans were made, which subsequently allows the Bank sector to make new loans.

3.2 Dynamic Recursions

Our goal is to simulate the path of the state variables in the model rather than just solve static equations for an equilibrium. We therefore seek update equations that move the state of the system at discrete time t to the state at time $t+1$. Those updates represent an aggregation of actions taken by the collection of agents within each sector during each period. The order of the updates is meant to approximate the aggregate effects of asynchronous transactions between individual agents across sectors that would occur over a single period. Table 1 lists the main program state variables. References to model flow variables will be capitalized (Savings (S), Escrow (E), Consumption (C), etc.).

From a formal point of view, if we represent the system variables at period t by the state vector $\vec{x}(t)$, we seek the state-transition rule

$$\vec{x}(t+1) = \Phi(\vec{x}(t)) \ . \tag{3}$$

The response to shocks in the baseline model can then be simulated by varying the input parameters to the model shown in the lower section of Table 1. An equilibrium then corresponds to a fixed point $\vec{x} = \Phi(\vec{x})$. We defer the discussion of initialization—it is in fact an interesting question exactly what initial values of the state variables across the economy correspond to realistic and reachable states.

Loan (balloon) repayments: The period begins by distributing the total interest on loans due in the current period. That will enable the households to determine total wealth and aggregate consumption for the period. We define the *loan array* $L(\tau,t)$, $\tau = 1,\cdots,\tau_m$, as aggregate loans made at time t with loan duration τ. With some abuse of notation, we will denote the *total* loans made at period t as

$$L(t) = \sum_{\tau=1}^{\tau_m} L(\tau,t). \tag{4}$$

We assume that $L(\tau,t) = w_\tau L(t)$, where $\{w_\tau\}$ is the distribution of loan terms, satisfying $\sum_\tau w_\tau = 1$ and $w_\tau \geq 0$, $\tau = 1,\cdots,\tau_m$. This can be interpreted as the result of randomly selected loan terms in the limit as the number of CG firms becomes large. For the simplest case $\tau_m = 1$ all loans taken out at time t are repaid with interest at time $t+1$. The system is then memoryless since the system state at time t depends only on aggregate flows and the interest rate at time t. The system memory increases with τ_m.

The loan repayment agreement is modeled after a zero-coupon bond: total principal and interest are due when the loan term expires. No payments are required in the interim, although we will assume that to avoid covering the loan balloon payments from their revenue

r	loan interest rate
s	return on investment, sometimes savings rate
$\mathcal{W}$	total household wealth
D_{lf}	demand for loanable funds
S_{lf}	supply of loanable funds
G	total cash in the economy, usually fixed at G_0
W_C	wages of labor in consumer goods sector
W_K	wages of labor in capital goods sector
S	household savings
B	demand-deposit Household account
E_{tar}	target demand-deposit Escrow account
E	demand-deposit Escrow account
C	household consumption
L	loans from Bank sector to finance capital
U_{cg}	unspent funds held by CG
R	bank cash reserves
I	interest earned by Household deposits
K_{tar}	target capital expenditures, per period
K_c	actual capital expenditures, per period
L_{tot}	continuing debt, total outstanding loans
B_c	total current Bank cash
Φ	total debt of Household sector
f	fractional reserve requirement
τ_m	maximum loan term, (minimum is 1)
e	propensity-to-save parameter
β_K	output elasticity of capital
β_L	output elasticity of labor
k_r	adaptation coefficient for loan rate adjustment
c	fraction of CG wages in cash
c_k	fraction of loans in cash

Table 1: The main state variables in the baseline model. Structural parameters are in the lower section.

streams, CG firms make per-period payments into an Escrow savings account. The payments are determined so that for a fixed return on savings, the amount in Escrow accumulates to the desired balloon payment when the loan becomes due. Other loan structures could, of course, be modeled. The zero-coupon format is assumed here since this typically applies to corporate bonds, and also gives rise to extensive credit expansions and contractions due to the creation of CG Escrow accounts.

Balloon payments on CG loans due at time t are

$$L_{BP}(t) = \sum_{\tau=1}^{\tau_m} [1 + r(t-\tau)]^{\tau} L(\tau, t-\tau) \tag{5}$$

$$L_{BPP}(t) = \sum_{\tau=1}^{\tau_m} L(\tau, t-\tau) \tag{6}$$

$$L_{BPI}(t) = L_{BP}(t) - L_{BPP}(t) \ . \tag{7}$$

where $r(t)$ is the loan interest rate at time t, L_{BP} is the total balloon payment, and L_{BPP} and L_{BPI} are the associated principal and interest, respectively. The loan repayments are included in total Bank assets, which are next distributed across the Savings and Escrow accounts.

Distribution of interest across Savings, Escrow accounts: Loan repayments are a transfer from the accumulated amount in Escrow $E(t)$ to Savings $S(t)$, and also back to Escrow since Escrow accounts are also used to generate credit. The redistribution across Savings and Escrow is in proportion to their size, since this allocation is a return on investment. Note, however, that only the interest on repayments needs to be explicitly allocated since the principal amounts are already included in $S(t)$ and $E(t)$. According to the proportional allocation rule, we have

$$\mathcal{B}(t) = S(t-1) + E(t-1) \tag{8}$$

$$\mathcal{L}(t) = \mathcal{B}(t) + L_{BPI}(t) \tag{9}$$

$$S'(t) = \mathcal{L}(t)\frac{S(t-1)}{\mathcal{B}(t)}, \qquad E'(t) = \mathcal{L}(t)\frac{E(t-1)}{\mathcal{B}(t)} \tag{10}$$

where $\mathcal{B}$ and $\mathcal{L}$ are, respectively, total Bank liabilities before and after the loan repayments, and S' and E' are intermediate values for Savings and Escrow, which will be needed to obtain $S(t)$ and $E(t)$ following withdrawals and deposits.

Return on investment: This is the fractional interest earned per period, and is determined de facto:

$$s(t) = \frac{L_{BPI}(t)}{\mathcal{B}(t)} = \frac{\mathcal{L}(t) - \mathcal{B}(t)}{\mathcal{B}(t)}. \tag{11}$$

This reflects the fact that the Bank sector is zero-profit, and the Household sector receives the distribution of all income from financing operations as their return on savings. An alternative adaptive version for determining $s(t)$ is discussed in Section 6 and Appendix D. Those incorporate knowledge of the loan array, and result in smoother dynamics, but do not affect the equilibrium.

Loan rate: The loan rate $r(t)$ is adjusted each period with the update

$$r(t) = r(t-1) + k_r \cdot (D_{lf}(t-1) - S_{lf}(t-1))/G_0 \ . \tag{12}$$

$D_{lf}(t)$ and $S_{lf}(t)$ are DLF and SLF, respectively, k_r is a constant step-size, and G_0 is the monetary base (total cash), fixed in the Baseline model. This is a simple tâtonnement adjustment at the bank that moves the rate up when the demand for loans exceeds supply, and vice versa. The step-size is normalized by the (for now fixed) size of the monetary base, G_0, and is governed by k_r. It governs the speed with which the banking sector adjusts the loan rate. In general, when k_r is too large the economy as a whole becomes unstable, and when it is too small the convergence to equilibrium is correspondingly sluggish. Ultimately, its determination is part of the calibration process.

Other methods for updating $r(t)$ could be used. For example, it is possible to adjust $r(t)$ myopically to balance $S_{lf}(t)$ and $D_{lf}(t)$ at each iteration. That, however, can lead to instability.[26] This is effectively due to a mismatch in the adjustment rate for $r(t)$ and the speed with which the loan portfolio can adapt to shocks. While adding such features will affect dynamics, they will not change the main results we present here.

Household wealth and Savings-Consumption split: Household wealth consists of all assets owned by households, namely,

$$\mathcal{W}(t) = W_C(t-1) + W_K(t-1) + S'(t) \ . \tag{13}$$

where W_C and W_K are total wages to households working in the CG and K sectors, respectively. Savings and consumption are then given by

$$S(t) = g(s(t)) \cdot \mathcal{W}(t) \tag{14}$$
$$C(t) = [1 - g(s(t))] \cdot \mathcal{W}(t) \ , \tag{15}$$

where $g(\cdot)$ is the propensity-to-save function, assumed to be a nonnegative and nondecreasing function of the current savings rate $s(t)$. In the examples given later in this paper we will use the simple form

$$g(s) = \frac{es}{1+es} \ , \tag{16}$$

where the positive constant e is the propensity to save; this ensures that $g(s) \leq 1$, and $g(s) \approx es$ for small es. The savings *deposit* at period t is then

$$\Delta S(t) = S(t) - S'(t). \tag{17}$$

Note that $\Delta S(t) < 0$ signifies a withdrawal of savings for consumption. The total Revenue received by the CG sector is equal to Household Consumption,

$$R_v(t) = C(t). \tag{18}$$

As noted earlier, we will distinguish between *cash* and *credit* transactions. The fraction of cash versus credit flowing between sectors will depend on the mix of cash and credit in loans

[26]It may be possible to maintain stability by forecasting other state variables, although that would not eliminate the transient behavior.

issued by the Bank sector. The *cash* portion of the total amount deposited by Households is given by

$$\Delta S_c(t) = W_{cash}(t-1) - c \cdot C(t) \tag{19}$$

where $c \cdot C(t)$ is the cash component of consumption and W_{cash} is the cash part of total (K and CG) wages. Note that $|\Delta S_c(t)| \leq |\Delta S(t)|$. The fraction of cash spent on consumption shown here can be different from the fraction of cash making up loans c_k, although in the numerical examples to be presented we will assume $c = c_k$.

Labor-Capital split: The firms that produce consumer goods must decide on the split between expenditures for labor and investment in capital. This depends on the production function $Y(N_L, N_K)$, where N_L is the quantity of labor and N_K is the quantity of capital, which are functions of t, their sum normalized to unity. We will assume

$$Y(N_L, N_K) = N_L^{\beta_L} N_K^{\beta_K} \tag{20}$$

where β_L and β_K are the output elasticities for labor and capital, respectively, and $\beta_L + \beta_K = 1$.[27] The amounts $N_L(t)$ and $N_K(t)$ are the only resources being allocated in the economy, and are determined by the relative amounts of wages for CG labor $W_C(t)$, and wages for Capital labor $W_K(t)$. That is, $N_L(t) = W_C(t)/[W_C(t) + W_K(t)]$, and $N_k(t) = 1 - N_L(t)$. The amount of capital is therefore memoryless in the sense that it varies instantaneously with the amount of labor allocated to the Capital sector, as discussed in the previous subsection.

The CG firms split their revenue across labor wages and capital purchases to minimize their combined labor plus capital cost to produce a given output. Given the preceding assumptions, this implies that the firms spend a fixed fraction of revenue on capital:

$$K_{tar}(t) = h \cdot C(t) \ , \tag{21}$$

where $K_{tar}(t)$ is the *target* capital expenditure and $h = \beta_K/(\beta_L + \beta_K)$. We assume that capital purchases are financed[28] so that $K_{tar}(t)$ represents payments on loans taken out in previous periods (to be discussed next). In equilibrium, actual capital expenditures $K_C = K_{tar}$ each period, and the loans issued to the CG sector each period must result in escrow payments K_C per period.

Demand for loans and Escrow deposits: Each period the CG sector computes its Demand for Loans, $D_{lf}(t)$, in order to meet its target capital expenditures $K_{tar}(t)$ in the future. Specifically,

$$D_{lf}(t) = K_{tar}(t)/\Theta(r, s) \tag{22}$$

where $\Theta(r, s)$ is the discount factor associated with finance charges, i.e., the averaged per-unit, per-period cost of financing a loan. The discount factor must take into account the assumption that loans are paid off through a series of deposits to an Escrow savings account that accumulates interest according to the return on savings. In equilibrium D_{lf} is chosen so that the Escrow deposits per period match K_{tar}. The calculation of Escrow payments is given in Appendix A.

[27] Recall that we are not explicitly modeling growth in this model.

[28] During a transient CG may have "unspent funds" that can be used to purchase capital directly, to be discussed.

Specifically, the total accumulation in Escrow after i payments due to a unit loan with term τ is, from (80),

$$\mathcal{A}(r,s,i,\tau) = (1+r)^{\tau}\frac{(1+s)^i - 1}{(1+s)^{\tau} - 1} \ , \tag{23}$$

where r and s are evaluated at the time the loan is initiated, assuming contractual rates at that time. The target amount needed in Escrow at time t is then

$$E_{tar}(t) = \sum_{\tau=1}^{\tau_m}\sum_{i=1}^{\tau}\mathcal{A}[r(t-i), s(t-i), i, \tau]L(\tau, t-i). \tag{24}$$

This includes the loan repayments for the current iteration, which are subsequently subtracted from the Escrow accounts. (See (26).) To meet this target the CG sector makes the Escrow deposit

$$\Delta E(t) = E_{tar}(t) - E'(t) \ . \tag{25}$$

We will also need the cash portion, which is $c \cdot \Delta E(t)$. The updated Escrow account after the withdrawal from Savings (17) and Escrow deposit (25) is

$$E(t) = E'(t) + \Delta E(t) - L_{BP}(t). \tag{26}$$

Returning to the determination of $D_{lf}(t)$ in (22), the discount factor $\Theta(r,s)$ can be computed by adding up payments per period over all loans (see (82) in Appendix A). In equilibrium, the corresponding capital expenditures per period $K_c(t) = \Delta E(t)$, the Escrow deposit in (25).

CG Wages, unspent capital budget: In analogy with (21), the target amount CG wishes to spend on labor is $(1-h)\cdot R_v(t)$. In equilibrium, this will be equal to $R_v(t) - \Delta E(t)$ (revenue minus current capital expenditures). However, because $\Delta E(t)$ depends on loans from previous periods, in general it will not be the same as the target capital expenditure during a transient. We therefore let

$$W_C(t) = \min\{(1-h)\cdot R_v(t), R_v(t) - \Delta E(t)\}. \tag{27}$$

That is, if $\Delta E(t)$ exceeds the target capital expenditures $h \cdot R_v(t)$, then the CG sector first makes its loan payments, and then passes along the remaining revenue to the Household sector as CG wages.[29] Otherwise, if $\Delta E(t) \le h \cdot R_v(t)$, then $W_C(t)$ is given by the first term (target wages). CG then retains an additional (unspent) amount for capital:

$$U_{cg}(t) = R_v(t) - \Delta E(t) - W_C(t). \tag{28}$$

We assume that the CG sector uses this to purchase additional capital directly (without financing), so that this amount is added to capital wages (see (38)). Of course, in equilibrium $U_{cg}(t) = 0$.[30]

[29]It is possible that in a severe credit contraction $\Delta E(t) > R_v(t)$ in which case the CG sector goes bankrupt. The recursions can be re-initialized by defaulting on loans or by having a CB inject cash, as will be discussed in Section 6.

[30]CG could instead spend $U_{cg}(t)$ to purchase additional labor. This has the property, perhaps unrealistic, that CG under-invests in capital, relative to labor, during a credit expansion when loan obligations are increasing.

Bank assets, liabilities, and the supply of loans: The supply of loans, $S_{lf}(t)$, is determined by applying the fractional reserve constraint to total Bank reserves (cash), accounting for current liabilities. The Bank cash is given by

$$B_c(t) = [B_c(t-1) - c_k L(t-1) + \Delta S_c(t) + c \cdot \Delta E(t)]^+, \tag{29}$$

where c_k is the fraction of loans issued in cash. The CG sector pays a fraction c of all its expenditures (labor and capital) in cash, so that $c \cdot \Delta E(t)$ is the cash part of the CG escrow payment (capital expenditure). Total Bank *assets* at period t are then $B_c(t)$ plus the sum of all loans in the loan portfolio, which is given by

$$L'_{tot}(t) = \sum_{\tau=1}^{\tau_m} \sum_{i=2}^{\tau} L(\tau, t-i+1), \tag{30}$$

where we exclude loans due in the current period t, since those were repaid at the beginning of the period. In equilibrium we have $L'_{tot}(t) = \Gamma' L(t)$, where the loans per period $L(t)$ remains constant and

$$\Gamma' = \sum_{\tau=1}^{\tau_m} \tau w_\tau - 1. \tag{31}$$

We will distinguish L'_{tot} from the total loans in the loan portfolio at the *end* of the period,

$$L_{tot}(t) = \sum_{\tau=1}^{\tau_m} \sum_{i=1}^{\tau} L(\tau, t-i+1), \tag{32}$$

where this now includes the loans made during the current period (see (37)). In equilibrium, $L_{tot}(t) = \Gamma L(t)$, where

$$\Gamma = \Gamma' + 1 = \sum_{\tau=1}^{\tau_m} \tau w_\tau. \tag{33}$$

Total Bank *liabilities* at time t consist of its total credit obligations to the other sectors, and is given by $S(t) + E(t) + (1-c) \cdot W_C(t) = \mathcal{L}(t) + (1-c) \cdot W_C(t)$. That is, total Bank liabilities are the sum of all Savings and Escrow accounts credited to households and CG firms, respectively, plus the additional credit that is circulating as wages to the CG sector. The latter must be included since those credit transactions represent transfers between demand deposit accounts owned by the CG and Household sectors. Those accounts must be held by the Bank, hence add to the Bank's credit obligations. We will assume that Bank assets are always equal to Bank liabilities. This need not be the case in practice, and in fact, it is possible to add an offset to Bank assets or liabilities, which then persists in equilibrium. This is discussed further in Section 4.

The net amount of Bank reserves, which serves as the base for issuing credit (loans), is therefore

$$\rho(t) = B_c(t) - f \cdot [S(t) + E(t) + (1-c) \cdot W_C(t)]. \tag{34}$$

Consider first the all-cash model ($c = c_k = 1$). Loans in that scenario are made by deducting the loans from Bank reserves, and adding that cash to CG accounts. Hence $\rho(t)$ is the amount of "excess reserves", all of which can be loaned out, so $S_{lf}(t) = \rho(t)$. With $c = c_k = 1$ (all-cash model), and ignoring loan repayments, Bank liabilities build in successive periods (i.e., credit expands) by loaning out increments of reserves according to the classical description of fractional reserve banking. (That is, $\rho(t+i) = (1-f)\cdot\rho(t+i-1)$, $i \geq 1$, and summing over i gives the asymptotic amount of credit (total loans) as $(1/f-1)\cdot\rho(t)$.) The actual amount of outstanding credit is reduced by loan repayments, and in equilibrium the credit revolves so that all expiring loans in period t are replaced by new loans with total amount S_{lf}. Credit expansion in the all-cash model introduces some inertia, since credit builds over successive periods. However, the associated time-constant is relatively small compared with that associated with the loan portfolio (say, for $\tau_m > 50$). This time-constant inherently limits the velocity of money.

Now consider the case $c = c_k = 0$ (all-credit model). Here the Bank sector keeps $B_c(t)$ in reserves, and can create new accounts with total amount $\rho(t)/f$. Hence $S_{lf}(t) = \rho(t)/f$, the reserve requirement is binding, and the Bank cannot issue more credit in successive periods until loans are repaid. This eliminates the inertia associated with credit expansion in the all-cash model. Numerical examples comparing the all-cash and all-credit models show that the all-credit model generates substantially more credit in equilibrium. Although this has a noticeable effect on equilibrium and dynamics for the particular cases considered, overall both models seem to exhibit qualitatively similar behavior.

In practice, a mix of cash and credit circulates in the economy, so that we allow c and c_k to take any value between zero and one. It can be shown that with such an arbitrary mix of cash and credit, the supply of loans becomes

$$S_{lf}(t) = \frac{1}{f + c_k(1-f)}\rho(t). \tag{35}$$

An alternative form for $S_{lf}(t)$ can be derived by using the fact that Bank liabilities equals assets, given by $B_c(t)$ plus total outstanding loans, $\Gamma' L(t)$, where Γ' is given by (31). Substituting in (35), replacing $L(t)$ with $S_{lf}(t)$, which holds in equilibrium, and solving for $S_{lf}(t)$ gives

$$S_{lf}(t) = \frac{(1-f)\cdot B_c(t)}{f + c_k(1-f) + f\cdot\Gamma'} \ . \tag{36}$$

This is equivalent to (35) in equilibrium, and assumes that the Bank knows the equilibrium loan window distribution $\{w_i\}$. Numerical examples indicate that using (36) instead of (35) tends to reduce volatility during transients.

Loans, Capital wages: The amount of loans issued at period t is

$$L(t) = \min\{S_{lf}(t), D_{lf}(t)\} \ . \tag{37}$$

The loans are passed through the Capital sector as wages, so that capital wages are

$$W_K(t) = L(t) + U_{cg}(t) \tag{38}$$

where U_{cg} is the unspent CG revenue available to purchase additional capital. The cash portion of total wages is then

$$W_{cash}(t) = c_k W_K(t) + c W_C(t) + c_k U_{cg}(t) \ , \tag{39}$$

which appears in (19).

Unemployment and GDP Measurements: Certain variables of interest can be considered *measurements*, in the sense that they relate directly to commonly used metrics, but do not directly affect the economy's evolutionary path, at least in the current development. These include the allocation of labor across the CG and K sectors, CG production $Y(t)$, given by (20), GDP, and unemployment. To determine unemployment, we note that in the absence of wage frictions, the *target* wage rate is determined by the total wage flows as

$$w_{tar}(t) = \frac{W_C(t) + W_K(t)}{N_L}. \tag{40}$$

With downward friction the wage rate becomes

$$w(t) = \max\{w_{tar}(t), w(t-1) + k_L \cdot [w_{tar}(t) - w(t-1)]\} \tag{41}$$

where k_L is the wage-smoothing constant. The quantities of labor allocated to CG and K are then, respectively,

$$\tilde{N}_L = \frac{W_C(t)}{w(t)} \qquad \tilde{N}_K = \frac{W_K(t)}{w(t)}, \tag{42}$$

and the fraction of unemployed labor is

$$\tilde{N}_U/N_L = 1 - (\tilde{N}_L + \tilde{N}_K)/N_L. \tag{43}$$

GDP can be measured in various ways, for example, as Consumption $C(t)$, or Consumption plus Capital purchases $C(t) + L(t)$. Normalization by the aggregate price of CG, $C(t)/Y(t)$, then accounts for credit fluctuations. For the numerical results in Section 7, only CG output is shown.

Initialization: The state variables can be initialized in different ways depending on how the cash is initially distributed across the sectors. For example, one possibility is to let $B_c(0) = B_0$ and $W_{cash}(0) = W_0$. The total cash (monetary base) is $G_0 = B_0 + W_{cash}(0)$. All other state variables can be initialized at zero. Provided that the loan rate step-size k_r is not too large, the state variables then typically converge to their equilibrium values when iterating the recursions as t becomes large. To simulate the response to a parameter shock, it is possible, of course, to initialize all state variables to their equilibrium values, as computed in Section 4. This also provides a good consistency check on the predicted equilibrium.

Remarks: We conclude this section with the following observations:

- The duration of a period can be calibrated against the loan window distribution. For example, if we assume that the maximum loan duration is five years (after which longer-term loans are refinanced or traded), then taking $\tau_m = 60$ implies that each period corresponds to one month.

- Each state variable appears on the left-hand side of a dynamic equation in the (temporally ordered) loop exactly once, which means that a state variable $x(t)$ is assigned its value exactly once. The state variable's *equilibrium value* is then $x = \lim_{t\to\infty} x(t)$, if that limit exists.

- The total cash at the end of each period is $B_c(t) + cW_C(t) + cU_{cg}(t)$, which can be shown to equal G_0.

3.3 Return on investment s and interest rate r

The return on investment s, given by (11), is clearly a function of the loan interest rate r. Relations among state variables in equilibrium for the baseline model will be discussed and illustrated in the next two sections. Here we point out that s is generally not the same as r in equilibrium, and consider the implications of the corresponding relation. In fact, it is possible that $s > r$, which on the surface seems problematic. The technical reason for this becomes clear when we write the return on investment normalized by total outstanding loans L_{tot}, given by (32). That is, if the total loan amount per period is L, then the total interest accumulated in equilibrium is

$$\begin{aligned} L_{BPI} &= \sum_{i=1}^{\tau_m} [(1+r)^i - 1] w_i L \qquad (44) \\ &\geq \sum_{i=1}^{\tau_m} r i w_i L \;=\; r \Gamma L \;=\; r L_{tot} \; , \end{aligned}$$

where we use Bernoulli's inequality, Γ from (33), sum over all loans that become due in the current period, and ignore the reserves that the bank keeps. Thus

$$s' = L_{BPI}/L_{tot} \geq r, \qquad (45)$$

with equality as $r \to 0$. This is due to the compounding of interest on existing loans that will be paid off in the future. Once again, the root cause is the memory in the loan portfolio.

If we now account for the bank reserves, this means that the total amount in Savings plus Escrow (total base on which s is actually computed) is greater than L_{tot}. Hence from (11), $s \leq s'$ with equality when there are no reserves ($f = 0$). Increasing f from zero, then, means s decreases, relative to r, and consequently there is an f^* where $s = r$.

Consider the scenario where $f < f^*$, so that $s > r$. CG firms then have an incentive to take out loans at rate r (increasing DLF), and put those back in savings accounts to earn interest s, adding to bank liabilities. That shifts loans from capital investment to bank liabilities, which reduces SLF. Hence r rises relative to s, and this arbitrage will continue until $r = s$. The net effect of this arbitrage is then to *curtail* credit expansion, which accomplishes the same effect as increasing f until $r = s$. This has the interesting implication that even in the absence of uncertainty and risk, as considered here, there is an upper limit on capital investment enabled by credit expansion, which corresponds to a strictly positive fractional reserve rate f^*. Attempting to expand credit beyond this point by lowering f only serves to provide opportunities for financial arbitrage. (The two scenarios, defined by $f = f^*$ and $f < f^*$ with arbitrage, may not give exactly the same equilibria, but both serve to limit capital investment.)

Conversely, if $f > f^*$, then $s < r$ and households have an incentive to withdraw savings in order to make loans directly to the CG sector. That increases SLF, causing r to fall relative to s. The net effect of that arbitrage is then to *expand* credit until $s = r$, accomplishing the same effect as *lowering* f. Hence we conclude that for the scenario considered with no risk, the Bank sector would have an incentive to set f *endogenously* so that $r = s$, eliminating opportunities for arbitrage. Of course, we emphasize that this does not account for risk, which in practice would likely be the dominant factor in determining total credit expansion. Hence for the numerical results presented in subsequent sections we will set f exogenously and leave the possibility of dynamically adapting f for future work.

4 Equilibrium of the Baseline Model

The equilibrium of the model just described can be calculated analytically in a straightforward way, but the derivation is not without some subtleties. The details are laid out in Appendix B, and we summarize the results here.

The key fixed-point condition is, of course, the equality of supply and demand of loanable funds, S_{lf} and D_{lf}. This equality determines the equilibrium loan rate r, the single price in the central money market of the model. However, when the consequences of this fixed-point condition are worked out, we learn that there is one more variable than equation. That is, up to this point, we have exactly one extra degree of freedom left in the model. Thus, if we consider the baseline model with no further condition, there is a one-dimensional manifold of equilibria, a fact that is readily observed in simulations.

Adding a constraint results in an equal number of equations and variables, and we choose to specify a quantity that has a natural interpretation—the net debt (or, if negative, assets) of the Household sector, which we denote by Φ.[31] Because the balance of money and credit flow is enforced at every period, Φ is constant throughout a simulation. In the terminology of computer science, Φ is a *loop invariant*, and it is established by the initial conditions of the model. For the purposes of this paper, we choose $\Phi = 0$, although it will be of interest in future work to study how Φ affects the equilibrium.

When we do add the condition that household debt is zero, it turns out that the resulting numerical problem reduces to that of solving two nonlinear equations in the two unknowns r and s, the loan rate and return on investment, respectively. The two equations are (178) and (179) in Appendix B.

5 Comparative Statics

The ability to compute the baseline equilibrium without simulating the recursions makes it easy to study the comparative statics of the model, and there are several reasons for doing this:

1. They can confirm our intuitive expectations of how equilibrium state variables depend on parameters, thus lending some confidence that the model captures realistic trends.

2. They provide an independent consistency check on the dynamic recursions.

3. They can provide starting points for simulations, thus saving the time and possible difficulty of arriving at an equilibrium from a cold start.

4. They facilitate calibration.

In the next three subsections we present some representative studies of comparative statics in line with the first item, illustrating the dependence of equilibrium variables on the fractional reserve rate, the fraction of transactions in cash, and the maximum loan term.

[31] In our many equilibrium calculations and corresponding simulations we have always found the resulting equilibria to be unique. A proof of uniqueness for the special case $\tau_m = 1$ is given in Appendix C.

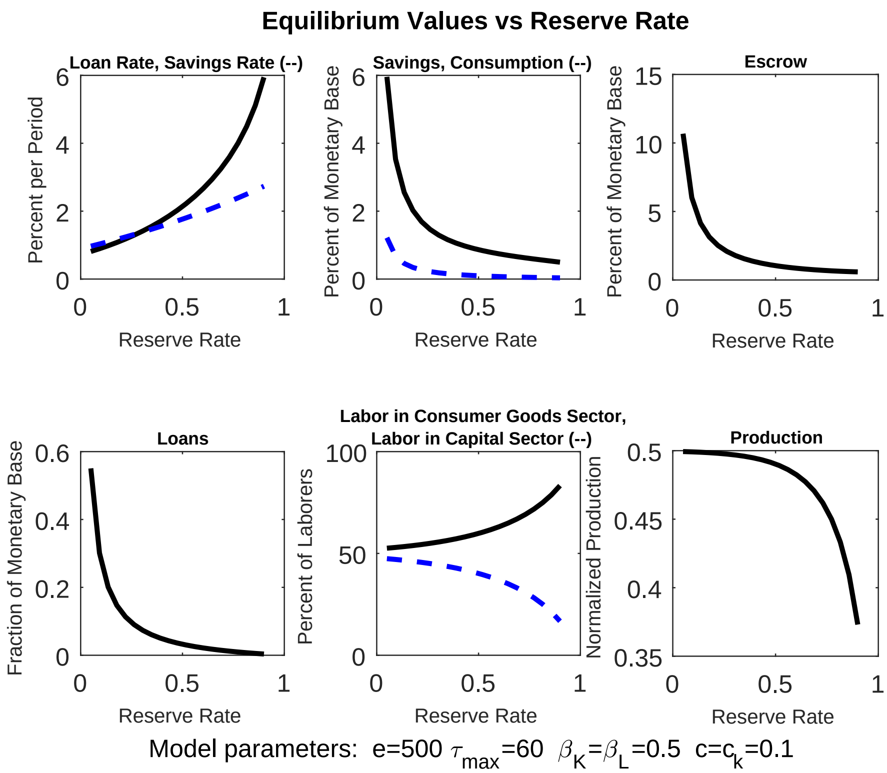

Figure 2: State variables and production versus the fractional reserve rate f.

5.1 State variables vs. fractional reserve rate

The first set of plots, shown in Fig. 2, shows the effects of varying the fractional reserve rate f on several important system variables. The results are in complete conformity with our intuition: as the reserve rate increases, available credit contracts, the loan rate and return on investment therefore increase; savings, consumption, escrow, loans, and production as fractions of the monetary base decrease; and labor shifts away from the capital labor sector as investment shrinks.

Note that the savings rate (return on investment) in the upper-left plot in Fig. 2 exceeds the loan rate for small (but reasonable) values of reserve rate. As discussed in Section 3.3, this always occurs for sufficiently small values of f. The reason is that the loan rate is a nominal, contractual rate set by the (zero-profit) Bank, whereas what we term the savings rate is actually the ex post return on investment received by the Household and CG sectors, and therefore benefits from compounding, especially for small reserve rates and long loan durations, when more credit is available.

5.2 Effects of varying the cash to credit ratio

The next two plots illustrate the effects of changing the fraction of transactions in cash versus credit c, assuming $c = c_k$. Figure 3 shows household wealth and total loans (left), and loan rate r and return on investment s (right) as functions of c. Once again this confirms our intuition. As more cash is used for transactions, available credit contracts, and household wealth and loans contract. What is a bit surprising, perhaps, is that the change in equilibrium

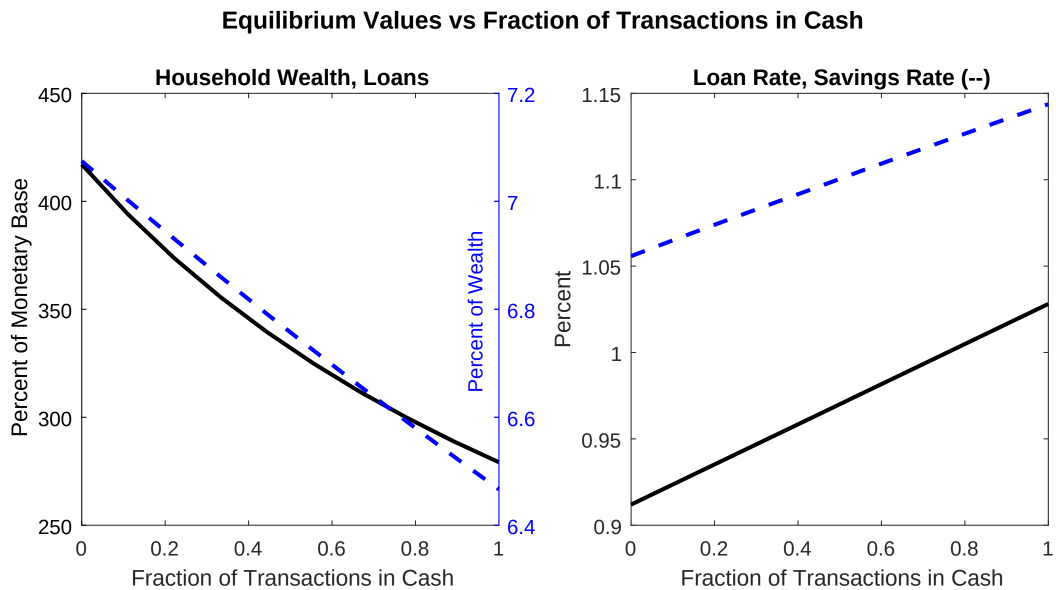

Figure 3: Total wealth and loans (left) and loan rate and return on investment (right) versus fraction of transactions in cash c. The parameters are $e = 500$, $\beta_L = \beta_K = 0.5$, $\tau_m = 60$, and $f = 0.1$.

loan rate in going from all-credit $c = 0$ to all-cash $c = 1$ is relatively small, about 0.9% to about 1%.

Figure 4 shows loan rate (left) and total loans (right) versus the maximum loan duration τ_m. Two plots are shown in each figure corresponding to $c = 0$ (all credit) and $c = 1$ (all cash). This figure shows that the model variables become much more sensitive to c when τ_m is small. For example, when $\tau_m = 1$ in the all-credit model, (36) shows that total loans in equilibrium normalized by the monetary base is $S_{lf}/G_0 = S_{lf}/B_c = (1-f)/(2f)$ since all cash is held at the Bank. For the all-cash model, $S_{lf}/G_0 = (1-f)/(1+f) \cdot (B_c/G_0)$, where B_c/G_0 is the fraction of cash held at the Bank as reserves. This suggests that the difference in credit generated in the two regimes can become arbitrarily large as f decreases to zero. While that is true in equilibrium, the loan rate r becomes negative when f decreases below a threshold f', which is observed to increase as τ_m decreases. In this example, $f = 0.5$, as opposed to $f = 0.1$ in Figure 3, so that r remains positive for the entire range of τ_m shown. At $\tau_m = 5$, Figure 4 shows that the transition from all cash to all credit reduces the loan rate r from about 1% to less than 0.5%. As τ_m becomes large, the all-cash model is able to generate the same amount of credit as the all-credit model. This is due to the ability of cash to re-circulate multiple times in Escrow, allowing the Bank to issue more loans.

5.3 Loan rate vs. maximum loan term

The last example in Fig. 5 shows plots of the equilibrium loan rate vs. the maximum loan term τ_m, where, as usual, loan terms are uniformly distributed from 1 to τ_m. To explain the observed increase, peaking, and ultimate decrease of the loan rate, consider first what happens if the maximum loan term increases suddenly from a low value. The immediate effect, off equilibrium, is that actual capital expenditures $K_C(t)$ suddenly decrease, because payments to Escrow are now spread over a longer span. This means that the CG sector has

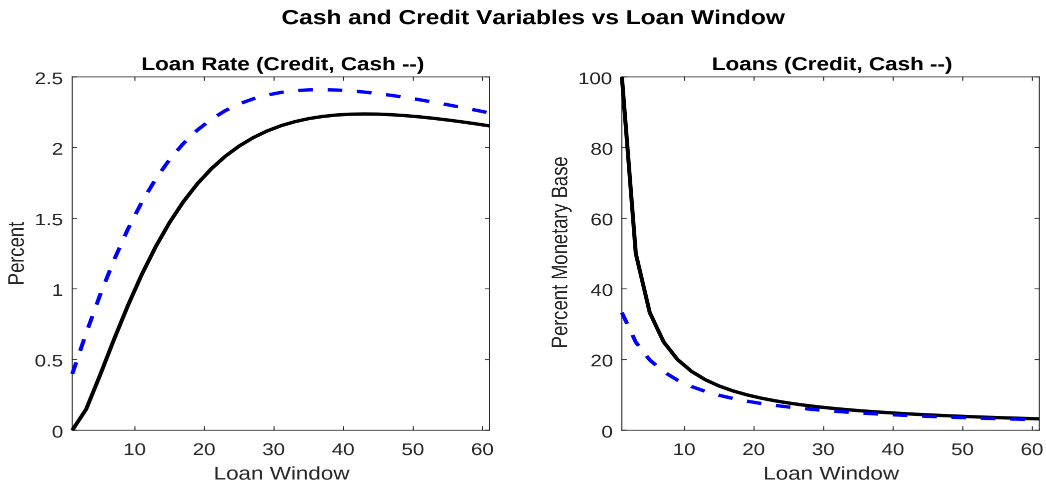

Figure 4: Loan rate (left) and total loans (right) versus maximum loan window τ_m. The parameters are $f = 0.5$, $e = 500$, $\beta_L = \beta_K = 0.5$, $c = c_k = 1$ in the all-cash case, and $c = c_k = 0$ in the all-credit case.

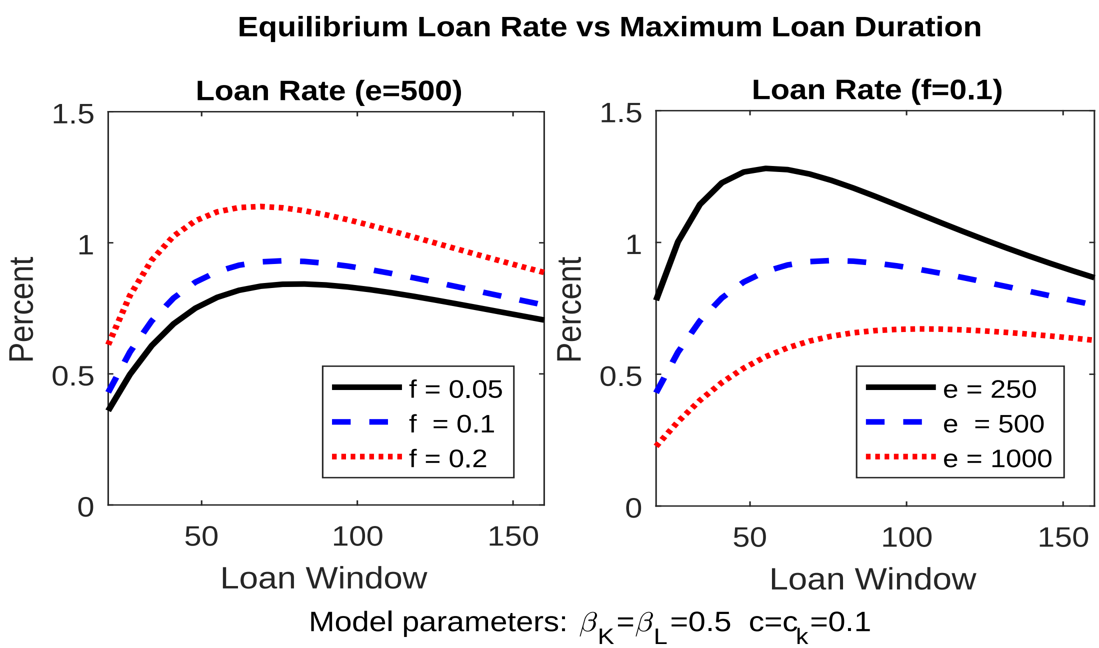

Figure 5: Equilibrium r vs. τ_m for: (left) various values of fractional reserve rate f, $e = 500$; (right) various values of propensity-to-save e, $f = 0.1$.

more to spend on capital to meet its target, $K_{tar}(t)$, thus increasing demand for loans and putting upward pressure on the loan rate. With no initial corresponding first-order increase in the supply of loanable funds, the loan rate rises, and eventually reaches a new, higher equilibrium.

At some point, however, as τ_m continues to increase, the part of the balloon payments

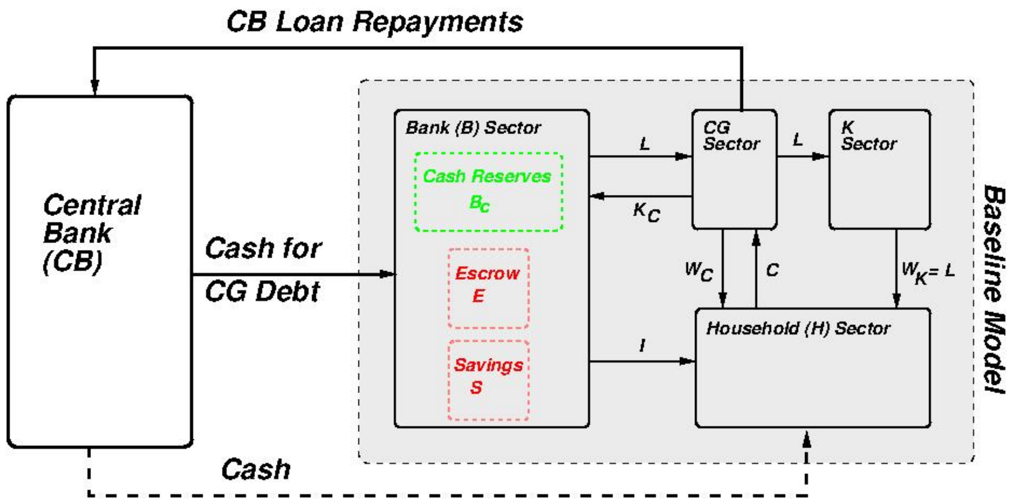

Figure 6: Money transfers in the model with central bank.

that is interest becomes comparable to the part that is principal. This is easy to see from the fact that $L_{BPI}/L_{BPP} = X - 1$,[32] where $X = (1/\tau_m)\sum_{i=1}^{\tau_m}(1+r)^i$ grows with τ_m. This causes the supply of loanable funds to increase faster than the demand, whose rate of increase is slowing, and the equilibrium loan rate then turns around, so that there is a value of τ_m at which the loan rate peaks.

The plots show curves for various values of reserve rate for fixed propensity-to-save, and various values of propensity-to-save for fixed reserve rate. The displacements of the curves are all consistent with expectations: the rate goes up with increasing reserve rate, and down with increasing propensity-to-save (when the supply of loanable funds increases).

6 Model with Central Bank

We now expand the baseline model by introducing injections of new money into the economy by a CB. This is illustrated in Fig. 6, which shows the cash flows to and from the CB. The CB can inject cash either through the Bank, by buying CG debt, or by giving it to the Household sector. Loans owned by the CB are then repaid by the CG sector along with other loans owned by the Bank. The following recursions assume that the CB expands the monetary base at a fixed rate by injecting an amount of cash equal to a fixed fraction κ of the monetary base each period through purchases of CG debt. The amount that is repaid to the CB can be varied, so that part of the injected cash is a gift to the CG sector (corresponding to a "helicopter drop"). We also introduce a constant loan default rate δ, which applies uniformly to all loans. Policy experiments can then be simulated by changing these parameters either across runs to examine steady-state behavior,[33] or within a simulation to study dynamic behavior.

[32]See Appendix B.

[33]With CB injections we refer to "steady-state", as opposed to "equilibrium", to emphasize that if the system does reach a steady-state, then all state variables representing monetary flows increase at the injection rate, as opposed to remaining constant. As we later emphasize, the corresponding *normalized* state variables, relative to the monetary base, converge to constants in steady-state. Also, as in the baseline model, we assume Household debt is zero, so that a unique steady-state is always observed.

Initialization As in the baseline model, there is an initial distribution of cash among sectors:

$$B_c(0) = B_0, \quad W_{cash}(0) = W_0, \quad G(0) = B_0 + W_{cash}(0) \tag{46}$$

where $B_c(0)$ is the initial cash, which the Bank keeps as reserves, $W_{cash}(0)$ is the additional amount of cash in the economy flowing between sectors (or held in the CG sector), and $G(0)$ is the initial (cash) monetary base. For the policy experiments in the next section, all other state variables are initialized at zero.

The following iterations for the model state variables are then computed for $t = 1, 2, \cdots$.

Loan defaults: If a loan defaults, then that loan is deducted from the corresponding value of the loan array $L(\tau, t-i; t)$, where τ is the loan term, and $t - i$ is the time at which the loan was made. We have added dependence on t, since with defaults existing elements of the loan array can change over time. Here we also have a loan array for loans owned by the CB, $L_{CB}(\tau, t-i; t)$. In aggregate we will assume that a fraction δ of all loans default each period, so that

$$L'(\tau, t-i; t) = (1-\delta)L(\tau, t-i; t-1) \tag{47}$$

$$L'_{CB}(\tau, t-i; t) = (1-\delta)L_{CB}(\tau, t-i; t-1) \tag{48}$$

for each $i = 1, 2, \cdots, \tau$ and $\tau = 1, \cdots, \tau_m$. We define the intermediate variables L' and L'_{CB} since the loan arrays will be subsequently modified due to the purchase of loans by the CB. In general, δ could depend on i and τ; however, for simplicity, we will assume that defaults occur uniformly over all elements of the loan array. Since a loan can default at an arbitrary time within the duration of the loan, it is assumed that the CG sector makes escrow payments up until the time of default. The bank then seizes the escrow account and distributes what has accumulated in that account across Savings and Escrow. The amount that is seized must be less than the full amount that the CG sector owes to the Bank sector for the loan that defaulted.

Here we assume that loans default at a constant rate δ. That is, at each iteration a fraction δ of *all* loans default. With this assumption the aggregate amount left in Escrow from loans that default at time t is $\delta \cdot E(t-1)$. The return on household savings will then include the balloon repayments for all non-defaulting loans plus $\delta \cdot E(t-1)$.

Loan (balloon) repayments: With CB injections loans are split between the two loan arrays $L(\tau, t-i; t)$ and $L_{CB}(\tau, t-i; t)$. Loan repayments to the Bank are the same as before,

$$L_{BP}(t) = \sum_{\tau=1}^{\tau_m} [1 + r(t-\tau)]^{\tau} \cdot L'(\tau, t-\tau; t) \tag{49}$$

$$L_{BPI}(t) = \sum_{\tau=1}^{\tau_m} \{[1 + r(t-\tau)]^{\tau} - 1\} \cdot L'(\tau, t-\tau; t) \tag{50}$$

where $L_{BP}(t)$ is the total amount of loan repayments, and $L_{BPI}(t)$ is the corresponding interest.

Similarly, loan (balloon) repayments for CB-owned loans are given by

$$L_{CBP}(t) = \sum_{\tau=1}^{\tau_m} [1 + r(t-\tau)]^{\tau} \cdot L'_{CB}(\tau, t-\tau; t) \tag{51}$$

$$L_{CBI}(t) = \sum_{\tau=1}^{\tau_m} \{[1 + r(t-\tau)]^{\tau} - 1\} \cdot L'_{CB}(\tau, t-\tau; t) \tag{52}$$

where $L_{CBP}(t)$ is the total repayment and $L_{CBI}(t)$ is the corresponding interest. The loans are paid back to the CB *in cash*, reflecting the fact that the CB may attempt to control the amount of cash reserves in the economy. Also, we assume that CG pays the CB out of current *revenues*, so does not keep an escrow account for loan repayments to the CB. This corresponds to the scenario where the CB buys government debt (here included in total CG debt), and loan repayments come from current tax revenues. Since the revenue contains a mix of cash and credit (according to the parameters), the CB cashes in the credit part, which is removed from Bank reserves. The fractions of the principal and interest to be paid back to the CB are denoted by ζ_P and ζ_I, respectively, so that the cash flow to the CB is given by

$$B_{out}(t) = \zeta_P \cdot L_{CBP}(t) + \zeta_I \cdot L_{CBI}(t). \tag{53}$$

In particular, $\zeta_P = \zeta_I = 0$ means that the CB injects cash without repayments (monetizing all of the debt it buys), and $\zeta_P = \zeta_I = 1$ means that all injected cash is eventually paid back to the CB with full interest, whereupon that cash disappears from the economy.

Distribute Bank assets (liabilities) across Savings and Escrow: Defaults on the bank assets side are subtracted from the total amount of outstanding loans, excluding those owned by the CB, given by

$$L_{tot}(t) = \sum_{\tau=1}^{\tau_m} \sum_{i=1}^{\tau} L(\tau, t-i; t). \tag{54}$$

Note that this does not change the Bank balance (assets less liabilities) since total assets are distributed across Savings and Escrow accounts as liabilities. In what follows, $\mathcal{L}(t) = S(t) + E(t)$ denotes total bank liabilities for Savings and Escrow *only*, and does not include liabilities for the credit part of Wages. Collecting those liabilities (corresponding to assets) and distributing proportionally across Savings and Escrow gives

$$\mathcal{B}(t) = S(t-1) + (1-\delta) \cdot E(t-1) \tag{55}$$

$$\mathcal{L}(t) = \mathcal{B}(t) + L_{BPI}(t) - \delta \cdot L_{tot}(t) + \delta \cdot E(t-1) + p_{CB}(t-1) \tag{56}$$

$$S'(t) = \mathcal{L}(t)\frac{S(t-1)}{\mathcal{B}(t)}, \qquad E'(t) = \mathcal{L}(t)\frac{(1-\delta) \cdot E(t-1)}{\mathcal{B}(t)} \tag{57}$$

where $\mathcal{B}(t)$ represents total liabilities (or assets) before interest payments are made (including defaults), and $p_{CB}(t)$ is the premium the CB pays to purchase loans, in addition to the loan principal, which covers the accrued interest up to that time (see (77)).

Note that these updates do not include loans owned by the CB. Any CB-owned loans that default are simply not repaid, i.e., reduce the value of B_{out} in (53), resulting in an effective gift to the CG sector. The defaults are therefore combined with government bonds, which are not repaid from revenue streams; rather the treasury prints the money and transfers it directly to the CB.

Return on investment: As in the baseline model,

$$s(t) = \frac{\mathcal{L}(t) - \mathcal{B}(t)}{\mathcal{L}(t)}, \tag{58}$$

but where $\mathcal{B}(t)$ and $\mathcal{L}(t)$ are given by (55) and (56). Because (58) is a myopic estimate, short-term fluctuations can lead to high volatility and instability, especially when the denominator (corresponding to outstanding credit) is small. In those scenarios, the trajectory of $s(t)$ can be smoothed, which can reduce volatility, or the alternative update in Appendix D can be used.

Loan rate: The update for loan interest rate is

$$r(t) = \max\left\{0, r(t-1) + k_r \frac{[D_{lf}(t-1) - S_{lf}(t-1)]}{G(t-1)}\right\} \tag{59}$$

where G_0 in (12) is replaced by $G(t)$, the total cash in the economy (bank cash plus cash flowing among and held by other sectors), which now varies with period t.

Household wealth and Savings-Consumption split: These are the same as for the baseline model (see (13)-(16)) except that the *nominal* return on savings $s(t)$ is now replaced by the *real* return on savings $\tilde{s}(t)$. Specifically, $\tilde{s}(t)$ satisfies

$$1 + \tilde{s}(t) = \frac{1 + s(t)}{\Pi(t)}, \tag{60}$$

where $\Pi(t)$ is the current *inflation multiplier* (1 + inflation rate in loans). With this substitution the savings *deposit* is then given by (17). Total revenue received by the CG sector is again given by (18) and the *cash* portions of the total amounts spent on consumption and deposited by Households are given by (19).

Inflation multiplier: The real return on savings depends on $\Pi(t)$, which is the rate of increase of loans. Similarly, the update for $D_{lf}(t)$ will depend on the rate at which CG revenue increases. In *steady state*, $\Pi(t) = Z(t+1)/Z(t)$, where Z can be taken to be *any* model flow variable (Revenue, Loans, Wages, etc). This will be the same as the rate of increase in the price of CG goods (total revenue divided by quantity purchased) and the wage rate (total wages divided by the number of households). Although these inflation rates will generally be different outside of steady state, for simplicity we will assume a single *inflation rate* as the rate at which *Revenue* (Consumption) increases. We make this choice because this variable plays a central role in estimating $D_{lf}(t)$, and because the inflation rate for loans tends to track the inflation rate for Revenue, since D_{lf} is a fixed fraction of Revenue. Because we assume a closed economy, wage and price inflation is a direct consequence of the inflation in *aggregate* Wages and Revenue, and in that sense plays a secondary role in understanding the model behavior.

For policy experiments, standard techniques can be used to obtain an estimate of $\Pi(t)$, for example, log-smoothing of $R_v(t+1)/R_v(t)$. The associated time constant then represents another source of inertia. In some scenarios, shocks combined with such an inflation estimator can cause short-term volatility and instability. For the policy experiments in Section 7, we instead take $\Pi(t)$ to be a smoothed version of the actual rate of cash injections by the CB. This implicitly assumes households and CG firms have access to that information and forecast the effects by applying a smoothing filter.

Target capital expenditures, demand for loans, Escrow deposits: As before, target capital expenditures are determined by maximizing output given labor and capital costs. For the Cobb-Douglas production function this gives $K_{tar}(t) = h \cdot R_v(t)$ (see (21)). The demand for loans $D_{lf}(t)$ is then set so that the expected capital expenditures per iteration (escrow deposits plus repayments to the CB) in steady state is $K_{tar}(t)$, assuming that $D_{lf}(t)$ is in fact the amount of loans made in steady state.

In contrast to the scenario without a CB, here we need to distinguish three types of loans: (1) CG loans with escrow accounts (as before), (2) loans owned by the CB, which are paid back out of current revenues, and (3) loans owned by the CB, which are not paid back. Letting $\Lambda(t) = L_{BP}(t) + L_{CBP}(t) + L_{CBI}(t)$ be the total amount of loan repayments to both the bank and the CB, the fractions of loan repayments in each of these categories, are, respectively,

$$a_1 = \frac{L_{BP}(t)}{\Lambda(t)} \qquad a_2 = \frac{B_{out}(t)}{\Lambda(t)} \qquad a_3 = \frac{L_{CBP}(t) + L_{CBI}(t) - B_{out}(t)}{\Lambda(t)} \tag{61}$$

For category (1), a unit loan taken out each iteration in steady state results in an escrow deposit given by

$$d_{u,1}(t) = \sum_{\tau=1}^{\tau_m} \sum_{i=1}^{\tau} w_\tau P(r, s, \tau) \Pi^{-i}(t-1) \tag{62}$$

where w_τ is the fraction of loans with duration τ, and $P(r, s, \tau)$ is the escrow payment for a unit loan of duration τ made at interest rate r and return on investment s, given by (79). For category (2), a unit loan taken out each iteration in steady state results in loan repayments to the CB given by

$$d_{u,2}(t) = \sum_{\tau=1}^{\tau_m} w_\tau [1 + r(t-\tau)]^\tau. \tag{63}$$

Loans in category (3) do not add to escrow payments, and are therefore not included in the computation of $D_{lf}(t)$.

The demand for loans is then

$$D_{lf}(t) = K_{tar}(t)/[a_1 \cdot d_{u,1}(t) + a_2 \cdot d_{u,2}(t)] \tag{64}$$

Strictly speaking, this should apply only in steady state; however, we will assume that the CG sector always has access to the corresponding amounts of loan repayments, and computes D_{lf} in the same way even outside of steady state.[34]

The Escrow deposit is computed as before for the baseline model. Namely, the target amount in Escrow and the escrow deposit are given by (24) and (25), respectively, and the cash portion of the escrow deposit is $c \cdot \Delta E(t)$.

[34] In practice, the loan amounts that appear in (61) would likely be smoothed and forecast.

Updated Savings, Escrow: Savings and Escrow are now given by

$$S(t) = S'(t) + \Delta S(t) \tag{65}$$
$$E(t) = (1 - \delta) \cdot E'(t) + \Delta E(t) - L_{BP}(t). \tag{66}$$

This is the same as the baseline model, except that defaults are now subtracted from Escrow. Since all repayments to the CB are taken from the current revenue stream, the loan repayments from Escrow, L_{BP}, do not include loans owned by the CB. In practice, the Bank would own additional government bonds for which no escrow is kept. Here we assume that escrow is kept for all Bank-owned loans, and ignore the fact that there are many loans to the combined CG-government sector for which no escrow accounts would be held.

CG wages, unspent capital budget: These must be modified from the baseline model to account for payments to the CB. Specifically,

$$W_C(t) = \min\{(1 - h) \cdot R_v(t), R_v(t) - \Delta E(t) - B_{out}(t)\}. \tag{67}$$

If the amount of capital expenditures $\Delta E(t) + B_{out}(t)$ is less than the target, then the first term is smaller, and CG retains the unspent amount

$$U_{cg}(t) = R_v(t) - \Delta E(t) - W_C(t) - B_{out}(t). \tag{68}$$

We again assume that the CG sector uses this to purchase additional capital directly, so that this amount is added to capital wages (see (38)). In steady state $U_{cg}(t) = 0$.

Bank cash: As in the baseline model, the updated Bank cash accounts for loans made at the preceding iteration, savings withdrawals, and Escrow deposits,

$$B_c'(t) = [B_c(t-1) - c_k L(t-1) + \Delta S_c(t) + \Delta E_c(t)]^+ . \tag{69}$$

This assumes that none of the cash reserves are needed to repay CB-owned loans. That is, there is enough revenue to cover those repayments, or if not, then the CB allows those loans to default. The new intermediate variable B_c' is needed since the updated B_c will include cash from the CB.

CB injections, supply of loans: The CB is assumed to inject cash into the economy at a fixed rate by purchasing a constant fraction of CG debt each iteration. The amount injected, monetary base, and Bank cash are given by

$$F(t) = \kappa \cdot G(t-1), \qquad G(t) = G(t-1) + F(t) \tag{70}$$
$$B_c(t) = B_c'(t) + F(t). \tag{71}$$

As before (without CB injections), the supply of loans is then given by (35). This is a myopic estimate, which does not take into account side information the Bank may have. We can again include information about the existing loan array to obtain the more forward-looking alternative determination of S_{lf} (36). Here, however, the amount of total outstanding loans, which contribute to total Bank assets, is the sum of all loans in the CG loan array (excluding CB-owned loans), and in steady-state is $\Gamma'(t)L(t)$, where now

$$\Gamma'(t) = \sum_{\tau=1}^{\tau_m-1} \sum_{i=1}^{\tau} w_i[1-\delta(t)]^i \Pi^{-i}(t)[1-\eta(t)]^i, \tag{72}$$

and where $\eta(t)$ is the fraction of loans bought by the CB (see (74)). That is, total loans are reduced by the default rate across the loan array, the inflation rate, and the fraction of loans transferred to the CB. The same derivation then again gives (36), where $\Gamma'(t)$ is given by (72).

Fraction of loans bought by CB: For simplicity we assume that the CB purchases a fixed fraction of *all* CG loans that remain after defaults. Let $\pi(\tau, t-i; t)$ denote the amount paid for loans $L'(\tau, t-i; t)$. We determine $\pi(\tau, t-i; t)$ by setting the amount accumulated for the remainder of the loan duration $\tau - i$ at interest $r(t)$ equal to the balloon payment of the original loan:

$$[1 + r(t)]^{\tau-i}\pi(\tau, t-i; t) = [1 + r(t-i)]^{\tau} L'(\tau, t-i; t) \tag{73}$$

for $i < \tau$. We assume that the CB only purchases loans that expire after time t. Given that the CB wishes to spend a total of $F(t)$, the fraction of loans it buys is then

$$\eta(t) = F(t) / \left[\sum_{i=1}^{\tau_m-1} \sum_{\tau=i+1}^{\tau_m} \pi(\tau, t-i; t) \right] \tag{74}$$

where $\pi(\tau, t-i; t)$ is given by (73). (We assume that $F(t)$ never exceeds the entire amount of CG loans.) The new loans purchased by the CB are then transferred from the CG loan array to the CB loan array:

$$L(\tau, t-i; t) = [1-\eta(t)]L'(\tau, t-i; t) \tag{75}$$

$$L_{CB}(\tau, t-i; t) = L'_{CB}(\tau, t-i; t) + \eta(t) \cdot L'(\tau, t-i; t) \tag{76}$$

for $i = 1, \cdots, \tau_m - 1$ and $\tau = i+1, \cdots, \tau_m$.

The difference $\pi(\tau, t-i; t) - L'(\tau, t-i; t)$ is the *premium* the CB pays for purchasing the loans corresponding to $L'(\tau, t-i; t)$. Hence the total premium amount the CB pays at iteration t is

$$p_{CB}(t) = \eta(t) \sum_{i=1}^{\tau_m-1} \sum_{\tau=i+1}^{\tau_m} [\pi(\tau, t-i; t) - L'(\tau, t-i; t)]. \tag{77}$$

This is included as part of Bank liabilities in (56).

Loans, Capital wages: These are the same as for the baseline model, given by (37) and (38). The cash portion of total wages is again (39).

Measurements: Measurement variables such as GDP, the proportion of labor allocated to CG and K, and unemployment depend on the same state variables as in the baseline model, and hence can be computed in the same way, but with the following modifications. First, N_L in (40) is replaced $N_L(1-\delta)$, reflecting the assumption that a loan default means that the corresponding CG firm is no longer solvent, and that the fraction of loan defaults cause the same fraction of unemployment. Second, accounting for inflation, the downward wage friction (41) becomes

$$w(t) = \max\left\{w_{tar}(t), \Pi(t-1)\cdot w(t-1) + k_L \cdot [w_{tar}(t) - \Pi(t-1)\cdot w(t-1)]\right\}. \tag{78}$$

Measures of GDP can be normalized by the cash supply $G(t)$ to account for inflation.

7 Policy Experiments and Shocks

We now show a series of plots that illustrate some basic features of credit expansion and contraction associated with the model. We begin by discussing the choice of system parameters used to generate the plots, and subsequently show four sets of plots that are organized according to the main results discussed in Section 2.

7.1 Model Parameters

Our intent in this section is to show a representative set of experiments that illustrate some basic properties exhibited by the model. While there is a range of parameters that could be selected, we have chosen parameters that may correspond to a "reasonable" set of state variables (e.g., a loan interest rate in the range of 0.5 to 1% per period). Unless stated otherwise, the plots assume the parameters shown in Table 1. The parameters above the double line are sufficient to determine the equilibrium and steady-state values, whereas the parameters below the double line affect dynamics (transients). Changing the latter set of parameters can speed up or slow down transients, and mute or enhance transient swings in state variables due to shocks. As previously discussed, selection of those parameters has a critical effect on the duration of transients and stability. However, in practice the dominant influence on transient dynamics would likely be driven by traders in the financial sector, not modeled here. As for the Baseline model, the duration of a single time period is linked to the interpretation of loan durations. If we assume a maximum loan duration of five years (after which the loans are repaid, refinanced, or traded), then taking $\tau_m = 60$ means that each period represents approximately one month.

7.2 Transient and Steady-State Effects of Central Bank Injections

Fig. 7 shows the effects of injections by the CB. The CB injects money by buying outstanding CG sector debt from the Bank sector at period 100 after the system reaches an initial equilibrium. Thereafter, each period the amount of loans purchased by the CB is equal to 0.5% of the current monetary base. In this set of plots, the CB does not require repayments from the CG sector on the loans it purchases, so that the cash it injects adds to the existing monetary base and never leaves the economy. (The next figure illustrates what happens when the loans are paid back to the CB.) As a consequence, as shown in the first (upper

$f = 10\%$	fractional reserve requirement
$e = 500$	propensity to save
$\tau_m = 60$	maximum loan duration
$w_\tau = 1/\tau_m, \tau = 1, \cdots, \tau_m$	loan window distribution
$c = c_k = 0.1$	cash fraction of loans, consumption (mostly-credit)
$\beta_K = 0.5$, $\beta_L = 0.5$	return on capital, return on labor
$\kappa = 0.005$	CB injection rate (fraction of monetary base) (71)
$k_r = 1/1000$	step-size for loan rate
$k_s = 1/10$	step-size for smoothing return on investment
$k_C = 1/10$	step-size for smoothing consumption
$k_L = 1/4$	step-size for downward wage friction

Table 2: Default parameters used to generate the results in Section 7.

left) plot, both the amount of cash injected per period (amount shown on the left) and the monetary base $G(t)$ (amount shown on the right) increase exponentially.

The top right plot shows both nominal and real loan interest rates. The real interest rate is the nominal rate minus the estimated inflation rate, taken to be $\Pi(t) - 1$ where $\Pi(t) = G(t)/G(t-1)$. The CB injections create a transient in which the nominal rate *increases* and the real interest rate *decreases* relative to the initial equilibrium rate. (The initial drop in the real interest rate is due to the sudden onset of inflation.) The system reaches a steady-state in which the nominal and real interest rates each converge to their final values, different from the preceding equilibrium value. Hence this figure indicates that the CB injections have a real effect on the economy even in steady-state. This is further illustrated in the lower left plot, which shows the allocation of total labor capacity to the K and CG sectors. The decrease in the real interest rate makes capital less expensive, so the CG demand for capital increases. That increases the flow of total wages into the K sector relative to the flow of wages going to the CG sector, which causes laborers to shift from the CG sector to the K sector. This shift in turn causes a slight increase in production, as shown in the lower right plot. The increase in K production also shown in the figure is more pronounced than that for CG, since it is more directly influenced by the labor shift. The relative insensitivity of CG production shown in this example (and the other examples in this section) is partially due to the choice of parameters,[35] and is also due to the restrictive nature of the assumed Cobb-Douglas production function with constant returns to scale. (Nominal GDP shows much larger variations.)

The rise in nominal rate shown in the top right plot is due to inflation expectations. That is, demand for loanable funds $D_{lf}(t)$ accounts for the anticipated rise in revenues due to the CB injections. (Recall that $D_{lf}(t)$ is determined by the CG sector so that capital expenditures per iteration (escrow deposits) are a constant pre-determined fraction of revenue.) However, the increase in nominal rate from the preceding equilibrium value is *less than* the inflation rate, causing a drop in the real interest rate. In other words, if the real interest rate were held constant, then when injections commence, the supply of loanable funds $S_{lf}(t)$ would exceed $D_{lf}(t)$. The reason for this is that the CB injections effectively discount the demand-deposit obligations held as liabilities by the Bank, providing more room

[35]Decreasing the propensity to save e would accentuate the shift in labor-capital split, but would also significantly increase the interest rate r.

for the Bank to expand credit.

To illustrate the real change in interest rate with a simple example, suppose that $\tau_m = 2$, and that without CB injections the loan amount per period is L, split evenly between loan durations $\tau = 1$ and $\tau = 2$. At time t, loans initiated at time $t-1$ with $\tau = 1$ and loans initiated at time $t-2$ with $\tau = 2$ are repaid, so are not counted as liabilities when determining loans at time t. The total loan liabilities in equilibrium at time t are then $L/2$ from period $t-1$, corresponding to $\tau = 2$. To determine $S_{lf}(t)$ those liabilities are subtracted from the total amount of credit ξ the Bank can generate at time t, which is determined by the reserve requirement (see (35)). Hence in equilibrium the ratio of total loan liabilities to total credit generated is $L/(2\xi)$. Now suppose that the CB injects cash at a rate of $\Pi - 1$ per period. Then as t increases, in steady-state the loans per period and credit increase as $L(t) = L(t_0) \cdot \Pi^{t-t_0}$ and $\xi(t) = \xi(t_0) \cdot \Pi^{t-t_0}$, respectively, where $t > t_0$. At time t the total loan liabilities are then $L(t-1)/2 = L(t)/(2\Pi)$, and the ratio of total loan liabilities to total credit is $L(t)/[2\Pi\xi(t)] < L(t)/[2\xi(t)]$ when $\Pi > 1$. Hence the effect of the injections is to decrease the ratio of total loan liabilities to total outstanding credit, thereby increasing $S_{lf}(t)$ relative to $D_{lf}(t)$, and decreasing the real interest rate. This effect is accentuated as the duration of loans increases (*i.e.*, as τ_m increases).

We point out that an important assumption in this example is that the Bank *maximizes* its loans subject to the reserve requirement. The increase in $S_{lf}(t)$ due to injections therefore has a direct effect in increasing the loan amount. In practice, loan amounts will also depend on perceived risk of default, with the loans made at the margin incurring the largest risk. A steady-state influx of cash would therefore not necessarily induce the Bank to make more loans (relative to the monetary base) if it has already selected the loan target to optimize its risk-return tradeoff. Although that would seem to mute the real effects seen here, in that scenario the excess cash reserves accumulated by the Bank (above the legal limit) would in steady-state be passed along to shareholders in the Household sector and subsequently circulate in the economy, again causing a real effect. (Injections through the Household sector are illustrated in a subsequent example.) Finally, there may be reasons why an increase in S_{lf} would induce the Bank to increase loans even taking risk into account. One is deposit insurance, which may cause the Bank to discount the probability of defaults and bank runs, and another is the time constant needed to quantify the risk-return tradeoff over the loan portfolio once the injections commence.

Fig. 8 is analogous to Fig. 7, the difference being that all of the debt purchased by the CB is repaid with interest to the CB. The first plot shows that the CB again begins to buy debt at period 100 after the system reaches an initial equilibrium. The monetary base $G(t)$ then increases initially, but once the purchased loans come to term, $G(t)$ starts to fall since the interest on each of the loans purchased by the CB, which leaves the economy, must exceed the premium paid by the CB. Hence there is a net flow of money out of the economy and into the CB. The cash injected per period also falls, since each period the CB injects 0.5% of the decreasing monetary base.

The top right plot shows nominal and real loan interest rates. Here the effects of the CB injections are the opposite of what is shown in Fig. 7: the steady state nominal rate *decreases* and the real interest rate *increases* relative to the initial equilibrium rate. The decrease in nominal rate is now due to *de*flation expectations (as opposed to inflation expectations in Fig. 7) when computing $D_{lf}(t)$. Here the outstanding nominal debt shrinks as t increases, so that interest payments corresponding to prior loans issued at $t - t_0$ increase with t_0. Bank liabilities relative to the total credit generated then increase with t, which decreases $S_{lf}(t)$,

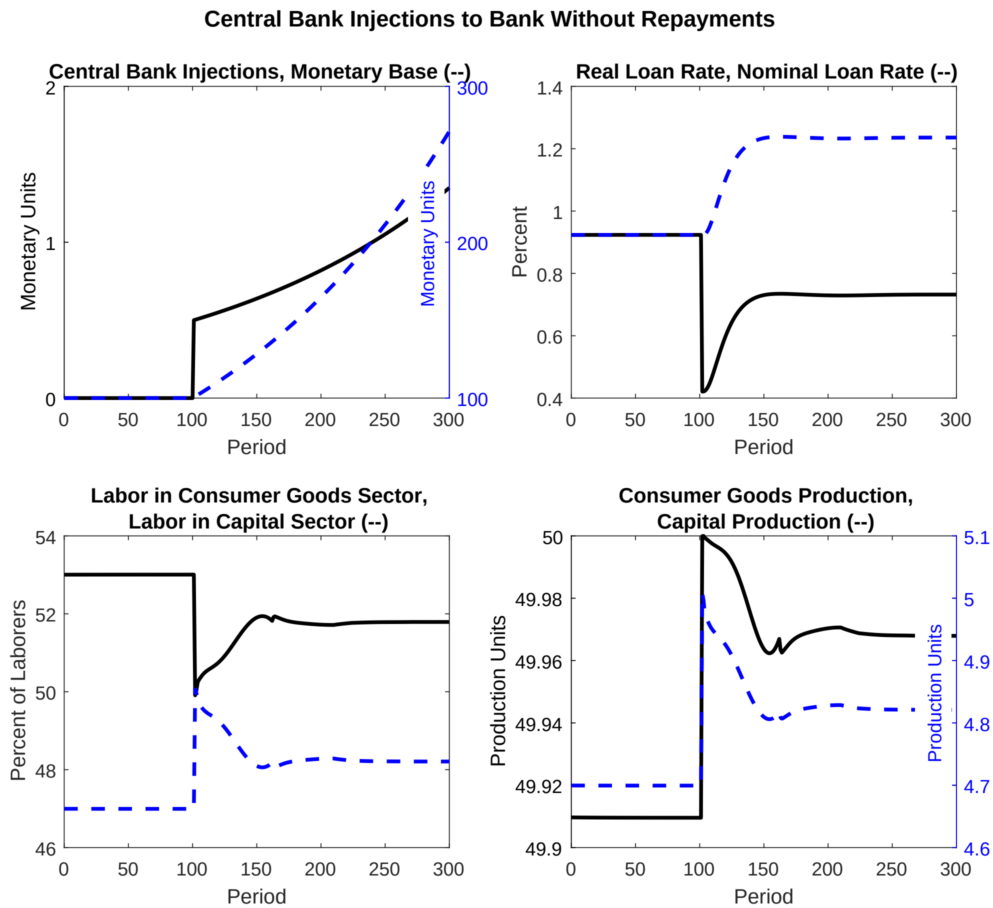

Figure 7: Transient and steady-state effects of CB injections with no repayments. Quantities are shown as a function of the period t. The CB begins buying CG debt at time $t = 100$ after the system has reached an initial equilibrium. The top left plot shows the monetary base (solid) and cash injections (dashed). The top right plot shows that the CB injections cause the real interest rate to decrease, which in turn affects the allocation of labor to the CG and K sectors (lower left) and production of consumer goods and capital (lower right). In each of the plots in this section, the y-axis for the solid (dashed) curve is shown on the left (right).

causing the real interest rate to rise. The difference between the nominal and real interest rate is again the estimated inflation rate $\Pi(t) - 1$, which approximately follows the slope of $G(t)$. Hence the nominal and real interest rates cross when $G(t)$ peaks. As in Fig. 7, the initial downward spike in real interest rate is due to the sudden jump in inflation.

The lower left plot shows $S_{lf}(t)$ and $D_{lf}(t)$, which determine the transient variations in interest rate. That is, when the CB begins injecting, $D_{lf}(t)$ shoots up due to inflation expectations, and exceeds $S_{lf}(t)$ so that the nominal interest rate rises (dashed blue curve). Subsequently $D_{lf}(t)$ falls, as inflation decreases, and crosses $S_{lf}(t)$ at which point the nominal interest rate decreases. The transient variations in $D_{lf}(t)$ are due to the changing inflation and also changes in the interest rate and return on savings. The bottom right plot shows the corresponding shift in labor/capital split. After the complicated transient, again due to the interaction between S_{lf} and D_{lf}, which determines loans, the (small) increase in the

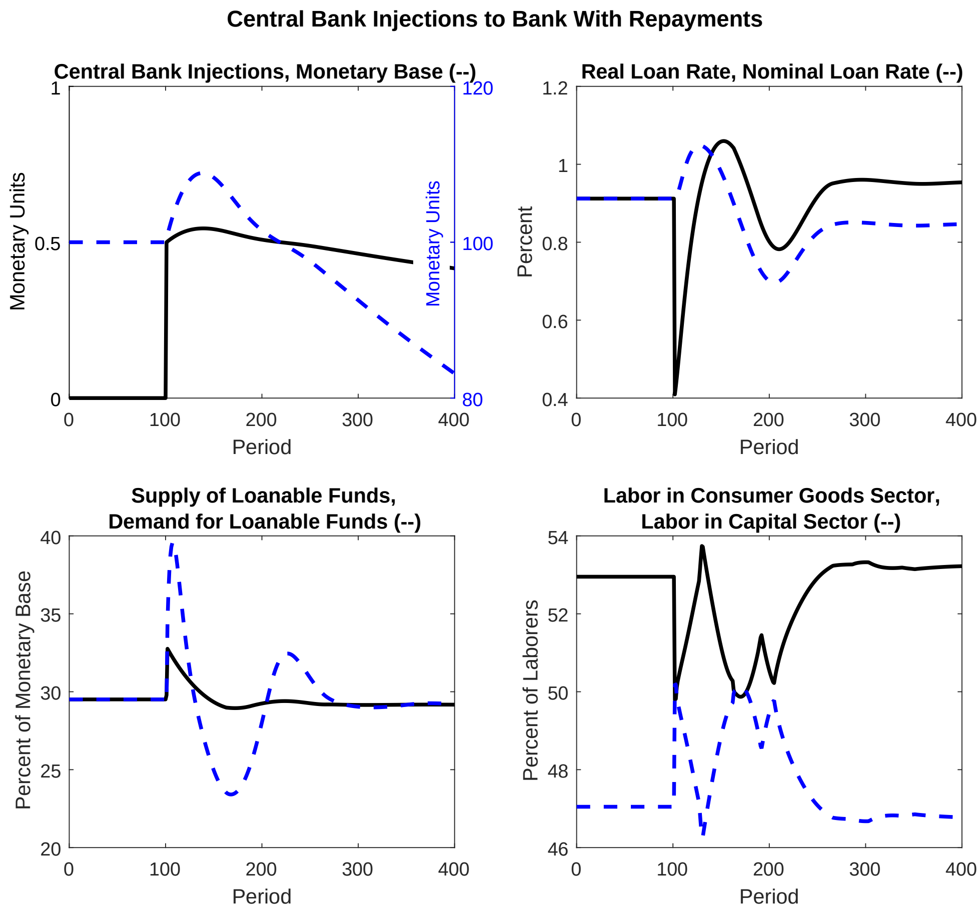

Figure 8: Transient and steady-state effects of CB injections with full repayments. Quantities are shown as a function of the period t. The CB begins buying CG debt at time $t = 100$ after the system has reached an initial equilibrium. The top left plot shows the monetary base (solid) and cash injections (dashed). The monetary base decreases in steady-state since the cash outflow including interest payments to the CB exceeds the cash inflow needed to purchase the CG debt. The top right plot shows that the CB injections cause the real interest rate to *in*crease in steady state. The lower left plot shows $S_{lf}(t)$ and $D_{lf}(t)$, which drive the interest rate dynamics along with the transient behavior of the labor/capital split shown in the lower right.

real interest rate makes capital somewhat more expensive, which decreases the flow into the Capital sector relative to total wages, causing some labor to switch from the Capital sector to the CG sector. (Again, the shifts in labor are slight for this parameter set due to the relatively small shift in interest rate.)

7.3 Credit Expansion and Contraction

Figs. 9-10 show the effects of credit expansion and contraction. In Fig. 9 the Bank initially expands and subsequently contracts credit by changing the credit multiplier $m = 1/f$. Specifically, in the first (upper left) plot m (solid curve) increases from 10 to 20 (f decreases from 10% to 5%) starting at period 100 after the system has reached an initial equilibrium.

After the system reaches a new equilibrium with expanded credit, m is subsequently reduced from 20 back to 10. Also shown is total Household money, or wealth (dashed), given by (13). The changes in credit are reflected in the nominal wealth, which respectively increases and decreases as credit expands and contracts. (The effect on real wealth, normalized by CG price, is related to the change in actual production.) The top right plot shows that the expansion of credit lowers the loan rate to a new equilibrium value, and the subsequent contraction increases the loan rate back to its initial equilibrium value. The overshoots can be muted by decreasing the step-sizes in Table 2, but then the durations of the transients increase.

The lower left plot in Fig. 9 shows total sector wages W_C and W_K given by (27) and (38). The credit expansion and contraction respectively increases and decreases total wages across CG and K sectors. The decrease in total wages associated with the credit contraction causes unemployment, shown as the dotted curve. Specifically, the fall in total wages creates a large spike in unemployment, which subsides once the system returns to the original equilibrium. We emphasize that because wages fall across *both* the CG and K sectors, unemployment is spread across the *entire* economy (*i.e.*, over both CG and K sectors). The plots also indicate that capital wages fall first, due to the contraction in loans, causing unemployment in the K sector before spreading to the CG sector. This causes the transient shifts in labor allocations shown in the lower right plots. The contraction causes a net downward spike in labor allocations, and an associated loss in productivity during the periods of unemployment.

The severity of the transients shown in this figure again depends on the choice of step-sizes in Table 2. In particular, the overshoot in wages due to the expansion shown in the lower left plot causes mild unemployment when the wages adjust downward. There is also a transient shift in which the allocation of labor across CG and K reverses (causing a mild transient decrease in production, not shown). While changing the step-size and unemployment parameters can mute these transients, the parameter changes do not affect the drop in wages across both sectors during the contraction and the associated loss in productivity.

Fig. 10 shows plots analogous to Fig. 9, but where the credit expansion or contraction is due to increasing or decreasing CB injections. The top left plot shows the CB injections (solid) and the monetary base (dashed). Injections of 0.5% of the monetary base per period are introduced at period 50, and subsequently terminated at period 350. The monetary base therefore increases exponentially during the steady-state injections and then levels out at a constant higher value when the injections stop. The top right plots show the real and nominal interest rates. The real interest rate decreases to a new steady-state value caused by the injections, and then increases back to its original (pre-injection) value. The bottom left plots show total CG (solid) and K (dashed) wages, along with unemployment (dotted). As in the previous figure, the credit expansion due to the injections increases nominal wages, which fall to their original level after the injections stop. The decrease in total wages combined with downward wage friction causes a spike in unemployment during the contraction, which is again across both the CG and K sectors. The lower right plot shows the labor-capital split. The injections cause labor to move from CG to K, corresponding to a drop in the real interest rate, which is reversed when the injections stop.

Both Figs. 9 and 10 show that the expansions and contractions are asymmetric. The contractions tend to be somewhat more volatile, even in the absence of downward wage friction. This is because when credit expands, loans are determined by D_{lf}, which increases to match $S_{lf}(t)$, and when credit contracts, loans are determined by the sudden drop in

$S_{lf}(t)$. Comparing Fig. 10 with Fig. 9, the CB injections in Fig. 10 cause a larger shift in the real interest rate than that shown in Fig. 9. Hence the CB injections have a somewhat more pronounced effect on the labor-capital split. (We show in a subsequent plot, Fig. 14, that increasing the expansion beyond $m = 20$ in Fig. 9 does not significantly increase the drop in interest rate.) In contrast, the unemployment spike during the contraction is *less* pronounced in Fig. 10 than in Fig. 9. That is because during the contraction, wages in Fig. 9 fall by approximately 50%, whereas in Fig. 10, wages fall about 10%. The increase and decrease in wages in Fig. 9 correspond directly to the increase and decrease in the credit multiplier m, whereas the increase and decrease in wages in Fig. 10 appear subdued since wages are normalized by the monetary base, which increases during the expansion, and levels out during the contraction. Again we emphasize that the particular dynamics shown are affected by the particular choice of system parameters.[36]

We next contrast the preceding credit expansion and contraction examples with Fig. 11, which shows the effect of an output elasticity shock. The top left plots show that the labor elasticity β_L (solid) is raised from 0.5 to 0.6 starting at period 100, which means β_K (dashed) is lowered from 0.5 to 0.4. Hence the return on capital decreases, the demand for loans decreases relative to the supply, the loan rate decreases, as shown in the upper right plot, which in turn shifts the labor-capital split, as shown in the lower right plot. That is, labor shifts from the K sector to the CG sector. (The labor-capital split which maximizes output becomes 60% labor / 40% capital instead of 50-50.) The lower left plot again shows wages and unemployment. In contrast to Figs. 9 and 10, here the fall in wages occurs entirely in the *CG sector* (ignoring the small overshoot in K wages after the shock). Hence unemployment is confined to the CG sector and does *not* occur in the K sector, in contrast with the preceding two figures. This example then serves to emphasize the important distinction between the effects of a credit contraction, which spread from the K sector (by reducing loans) to the CG sector, and the effects of other parameter shocks, which may have different qualitative effects on the different sectors in the economy.

7.4 Loan Defaults

We now illustrate the effects of loan defaults in Figs. 12-13. In both figures defaults are raised exogenously to 0.5% of all loans from an initial equilibrium with no defaults. The defaults then persist for the remainder of the example. Once the system reaches an equilibrium with defaults, a credit contraction is introduced. Specifically, the top left plots in Fig. 12 show that defaults (dashed) are introduced at period 100, and that the credit multiplier m (solid) is initialized at 15, and is reduced to 10 at period 400. This is motivated by a scenario in which a large credit multiplier causes the Bank sector to make excessively risky loans, eventually increasing the fraction of loans that default in equilibrium. In response to the defaults, the Bank subsequently increases the reserve rate, causing a contraction in credit. In practice, the credit contraction would likely occur along with the defaults, whereas here we separate these events. This is to illustrate the different effects caused by these events, which would be difficult to observe if the events overlapped.

The top right plots in Fig. 12 show the loan rate r (solid, y-axis on left), and total loans per period (dashed, y-axis on right). It is perhaps surprising that the introduction of defaults

[36] In particular, increasing the step-sizes in Table 2 can produce instabilities with long-term oscillations and crashes in which the CG sector debt exceeds its revenues.

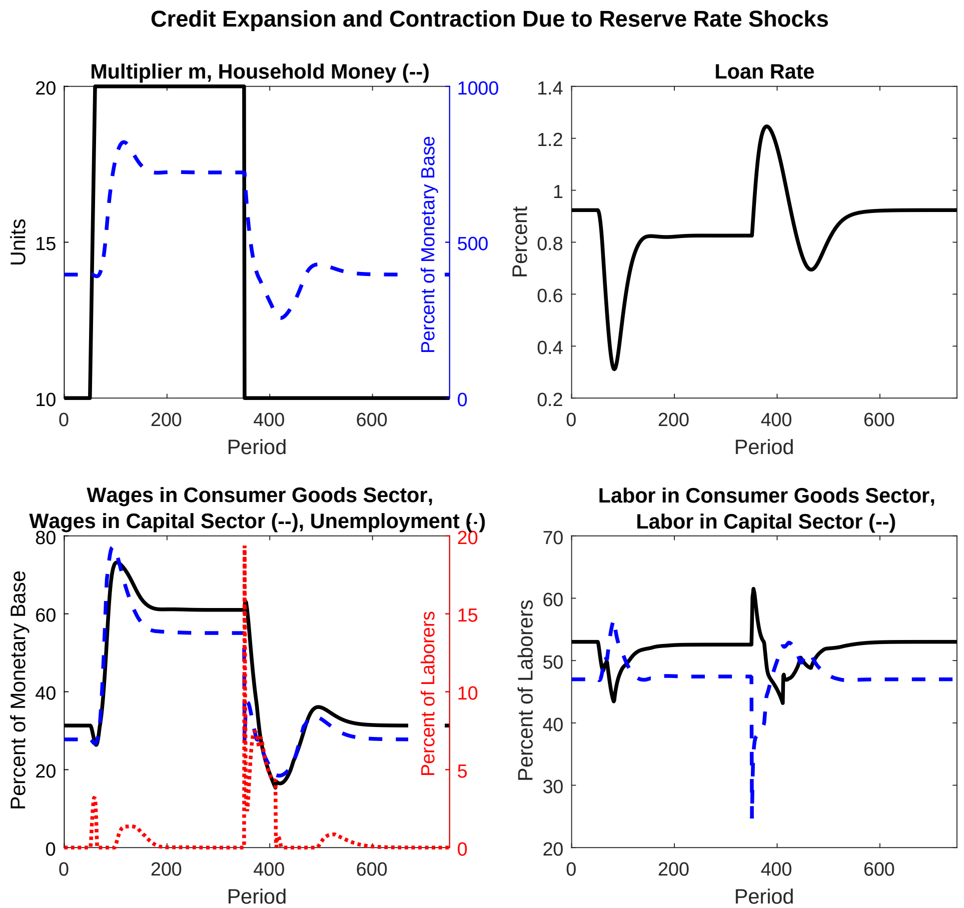

Figure 9: Dynamic effects of Bank credit expansion and contraction. The top left plots show the credit multiplier $m = 1/f$ (solid) and total Household money, or wealth (dashed). The multiplier m increases from 10 to 20 at period 50, and decreases from 20 to 10 at period 350. The top right plot shows the loan rate r, which decreases to a new equilibrium after m is increased, and increases back to its original value when m is decreased. The lower left plots show total wages W_C (solid) and W_K (dashed), and also unemployment (dotted, y-axis on right). Unemployment is caused by a decrease in total wages. The lower right plot shows the fraction of total labor allocated to the Consumer Goods and Capital sectors (labor-capital split). Unemployment causes the downward spike in total labor utilization.

causes loans per period to *increase*. The reason for this is that when a loan defaults, and is deleted from Bank assets, the Bank also deletes the equivalent amount of savings liabilities. Those deleted liabilities are no longer subtracted from the total credit the Bank can generate when determining $S_{lf}(t)$ in (35). In other words, the Bank increases its loans to make up for the defaulting loans, so that it achieves its target credit expansion m. The return on investment s decreases, however, because the defaulted loans are not repaid in full. (Recall that the defaulted loans are partially repaid since the Bank seizes whatever has accumulated in Escrow and passes that on to the Households.) The decrease in s increases consumption relative to savings, which in turn increases CG revenue and therefore DLF. For the example shown, the initial increase in DLF due to defaults exceeds the increase in SLF, causing the loan rate r to rise. Of course, in practice, the Bank would not attempt to maintain the same

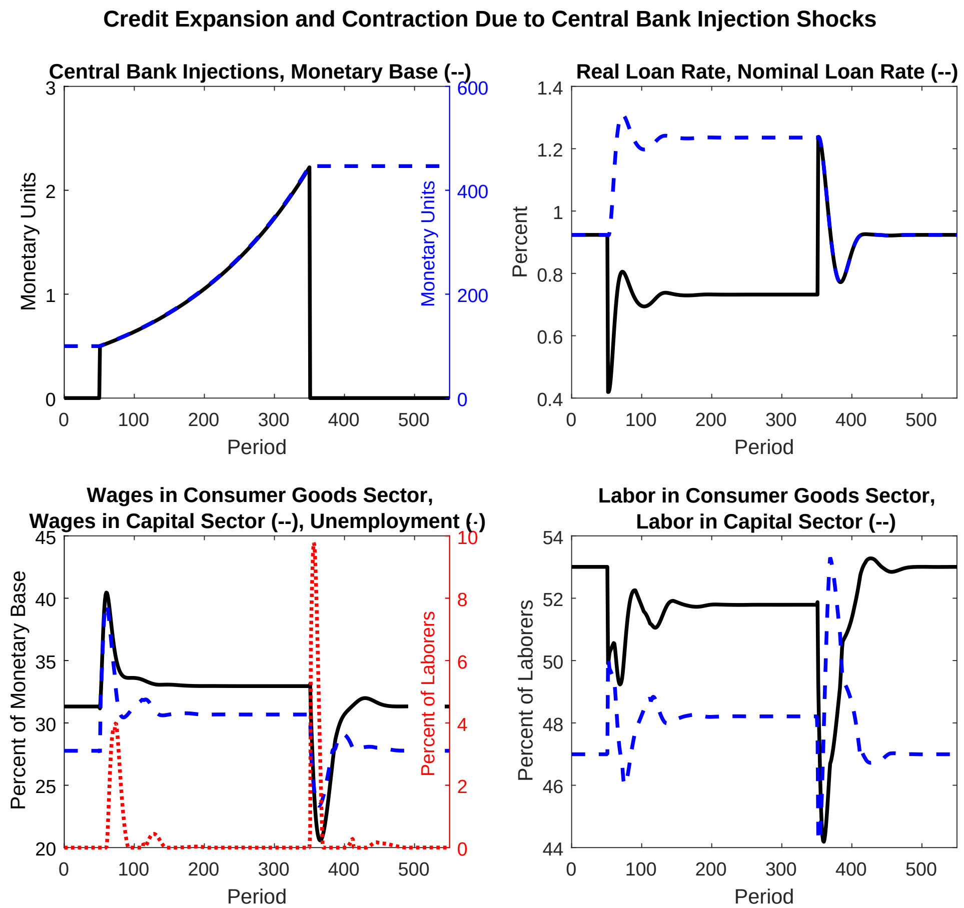

Figure 10: Dynamic effects of credit expansion and contraction due to CB injections. The top left plots show the CB injections (solid) and monetary base (dashed). Injections of 0.5% per period start at period 50, and then stop at period 350 when the system has reached a steady-state. The top right plot shows the real (solid) and nominal (dashed) loan rate r. The real loan rate decreases to a new equilibrium after the CB injects, and increases back to its original value when the CB stops injecting. The lower left plots show wages W_C (solid) and W_K (dashed), normalized by the monetary base, and also unemployment (dotted, y-axis on right). Unemployment is caused by a decrease in wages. The lower right plot shows the fraction of total labor allocated to the Consumer Goods and Capital sectors (labor-capital split). Unemployment causes the downward spike in total labor utilization.

target credit expansion m at the onset of defaults. Hence m is subsequently reduced causing the interest rate to rise, and loans per period to fall.

The bottom left plots show the labor-capital split (solid and dashed), and unemployment (dotted, y-axis on the right). The defaults introduce 0.5% unemployment (corresponding to the default rate), and shift the labor-capital split towards more capital due to the increase in loans. The bottom right plots show that the defaults cause a slight increase in the production of capital (dashed), but a decrease in the production of CG (solid). The subsequent credit contraction then causes a large spike in unemployment and downward spike in both CG and capital production. These results indicate that while the defaults themselves cause a moderate increase in unemployment and reduction in return to savings, the subsequent

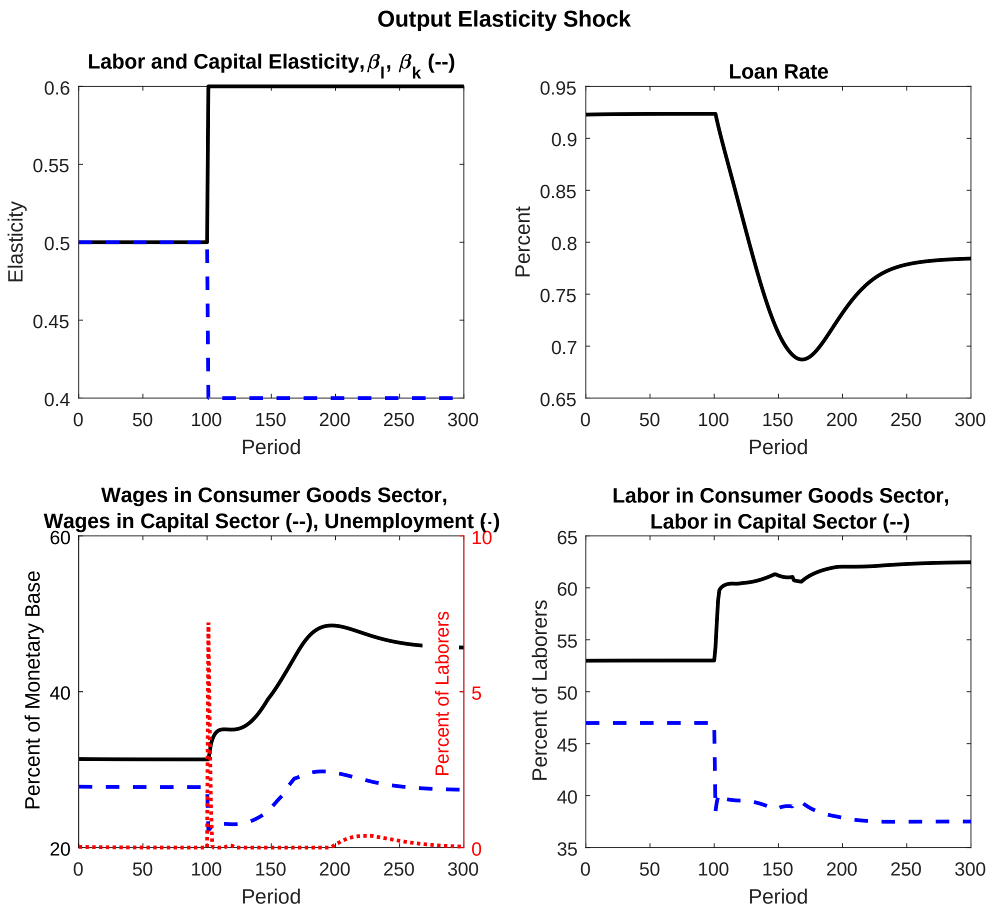

Figure 11: Effects of an output elasticity shock. The top left plots show output elasticities β_L (solid) and β_K (dashed), which are raised and lowered, respectively, starting at period 50. The upper right plot shows that the loan rate decreases to a new equilibrium value, and the lower right plots show the corresponding shift in the labor-capital split. The lower left plots show wages W_C (solid), W_K (dashed), and unemployment (dotted). Here the unemployment is caused by the decrease in W_K, and is therefore confined to the K sector.

credit contraction triggered by the defaults causes a much more significant disruption to productivity. Further experiments indicate that for this set of parameters the dynamics are insensitive to variations in the fractions of cash in the economy c, c_k.

Fig. 13 shows results analogous to Fig. 12, but with CB injections. The top left plots show that the CB injects 0.5% of the monetary base per period (solid), and that defaults are introduced at period 100 (dashed) after the system has reached an initial steady-state. This is motivated by a scenario in which the injections cause the Bank to increase the risk associated with its loan portfolio, leading to an increase in default rate (0.5% per period). The increase in default rate subsequently causes the Bank to reduce the credit multiplier m (dotted) from 15 to 10 starting at period 600. As in Fig. 12, we allow the system to converge to a new equilibrium with defaults before the credit contraction is introduced to highlight the associated effects.

The top right plots show the real loan rate r (solid, y-axis on left), and total loans per period (dashed, y-axis on right). Again the introduction of defaults causes loans per pe-

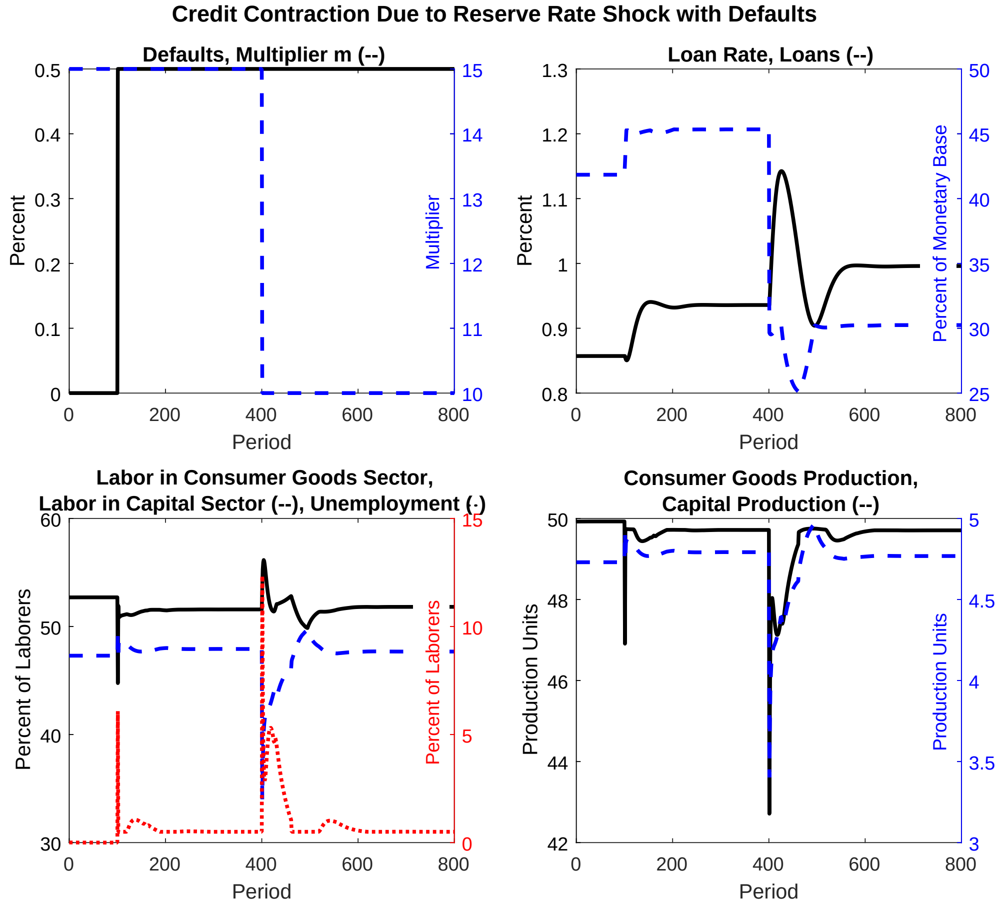

Figure 12: Effect of defaults and subsequent credit contraction. The top left plots show the default rate (dashed) and the credit multiplier m (solid). Defaults increase to 0.5% per period, and m is subsequently reduced from 15 to 10. The top right plots show the loan rate (solid) and loans per period (dashed). Due to the target credit expansion, defaults alone cause loans to *increase*. The bottom left plots show the labor-capital split (solid, dashed), and unemployment (dotted). Defaults cause 0.5% unemployment and the contraction causes the corresponding spike in unemployment. The bottom right plots show production of capital and consumer goods. Defaults cause capital production to increase slightly, but production of consumer goods to decrease.

riod to increase. The subsequent credit contraction then causes loans to fall by about 30% (similar to Fig. 12), and the interest rate rises accordingly. The effects on labor-capital split and unemployment (lower left) and production of consumer goods and capital (lower right) are qualitatively the same as in Fig. 12. Namely, the defaults again introduce 0.5% unemployment (corresponding to the default rate), and shift the labor-capital split towards more capital due to the increase in loans. This causes a slight increase in the production of capital, but a decrease in the production of CG. The subsequent credit contraction then causes a large spike in unemployment and downward spike in both CG and capital production.

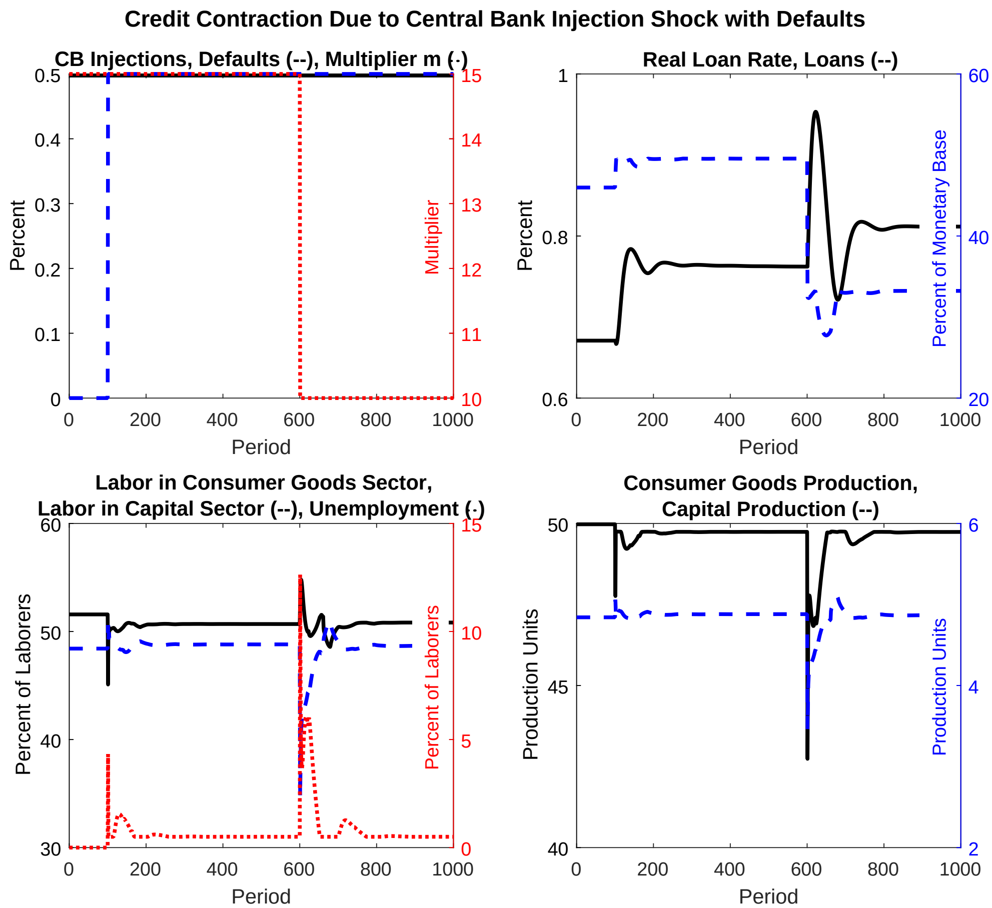

Figure 13: Effect of defaults and subsequent credit contraction with CB injections. The top left plots show the injection rate (solid), default rate (dashed), and the credit multiplier m (solid). Defaults increase to 0.5% per period, and m is subsequently reduced from 15 to 10. The top right plots show the real loan rate (solid) and loans per period (dashed), the bottom left plots show the labor-capital split (solid, dashed), and unemployment (dotted), and the bottom right plots show production of capital and consumer goods. The effects shown here with injections are analogous to those shown in Fig. 12.

7.5 CB Injections Through the Bank versus Household Sectors

The final set of plots compare the effects of CB injections through the Bank sector, as assumed in the previous section and experiments, versus injections (cash) given directly to the H sector. We first show in Fig. 14 how the equilibrium is affected by the credit multiplier m. We will contrast this with credit expansion through the CB in subsequent plots. As m increases, the Bank sector is able to make more loans with the same cash reserves. As in previous experiments, transactions are made mostly by exchanging credit (since $c = c_k = 0.1$), so almost all of the cash in the economy remains in bank reserves, making them roughly constant. As a result, the supply of loans, and therefore actual loans increase nearly linearly in the upper left plot. The upper right plot indicates that the loan rate decreases to a positive lower bound as m becomes large, even though loans continue to increase. The reason is that the additional credit issued by the bank circulates through the

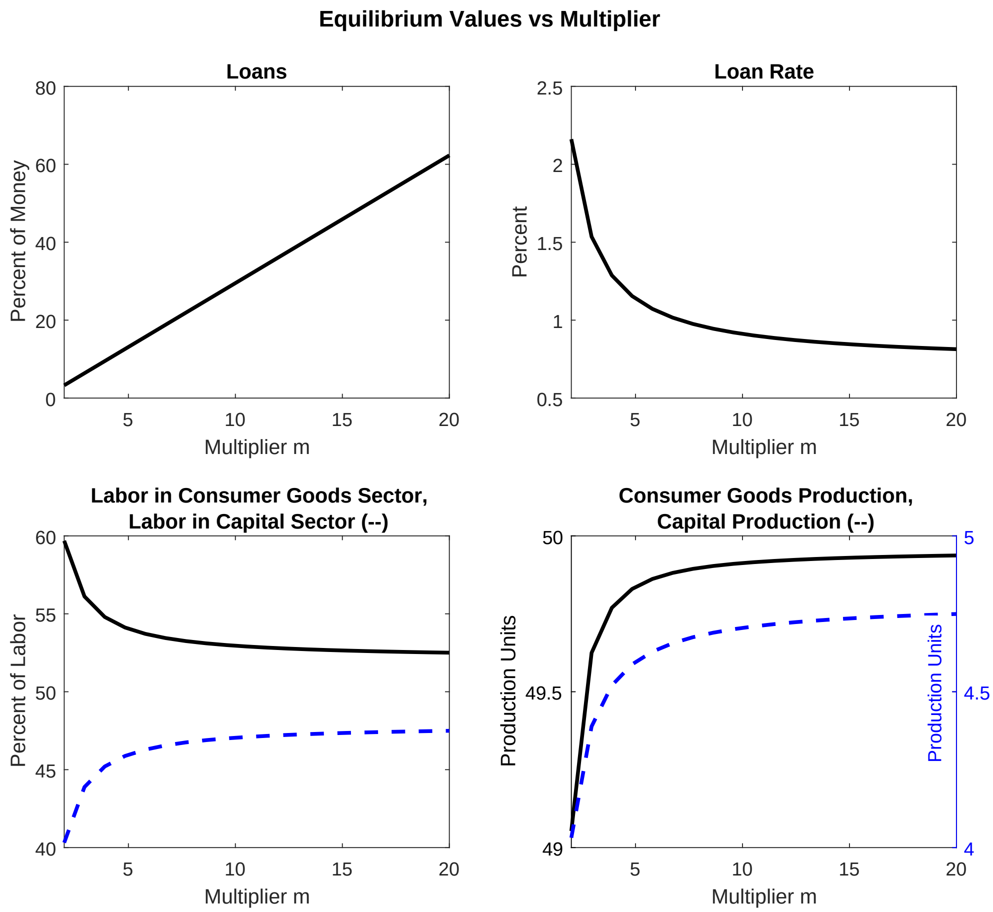

Figure 14: Equilibrium values versus credit multiplier m. The top left plot shows that loans increase with m (almost linearly for small m). The top right plot shows that the interest rate decreases with m and converges to a positive value ($> 0.5\%$) as $m \to \infty$. Hence the labor-capital split (lower left) does not converge to half-half, which would maximize production (lower right), as m becomes large.

economy to capital wages and CG revenue, which increases the demand for loans, diminishing the gap between supply and demand for loans as m becomes large. The increase in loans leads to a shift of labor from CG to the K sector, pushing the output closer to the maximum, as shown in the lower left and right plots.

Fig. 15 shows steady-state values versus the rate of CB injections introduced through the Bank sector. In the upper left plot, loans increase with these injections due to two effects: (1) the increase in cash reserves at the Bank, and (2) the devaluation of bank liabilities due to inflation, as discussed for Fig. 7. These effects are also present in Fig. 16, which shows a similar experiment, except the injections are introduced through the Household sector rather than through the Bank sector. Both figures (upper right plots) show that the real interest rate decreases due to the increase in supply of loans, whereas the nominal rate increases with the inflation rate. Here, however, the CB is able to drive the real interest rate much lower than the Bank sector can with credit expansion in Fig. 14. This is due to the devaluation of bank liabilities caused by inflation. Comparing the lower plots in Figs. 15 and 16, Fig. 15 shows a significantly larger shift in labor from CG to K as injections increase compared to

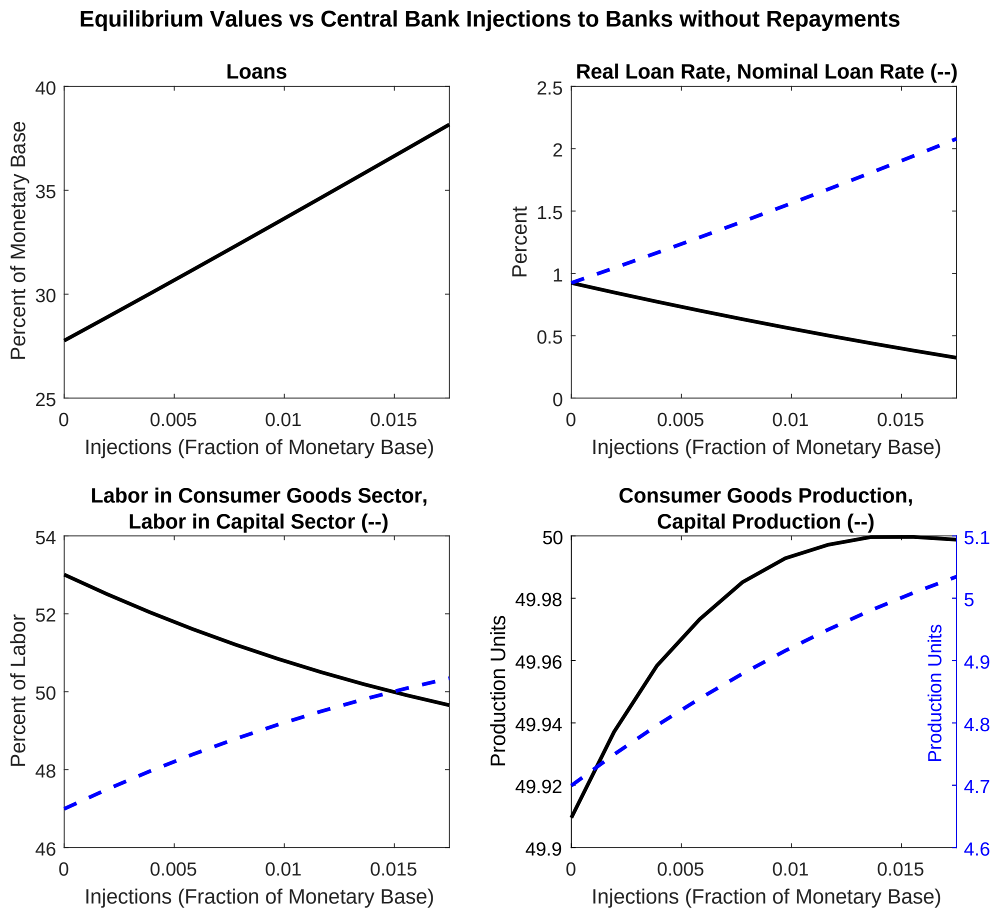

Figure 15: Steady-state values versus rate of CB injections through the Bank. The top left plot shows that loans increase nearly linearly with the injection rate. The top right plot indicates that the real interest rate (solid) can be driven to zero, in contrast to Fig. 14. Also shown is the nominal rate (dashed). The lower left plot shows the labor-capital split, and the lower right shows production of consumer goods (solid) and capital (dashed). As the injection rate increases, so do capital purchases, causing an excessive allocation of labor to the K sector (where the curves cross), and a decrease in production.

Fig. 16. In Fig. 16 the labor-capital split converges to half-half, which maximizes output. In Fig. 15 the shift of labor from CG to K becomes excessive as the injection rate increases. This causes the production (lower right plot) to peak at the point where the labor-capital allocation curves cross, in contrast to Fig. 16 where the production increases to the maximum.

The difference in results for these two experiments is due to the assumption that in Fig. 15, the CG sector does not repay any of the debt bought by the CB, including principal (corresponding to a “helicopter drop”). For a given loan portfolio this considerably reduces the total amount the CG sector pays for capital expenditures. Conversely, for a target capital expenditure this increases the amount of loans that can be carried. This effect is absent when the CB injects through the households. The result is a small decrease in loans in Fig. 16 relative to Fig. 15. (The absolute amount may appear insignificant, but because the ratio of total K wages to CG wages is nearly one in equilibrium, a relatively small change in either of those flows can cause a significant shift in labor.) In this case, loans become cheap enough

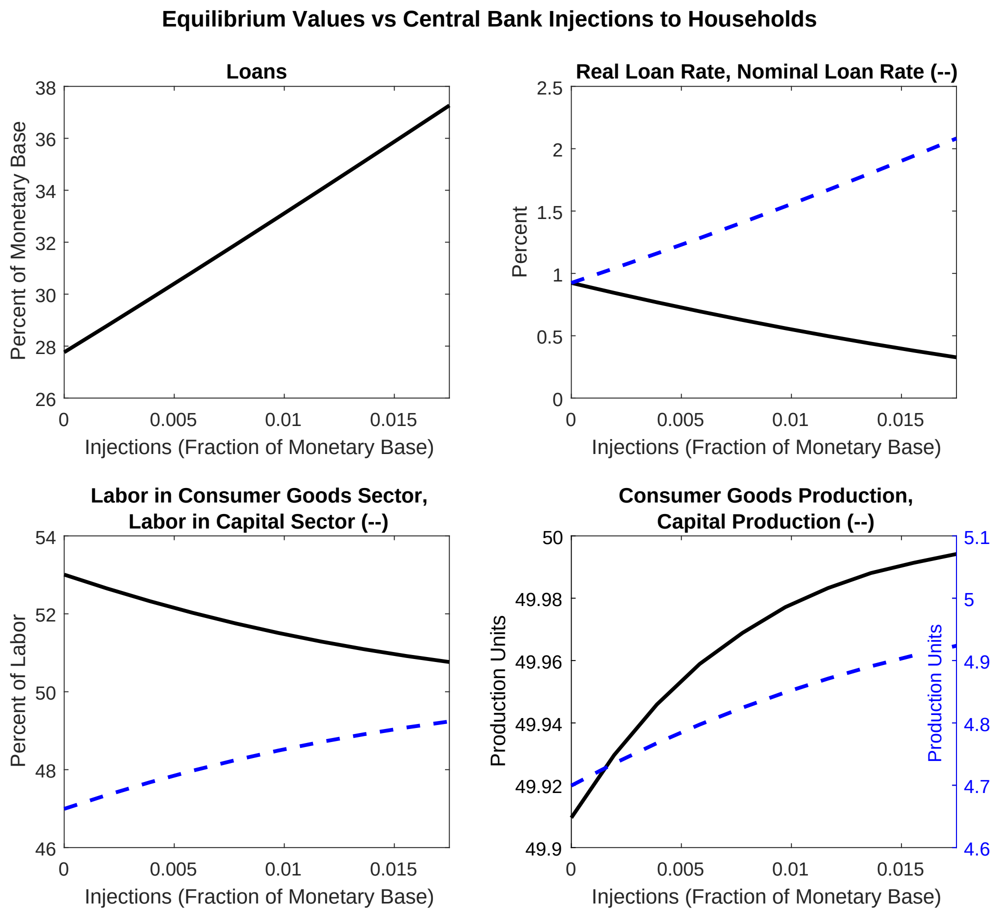

Figure 16: Steady-state values versus rate of CB injections through the Household sector. The top left plot shows that loans increase nearly linearly with the injection rate, although the maximum amount is somewhat less than that shown in Fig. 15. The top right plot again indicates that the real interest rate (solid) can be driven to zero. Also shown is the nominal rate (dashed). In contrast to Fig. 15, the labor-capital split (lower left) approaches half-half, which maximizes production (lower right).

in Fig. 15, relative to Fig. 16, so that as injections increase, the CG sector overinvests in capital, lowering production.

A key assumption here is that the CG sector only borrows to invest in *capital*, not labor. Hence as borrowing costs decrease, the CG sector continues to skew the labor-capital split more towards capital. In practice, the CG sector would eventually observe this overinvestment, and would instead use loans to finance labor costs as well as capital. Accounting for such a change in investment strategy would likely push the labor-capital split back towards the split that maximizes production, although it is not straightforward to analyze how that would effect the interest rate and other state variables.

8 Transient versus Steady-State Behavior

Examining Figs. 7-13, we see that transients associated with credit expansions and contractions can be quite long, i.e., at least τ_m iterations (60 in these examples), corresponding to a few years. Contractions tend to be somewhat longer-lasting and more volatile than expansions, due to the sudden decrease in loans and wages. This is evident in the transient leading to the deflationary scenario depicted in Fig. 8. This is in contrast to the shock in elasticity shown in Fig. 11, where the economy tends to adapt relatively smoothly. The dynamics are, of course, strongly influenced by the choice of step-size for adapting the interest rate along with the extent to which other variables are filtered (smoothed). The longer the loan durations, the more sensitive the dynamics are to the way variables are smoothed. Given insufficient smoothing, the transient response to a shock can be oscillatory, chaotic, or unstable, leading to a crash in which the CG sector cannot service its loans from revenues. Of course, in practice, additional smoothing of state-flow variables, such as S_{lf}, D_{lf} (along with capital expenditures), and savings S is likely to occur. This would introduce "viscosity" in flow-state variables (in turn causing price frictions), which is an additional source of inertia not explicitly included here.

The long transients shown here suggest a marked departure from the view that in practice, an economy is never far from an equilibrium, or steady-state, in which prices accurately reflect long-term fundamentals. Rather, shifting CB policies combined with changing fundamental (model input) parameters can cause state variables to deviate substantially from steady-state values for extended periods of time. This may have important implications for assessing the effects of CB policies on short-term traders. For example, traders following short-term trends may in fact amplify deviations from fundamentals during a long-lasting transient caused by a policy shift. Even traders with knowledge of fundamentals (i.e., model input parameters) may be motivated to follow inconsistent trends, when those trends are expected to persist.[37] Incorporating a financial sector with such traders, and studying the effects on dynamics is left for future work. We emphasize, however, that although traders will strongly influence dynamics, they cannot effectively suppress transients associated with re-populating the loan array in response to a policy shift.

The view that in practice, an economy at any particular time is more likely to be in a transient state, as opposed to near steady-state, complicates calibration of the model. That is, the model input parameters for the simulations shown here were selected to produce what might be considered reasonable steady-state behavior. However, while such steady-state behavior (specifically, a stationary loan portfolio with a fixed inflation rate) is a useful idealization, it is unlikely to occur in practice for a long enough time to enable reliable empirical measurements of model state variables. Hence the real *steady-state* effects of monetary policy indicated here may be difficult to identify in practice, since it is more likely that any real observed effects are associated with a non-equilibrium state.

9 Model Limitations and Future Work

We conclude by discussing some of the main limitations of the model, and how those might be addressed in future work. First, as previously discussed, to understand dynamics in more

[37] This complements explanations for bubbles and crashes given elsewhere, e.g., see Brunnermeier and Oehmke (2013).

detail, it will be necessary to add a financial sector with traders, thus allowing arbitrage, trend-based speculation, and carry trades. That could potentially include a bond market, in which traders buy/sell CG debt, and perhaps stock and commodities markets in which traders buy/sell particular sub-sectors of the CG sector. Aggregate prices would then be set to balance flows. This could be combined with models for account expectations, forecasting, and behavioral factors that lead to excessive leveraging (e.g., as modeled in Brunnermeier and Sannikov (2014)). An overall objective is to identify combinations of trading rules, parameters, and CB policies for which introducing traders either amplifies volatility, or enhances stability. In particular, pushing prices towards steady-state values before the loan array has settled may not necessarily dampen transients.

One of the main limitations of the model presented here is that it assumes a closed economy. Hence it cannot be used to assess the effects of foreign trade imbalances, tariffs, and differences in policies implemented by different CBs with associated effects on currency exchange. Foreign trade could be introduced by subdividing the CG sector into sub-sectors representing different classes of goods, and adding foreign sub-sectors with exogenous parameters (e.g., production and prices). More interesting and realistic would be to embed a foreign CG sub-sector in its own economy with a CB and an independent set of parameters. The flow of cash and credit across the two economies would then be determined endogenously and could be used to study the dynamic effects of varying model parameters and changing CB policies across the two economies.

In addition to including foreign trade, sub-sectors of CG could also include government, manufacturing, services, and materials. Similarly, the H sector can be divided into corresponding sub-sectors of labor along with a financial sub-sector. That might be done to gain insight into the effects of taxation, fiscal policy, allocation of resources and wealth, and movement of labor across economic sectors in response to shocks and policy decisions.

Another direction is to account explicitly for risk of loan defaults. That is, the Bank sector presented here maximizes loans subject to an exogenous reserve requirement. Different probabilities of default across different categories of loans in the array would then require the introduction of different risk premia, or credit spreads. That would allow the Bank to optimize its loan portfolio (both amount and loan window distribution) to maximize expected returns or a related objective (e.g., as in Benes et al. (2014)). Hence the fractional reserve requirement f would be determined endogenously, and could provide additional insight into the effect of leveraging and elevated risk on dynamics.[38]

Finally, there are, of course, many other ways in which the model can be refined and extended, which can potentially provide additional insight into both steady-state and dynamic behavior. For example, those include introducing viscosity in flow-state variables, refining the model for production (both CG and capital goods), including inventories, including leisure in the model for unemployment, and introducing random fluctuations in state variables. Random fluctuations, in particular, can be combined with particular types of trading algorithms, capturing a notion of limited-information expectations. Incorporating the features mentioned here represents no small task, but may provide further insights into macro-economic dynamics that cannot be directly obtained from prior models.

[38] Related DSGE models for studying the effects of credit spreads and reserve requirements on stability have been presented in Cúrdia and Woodford (2010) and Woodford (2016).

Appendix A Financing calculations

Suppose a unit loan for term τ at rate r is initiated at period 0, uniform payments of P per period are made to an Escrow account starting at period 1, and the Escrow account earns at the rate s per period.[39] We assume that when the loan is originated the rates r and s are contractually fixed at the then-current values for the entire term of the loan. After i periods, the firm owes $(1+r)^i$ to the Bank, and has an accumulation of $P\sum_{j=0}^{i-1}(1+s)^j = (P/s)[(1+s)^i - 1]$ in Escrow. Thus, for the firm to make the balloon payment of $(1+r)^\tau$ that is due at period τ, we require that the payments be

$$P = \frac{s(1+r)^\tau}{(1+s)^\tau - 1} \triangleq P(r,s,\tau)\ , \tag{79}$$

which is the generalization of the usual mortgage amortization formula to the case where there is a spread between the loan and savings rates. To check the end cases, when $s = 0$ this corresponds to setting aside $(1+r)^\tau/\tau$ per period with no reinvestment (the limit as $s \to 0$) for τ periods. When $s = r$ this corresponds to the usual mortgage formula, where the reinvestments earn at the Bank's lending rate. For reference the total accumulation in Escrow after i payments due to a unit loan with term τ is

$$\mathcal{A}(r,s,i,\tau) = (1+r)^\tau \frac{(1+s)^i - 1}{(1+s)^\tau - 1}\ . \tag{80}$$

The actual capital expenditures per period can now be written

$$K_C(t) = \sum_{\tau=1}^{\tau_m}\sum_{i=1}^{\tau} P(r(t-i), s(t-i), \tau)L(\tau, t-i)\ . \tag{81}$$

In equilibrium this yields the appropriate version of the functional Θ, the per-unit, per-period cost of loans:

$$K_C = L\sum_{\tau=1}^{\tau_m} \tau w_\tau P(r,s,\tau) \triangleq \Theta(r,s)L\ . \tag{82}$$

The balance in the Escrow account after t periods is

$$E(t) = \sum_{\tau=1}^{\tau_m}\sum_{i=1}^{\tau} \mathcal{A}(r(t-i), s(t-i), i, \tau)L(\tau, t-i)\ , \tag{83}$$

using (80) for the accumulation in Escrow. This expression includes payments of interest on non-mature and maturing loans but does not include new loans. The maturing loans will drop off the end of the summation and comprise the balloon payments. The functional J for equilibrium Escrow is

$$\begin{aligned} E &= L\sum_{\tau=1}^{\tau_m} w_\tau \frac{(1+r)^\tau}{(1+s)^\tau - 1}\sum_{i=1}^{\tau}\left[(1+s)^i - 1\right] \\ &= L\sum_{\tau=1}^{\tau_m} w_\tau \frac{(1+r)^\tau}{(1+s)^\tau - 1}\left[\frac{(1+s)^{\tau+1} - 1}{s} - \tau - 1\right] \\ &\triangleq J(r,s)L\ . \end{aligned} \tag{84}$$

[39] In this and the remaining technical appendixes we adhere strictly to the following convention: When we indicate the dependence on period t explicitly, as in "$r(t)$", we mean the dynamic variable. The plain symbol, as in "r", will refer to the equilibrium value.

The balloon payments at time t are in total

$$\begin{aligned} L_{BP}(t) &= \sum_{\tau=1}^{\tau_m} \mathcal{A}(r(t-\tau), s(t-\tau), \tau, \tau) L(\tau, t-\tau) \\ &= \sum_{\tau=1}^{\tau_m} (1 + r(t-\tau))^\tau L(\tau, t-\tau) \ , \end{aligned} \tag{85}$$

which is in equilibrium

$$L_{BP} = L \sum_{\tau=1}^{\tau_m} w_\tau (1+r)^\tau \ . \tag{86}$$

The balloon payments can be broken down into principal and interest parts:

$$L_{BPP}(t) = \sum_{\tau=1}^{\tau_m} L(\tau, t-\tau) \tag{87}$$

$$L_{BPI}(t) = L_{BP}(t) - L_{BPP}(t) \ , \tag{88}$$

and in equilibrium

$$L_{BPP} = L \tag{89}$$

$$L_{BPI} = L_{BP} - L \ . \tag{90}$$

We also introduce the functional X that gives us the equilibrium balloon payments:

$$L_{BP} = L \sum_{\tau=1}^{\tau_m} w_\tau (1+r)^\tau \triangleq X(r) L \ . \tag{91}$$

Thus, in equilibrium, the interest part of the balloon payments is

$$L_{BPI} = L_{BP} - L = (X-1)L \ . \tag{92}$$

Appendix B Baseline equilibrium calculation

B.1 Equilibrium calculation

In equilibrium,[40] the market for loanable funds equilibrates, so that we know that $D_{lf} = S_{lf} = L = W_K$. Begin by collecting the dynamic equations in subsection 3.2 with equilibrium

[40] We will make no distinction between an *equilibrium* and a *fixed-point*, and use the terms interchangeably.

values for the variables:

$$
\begin{aligned}
D_{lf} &= S_{lf} = L = W_K && (93)\\
s &= L_{BPI}/(S+E) && (94)\\
\Delta S &= g\mathcal{W} - S\,, \quad g(s) \text{ a known function} && (95)\\
R_v &= C = (1-g)\mathcal{W} && (96)\\
C_c &= c\,C && (97)\\
\Delta S_c &= W_{cash} - C_c && (98)\\
K_{tar} &= hC && (99)\\
E_{tar} &= \text{known linear function of } L && (100)\\
\Delta E &= E_{tar} - E && (101)\\
\Delta E_c &= c\,\Delta E && (102)\\
D_{lf} &= K_{tar}/\Theta(r,s) && (103)\\
W_C &= C - \Delta E && (104)\\
L_{BP} &= \text{known linear function of } L && (105)\\
L_{BPI} &= L_{BP} - L && (106)\\
B_c' &= B_c - c_k L + \Delta S_c + \Delta E_c && (107)\\
S' &= S + \Delta S && (108)\\
E' &= E_{tar} - L_{BP} && (109)\\
\mathcal{B} &= S' + E' && (110)\\
\mathcal{L} &= \mathcal{B} + L_{BPI} && (111)\\
S &= \mathcal{L}S'/\mathcal{B} && (112)\\
B_c &= B_c' && (113)\\
E &= \mathcal{L}E'/\mathcal{B} && (114)\\
S_{lf} &= \tfrac{1}{f+c_k(1-f)} \cdot [B_c - f\cdot(S+E+(1-c)W_C)] && (115)\\
W_{cash} &= c_k W_K + cW_C && (116)\\
\mathcal{W} &= W_C + W_K + S\,. && (117)
\end{aligned}
$$

We can now express all the state variables, except r and s, as functions proportional to L, and ordered in such a way that they can be computed sequentially. As we will see below, the value of L is determined by the monetary base, an arbitrary scale. First, though, it is easy enough to eliminate the variables C_c, ΔS_c, W_{cash}, ΔE_c, and B_c', by direct substitution:

$$
\begin{aligned}
D_{lf} &= S_{lf} = L = W_K && (118)\\
s &= L_{BPI}/(S+E) && (119)\\
\Delta S &= g\mathcal{W} - S && (120)\\
R_v &= C = (1-g)\mathcal{W} && (121)\\
K_{tar} &= hC && (122)\\
E_{tar} &= \text{known linear function of } L && (123)\\
\Delta E &= E_{tar} - E && (124)\\
D_{lf} &= K_{tar}/\Theta(r,s) && (125)
\end{aligned}
$$

$$
\begin{aligned}
W_C &= C - \Delta E & (126)\\
L_{BP} &= \text{known linear function of } L & (127)\\
L_{BPI} &= L_{BP} - L & (128)\\
S' &= S + \Delta S & (129)\\
E' &= E_{tar} - L_{BP} & (130)\\
\mathcal{B} &= S' + E' & (131)\\
\mathcal{L} &= \mathcal{B} + L_{BPI} & (132)\\
S &= \mathcal{L}S'/\mathcal{B} & (133)\\
E &= \mathcal{L}E'/\mathcal{B} & (134)\\
S_{lf} &= \tfrac{1}{f+c_k(1-f)} \cdot [B_c - f \cdot (S + E + (1-c)W_C)] & (135)\\
\mathcal{W} &= W_C + W_K + S \ , & (136)
\end{aligned}
$$

Observe that the bookkeeping for the bank cash B_c, (107), yields the equilibrium cash-balance condition

$$
B_c \;=\; B_c - c_k L + c_k W_K + c W_C - c\,C + c\Delta E \ , \tag{137}
$$

or

$$
c_k(L - W_K) = c(W_C - C + \Delta E) = 0 \ , \tag{138}
$$

which tells us nothing new, except that the cash ratios $0 \leq c, c_k \leq 1$ can be chosen arbitrarily.

We will use the following known functions of r, s:

$$
\begin{aligned}
L_{BP} &= L \textstyle\sum_{i=1}^{\tau_m} w_i (1+r)^i \triangleq XL & (139)\\
L_{BPI} &= (X-1)L & (140)\\
E_{tar} &= L \textstyle\sum_{\tau=1}^{\tau_m} w_\tau \sum_{i=1}^{\tau} \mathfrak{A}(r,s,i,\tau) \triangleq YL & (141)\\
\Theta &= \textstyle\sum_{\tau=1}^{\tau_m} \tau w_\tau P(r,s,\tau) & (142)\\
g &= \text{ given function of } s & (143)\\
Q &\triangleq \Theta/(h(1-g)) & (144)\\
Z &\triangleq (Y - 1 + gQ)/(Y - X + gQ) \ \ . & (145)
\end{aligned}
$$

We can now write all the state variables (except the rates) explicitly as multiples of L, using known factors, the factors, however, depending on the rates.

$$
\begin{aligned}
D_{lf} &= S_{lf} = L = W_K & (146)\\
E_{tar} &= YL & (147)\\
K_{tar} &= \Theta L & (148)\\
L_{BP} &= XL & (149)\\
L_{BPI} &= (X-1)L & (150)\\
C &= (\Theta/h)L & (151)\\
\mathcal{W} &= QL & (152)\\
E' &= (Y-X)L & (153)\\
S' &= gQL & (154)
\end{aligned}
$$

$$\begin{aligned}
\mathcal{B} &= (Y - X + gQ)L & (155)\\
\mathcal{L} &= (Y - 1 + gQ)L & (156)\\
S &= gQZL & (157)\\
E &= (Y - X)ZL & (158)\\
\Delta E &= [1 - gQ(1 - Z)]\, L & (159)\\
\Delta S &= [gQ(1 - Z)]\, L & (160)\\
W_C &= [Q(1 - gZ) - 1]\, L & (161)\\
s &= (X - 1)/(Y - 1 + gQ) \ . & (162)
\end{aligned}$$

We are thus left with the problem of calculating two unknowns: r and s. The explicit equation for s, (162) immediately gives us one condition:

$$\boxed{s = \frac{X - 1}{Y - 1 + gQ}} \quad \text{...Condition 1 .} \tag{163}$$

Notice that this gives, implicitly, the relationship between equilibrium s vs. r, and that this relationship does not depend on f, c, or c_k. We are thus left with one unknown—but, without further assumptions, we have run out of conditions.

B.2 Parameterizing the equilibrium manifold by r

It is particularly easy to resolve the extra degree of freedom in our model by simply choosing r, although that choice is not forced. This enables us to find the corresponding fixed point by solving (using binary search, for example) the one-dimensional search problem Condition 1, (163). Once the pair (r, s) is determined, the remainder of the state variables can be found by direct evaluation of the equilibrium conditions, (139)-(161).

We can solve Condition 1 numerically using binary search, writing (163) in the form "increasing function of s = left-hand side = g = right-hand side":

$$g = \frac{1}{1 + \frac{\Theta/h}{\frac{X-1}{s} - Y + 1}} \ , \tag{164}$$

using $Q = \Theta/(h(1 - g))$. The binary search narrows the search down to a region where the right-hand side is decreasing and crosses the left-hand side, after which locating the crossing point is very easy. Figure B17 shows the (increasing) left- and right-hand sides of (164) for the case $\tau_m = 60$, $r = 2\%$, $c = c_k = 0$, and $e = 500$. There is a pole in right-hand side at about 4%, which makes it necessary to exercise caution in the binary search. One way to make the search reasonably robust is to start with the right end of the initial search bracket very small, and expand it to the right (geometrically) until the value of the right-hand side is below that of the left-hand side, and then stop.

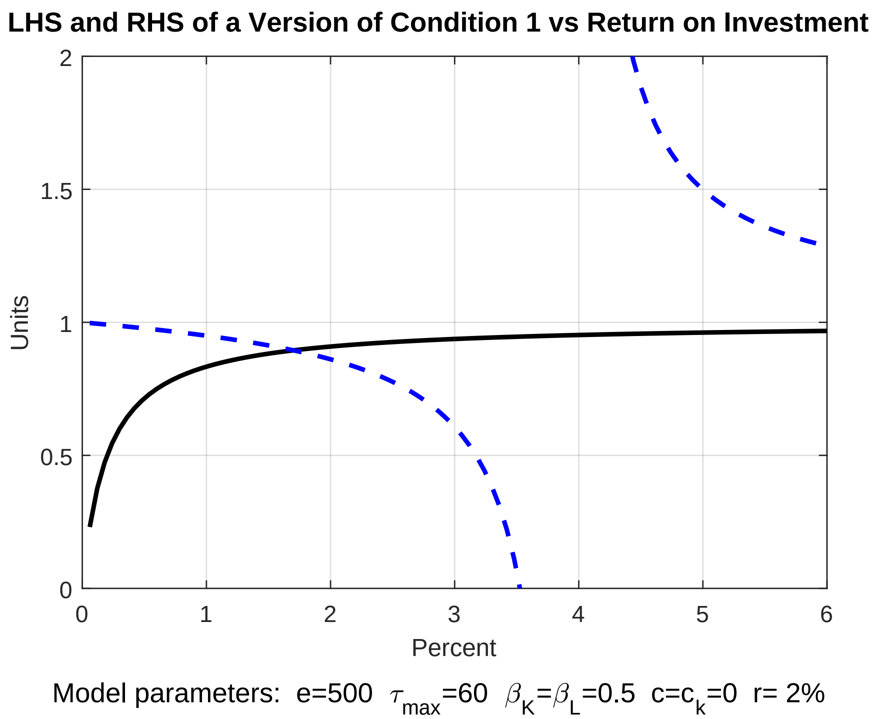

Figure B17: An example showing the left- and right-hand sides of (164), a version of Condition 1. There is a pole in the right-hand side, which we need to exclude from the initial binary search bracket.

B.3 Household debt

Households are in debt to the bank a total $\Phi(t)$ (normalized value $\varphi(t) \triangleq \Phi(t)/G_0$). As described below, we will choose the household debt exogenously (rather than the loan rate r) to render the equilibrium unique. We now calculate its value in terms of the other variables in the economy.

The total bank assets are

$$B_c(t) - c_k L(t) + L_{tot}(t) + \Phi(t) \;, \tag{165}$$

since $B_c(t) - c_k L(t)$ is the cash at the bank (at the point when $B_c(t)$ is defined within the iterated period), and we define $L_{tot}(t)$ to be the total outstanding loans at the *end* of period t, (which thus includes the loans made in that period):

$$L_{tot}(t) \triangleq \sum_{\tau=1}^{\tau_m} \sum_{i=1}^{\tau} L(\tau, t-i+1) \;. \tag{166}$$

The equilibrium value of $L_{tot}(t)$, used below, is

$$L_{tot} = L \sum_{\tau=1}^{\tau_m} \tau w_\tau \triangleq \Gamma L \;, \tag{167}$$

as we have defined it before. We note that the cash at the bank can also be written

$$B_c(t) - c_k L(t) \;\; = \;\; f(S(t) + E(t) + (1-c)W_C(t) + (1-c_k)W_K(t)) \;. \tag{168}$$

The corresponding total bank liabilities are

$$\mathcal{L}(t) + (1 - c_k)W_K(t) + (1 - c)W_C(t) \ . \tag{169}$$

We model the bank as a zero-profit sector, so its total assets and liabilities are equal at the end of every period. Therefore,

$$\Phi(t) \ = \ \mathcal{L}(t) + L(t) + (1 - c)W_C(t) - B_c(t) - L_{tot}(t) \ , \tag{170}$$

using $W_K(t) = L(t)$. Now use B_c from (135):

$$S_{lf} = \frac{1}{f + c_k(1 - f)} \left[B_c - f(S + E + (1 - c)W_C)\right] \ , \tag{171}$$

or

$$B_c = \left[f + c_k(1 - f)\right] L + f(S + E + (1 - c)W_C) \ , \tag{172}$$

to write Φ as

$$\begin{aligned} \Phi \ &= \ \mathcal{L} - L_{tot} - \left[f + c_k(1 - f)\right] L - f(S + E + (1 - c)W_C) + L + (1 - c)W_C \\ &= \ \mathcal{L} - L_{tot} - \left[f + c_k(1 - f)\right] L + (1 - f)(1 - c)W_C - f(S + E) + L \ . \end{aligned} \tag{173}$$

Using the equilibrium conditions (139)-(161), this can be written, as usual, as a multiple of L, yielding

$$\begin{aligned} \Phi/L \ &= \ (1 - f)(1 - c)\left[Q(1 - gZ) - 1\right] + (Y - 1 + gQ) \\ &\quad -\Gamma + 1 - \left[f + c_k(1 - f)\right] - f\left[(Y - X) + gQ\right] Z \\ &= \ (1 - f)(1 - c)\left[Q(1 - gZ) - 1\right] + (1 - f)(Y - 1 + gQ) \\ &\quad -\Gamma + 1 - \left[f + c_k(1 - f)\right] \ , \end{aligned} \tag{174}$$

simplifying using $[(Y - X) + gQ]Z = Y - 1 + gQ$, which follows from the definition of Z.

We may want to normalize Φ by G_0, not L, so we write G_0 in terms of L by combining the fact that $G_0 = B_c + cW_C$ and (172). The result is

$$G_0/L \ = \ (1 - f)c_k + f\left[Y + gQ + (1 - c)(Q - 1 - gQZ)\right] + c\left[Q(1 - gZ) - 1\right] \tag{175}$$

$$\triangleq \ V \ , \tag{176}$$

which, like X, Y, and Z, is a well defined function of r and s. We thus have $\varphi V = (\Phi/G_0)(G_0/L) = \Phi/L$, and we have φ in terms of known functions of r and s:

$$\varphi \ = \ \frac{(1 - f)\left[Y - 1 + gQ\right] + (1 - f)(1 - c)\left[Q(1 - gZ) - 1\right] - \Gamma + 1 - \left[f + c_k(1 - f)\right]}{V} \ . \tag{177}$$

B.4 The equilibrium manifold

So far, we have left one extra degree of freedom in our model, and this means there is a curve, a one-dimensional manifold, in the state space, along which all possible equilibria are constrained to lie.

As we shall see later, if the loan rate is shocked, the model economy is knocked off the equilibrium manifold. When the shock is removed, it returns to exactly the same point on the equilibrium manifold. Why? The answer can be found in the fact that net bank assets $\varphi(t)$, and hence the household debt, is a conserved quantity, and fixing φ pins down the equilibrium to a unique point.

Remark 1 *Household debt $\varphi(t)$ is constant for every t, so long as there is no exogenous injection or extraction of money. This follows from the fact that the bank sector balances accounts precisely at the end of every period. The economy will therefore return to the same point on the equilibrium manifold after state variables or parameters are perturbed, provided, of course, that system parameters are restored to their original values after the shock.*

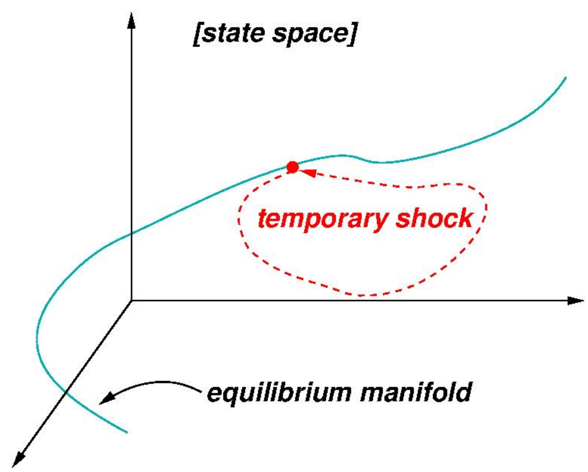

Figure B18: Illustrating the fact that the system will return to the same equilibrium after state variables or parameters are perturbed, provided that the parameters are returned to their original values. This is true regardless of how the equilibrium manifold is parameterized, and follows from the fact that, with no exogenous injections or extractions of money, household debt is invariant.

This is illustrated diagrammatically in Fig. B18.

Household debt will not play an important role in the policy experiments in this paper, and for simplicity we now stipulate that it is zero, thereby making the equilibrium unique. We then have two equilibrium conditions for the two unknowns r and s, the first from (163), and the second from the assumption that $\Phi = 0$. We collect them here:

$$\boxed{s = \frac{X-1}{Y-1+gQ}} \quad \text{...Condition 1 ,} \tag{178}$$

$$\boxed{\Phi/L = 0 \quad \text{from (174)}} \quad \text{...Condition 2 .} \tag{179}$$

To calculate equilibria numerically we now need to solve two equations in two unknowns, a two-dimensional search problem instead of the one-dimensional problem if we choose r. This is a consequence of our choosing Φ instead of r to disambiguate the equilibrium. We found the Nelder-Mead search algorithm[41] to be quite effective in this application.

Appendix C Uniqueness of fixed point for $\tau_m = 1$

While the functions of r and s that appear in Conditions 1 and 2 get complicated quickly for $\tau_m > 1$, they are simple enough for the case $\tau_m = 1$ to allow us to prove that, depending

[41] Nelder and Mead (1965).

on a reasonable condition involving f, h, c, and c_k, (assuming $0 \le f, h, c, c_k \le 1$ and $e > 0$), there is exactly one equilibrium positive r. We can thus conclude that with this condition, at least when $\tau_m = 1$, a fixed point with positive loan rate must be unique.

The proof is elementary and direct. When $\tau_m = 1$, the functions simplify to $X = Y = \Theta = 1 + r$, and $\Gamma = 1$. For reference, the fixed-point conditions become

$$\left(r + \frac{1+r}{h} es\right) s = r \quad \text{...Condition } 1' , \tag{180}$$

and

$$(1-c)(1+r)(1/h - 1) + \left(r + \frac{1+r}{h} es\right) - c_k - \frac{f}{1-f} = 0 \quad \text{...Condition } 2' , \tag{181}$$

respectively. We rewrite these using parameters $\alpha = e/h$, $\beta = c_k + f/(1-f)$, and $\gamma = (1-c)(1/h - 1)$, where we can assume that $\alpha, \beta, \gamma \ge 0$:[42]

$$(r + \alpha(1+r)s)s = r \quad \text{...Condition } 1'' , \tag{182}$$

and

$$\gamma(1+r) + r/s - \beta = 0 \quad \text{...Condition } 2'' , \tag{183}$$

using Condition $1''$ to simplify $2''$.

Solving Condition $2''$ for s and substituting in $1''$ gives us the quadratic in r

$$[\alpha - \gamma - \gamma^2]r^2 + [\alpha + (\beta - \gamma)(2\gamma + 1))]r - [\beta - \gamma]^2 = 0 \ . \tag{184}$$

Ordinarily α is much larger than β and γ, being proportional to e, and we can assume

Uniqueness Condition 1 $\alpha - \gamma - \gamma^2 > 0$ and $\alpha + (\beta - \gamma)(2\gamma + 1) > 0$.

With this assumption (ignoring the special case when $\beta = \gamma$), the sequence of signs of the coefficients is $+ + -$, and we can conclude by Descartes' Rule of Signs that there is exactly one positive equilibrium r.[43]

The all-cash special case for $\tau_m = 1$ We note that in the further special case when $c = c_k = 1$ (all-cash), $\tau_m = 1$, $\beta = 1/(1-f)$, $\gamma = 0$, and (183) immediately implies that $s = (1-f)r$.

The condition $s = r$ for $\tau_m = 1$ If, as discussed in Section 3.3, we require that $s = r$, for the case $\tau_m = 1$ and for general c and c_k, (182) yields a quadratic equation for $r = s$ (as a function of α only). Using this in (183) then gives us β and hence f.

[42] We have run out of convenient symbols and reuse α and β in this section. This should cause no confusion.

[43] We say nothing here about solutions for s, but the same technique leads to a quadratic equation and a similar condition for exactly one positive s.

Appendix D Adaptive Update for $s(t)$

An adaptive update is given by

$$s(t) = s(t-1) + k_s \left[R_{BPI}(t-1) - s(t-1) \cdot \mathcal{L}(t-1)\right] / G(t-1) \tag{185}$$

where k_s is the step-size, and R_{BPI} is an estimate of total returns to the bank in steady-state given the state variables at time t, including interest on loans from balloon payments, plus reclaimed credit from defaults. It is assumed that this aggregate information is known, or can be estimated and distributed to the banks. In steady-state $s(t)$ remains constant, and hence the second term on the right converges to zero. The total return to the bank is then given by[44]

$$\begin{aligned} R_{BPI}(t) =& \frac{L_{BPI}(t-1)}{L_{BPI}(t-1) + L_{CBI}(t-1)} \bar{L}(t-1) \left\{ \left(\sum_{i=1}^{\tau_m} w_i \Pi(t-1)^{-i+1} (1-\delta)^i [1 + r(t)]^i \right) - 1 \right\} \\ &+ \delta \cdot E(t-1) + p_{CB}(t-1) \end{aligned} \tag{186}$$

where $\bar{L}(t)$ is an estimate of the principle of all loans due at time t (owned by both the Bank and the CB), and $\Pi(t)$ is the inflation multiplier (1 + inflation rate in loans). The last two terms on the right in (186) are then reclaimed credit from defaults and p_{CB}, the total premium on loans, which the CB purchases, given by (77). The remaining (first) term including the large braces is then the expected return on all outstanding loans to be repaid to the bank. The terms inside the sum include discounts due to defaults and inflation. The fraction preceding $\bar{L}$ is the fraction of interest repaid to the Bank, relative to the total repaid to the Bank and the CB. In steady state, $\bar{L}$ becomes the loans per period, i.e., $\bar{L}(t) = S_{lf}(t)$, the supply of loanable funds, which equals $D_{lf}(t)$, the demand for loanable funds. However, in a transient this is no longer true, so that for purposes of estimating $R_{BPI}(t)$, we can take $\bar{L}(t)$ between the two values.

[44] This is used in the H sector savings calculation, but the de facto version in the baseline model is used everywhere else in the simulation because it tends to be more stable.

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