PROBLEM SET 8

ECE 4130: Introduction to Nuclear Science

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1. (a) *Problem 6-1.*

Let us denote by F the fuel and by S the coolant in the mixture. Then, Σ_a^F and Σ_a^S represent the absorption cross-section for the fuel and coolant, respectively, while Σ_a represents the nuclear cross-section of the mixture of the two (i.e. $\Sigma_a = \Sigma_a^F + \Sigma_a^S$). In our case, the fuel F is plutonium enriched to 3 w/o while the coolant S is liquid sodium. We know that for each constitutent element X of the mixture, the atom density N_X is given by

$$N_X = \frac{w_X \rho}{M(X)},$$

where w_X is the relative mass fraction of X in the mixture, $M(X)$ is the atomic weight of X, and ρ is the density of the substance from which the element is derived, namely the mixture. Furthermore, because this produces units of $\frac{\text{mol}}{\text{cm}^3}$, we can multiply by Avogadro's number N_A to produce units of $\frac{\text{atoms}}{\text{cm}^3}$ (i.e. units of atom density):

$$N_X = \frac{w_X \rho}{M(X)} N_A.$$

The absorption cross-section for an atom of each constituent element X can be found in a standard table, so we can compute the macroscopic absorption cross-section that each constituent element of the mixture contributes, recalling that the macroscopic absorptio cross-section Σ_a is given by $\Sigma_a = N\sigma_a$, where N is the atom density and σ_a is the nominal one-group absorption cross-section for X in a fast reactor. We proceed with finding the relative contributions for each constituent element X in the mixture:

$$\Sigma_a^X = N_X \sigma_a^X = \frac{w_X \rho}{M(X)} N_A \sigma_a^X.$$

We then calculate this value for each of the constituent elements of the mixture, namely sodium and plutonium, consulting Table 6.1 for the nominal one-group constants σ_a^{Na} and σ_a^{Pu}:

$$\Sigma_X^{Na} = \frac{w_{Na}\rho}{M(\text{Na})} N_A \sigma_a^{Na} = \frac{0.97 \cdot 1.0 \ \frac{\text{g}}{\text{cm}^3}}{22.9898 \ \frac{\text{g}}{\text{mol}}} \left(0.6022137 \times 10^{24} \ \frac{\text{atoms}}{\text{mol}}\right) (0.0008 \text{ b}) \left(\frac{10^{-24} \text{ cm}^2}{1 \text{ b}}\right) = 2.0 \times 10^{-5} \text{ cm}^{-1}$$

$$\Sigma_X^{Pu} = \frac{w_{Pu}\rho}{M(\text{Pu})} N_A \sigma_a^{Pu} = \frac{0.030 \cdot 1.0 \ \frac{\text{g}}{\text{cm}^3}}{239.0522 \ \frac{\text{g}}{\text{mol}}} \left(0.6022137 \times 10^{24} \ \frac{\text{atoms}}{\text{mol}}\right) (2.11 \text{ b}) \left(\frac{10^{-24} \text{ cm}^2}{1 \text{ b}}\right) = 1.6 \times 10^{-4} \text{ cm}^{-1}$$

Now, we find the fuel utilization f using the relation $f = \frac{\Sigma_a^F}{\Sigma_a}$:

$$f = \frac{\Sigma_a^F}{\Sigma_a} = \frac{\Sigma_a^F}{\Sigma_a^F + \Sigma_a^S} = \frac{1}{1 + \frac{\Sigma_a^S}{\Sigma_a^F}} = \frac{1}{1 + \frac{\Sigma_a^{Na}}{\Sigma_a^{Pu}}} = \frac{1}{1 + \frac{2.0 \times 10^{-5} \text{ cm}^{-1}}{1.6 \times 10^{-4} \text{ cm}^{-1}}} = \boxed{0.89}$$

We also compute the infinite multiplication factor k_∞ using the relation $k_\infty = \eta f$, where η is nominal one-group number of neutrons released in fission per plutonium neutron absorbed by a fissile nucleus in the fast reactor:

$$k_\infty = \eta f = (2.61)(0.89) = \boxed{2.31}$$

where we used η_{Pu} from Table 6.1. Since $k_\infty > 1$, we see that the reactor is $\boxed{\text{supercritical}}$.

(b) *Problem 6-9.*

i. We first find the *material* buckling of the reactor defined in Problem 6-1 using

$$B^2 = \frac{k_\infty - 1}{L^2} = \frac{k_\infty - 1}{\left(\frac{D}{\Sigma_a}\right)} = \frac{\Sigma_a}{D}(k_\infty - 1),$$

where D is the nominal one-group diffusion coefficient in the reactor, approximately given by

$$D \approx \frac{\lambda_{tr}}{3} = \frac{\left(\frac{1}{\Sigma_{tr}}\right)}{3} = \frac{1}{3\Sigma_{tr}} = \frac{1}{3\sum_X \Sigma_{tr}^X} = \frac{1}{3\sum_X N_X \sigma_{tr}^X},$$

where each X is a constituent element of the mixture. In our case, we can use the values from Problem 6-1 and Table 6.1 to find D if X ranges over sodium and plutonium:

$$D \approx \frac{1}{3\left[(0.0254 \times 10^{24}\ \text{cm}^{-3})(3.3 \times 10^{-24}\ \text{cm}^2) + (7.56 \times 10^{19}\ \text{cm}^{-3})(6.8 \times 10^{-24}\ \text{cm}^2)\right]} = 3.95\ \text{cm}.$$

We now use this in combination with the results from Problem 6-1 to find the material buckling:

$$B = \sqrt{\frac{\Sigma_a}{D}(k_\infty - 1)} = \sqrt{\frac{1.8 \times 10^{-4}\ \text{cm}^{-1}}{3.95\ \text{cm}}(2.31 - 1)} = 7.74 \times 10^{-3}\ \text{cm}^{-1}.$$

The results of Section 6.3 give us that the first-order *geometric* buckling of a bare spherical reactor is given by

$$B_1 = \frac{\pi}{\widetilde{R}},$$

where $\widetilde{R}$ is the critical radius. Then, we can rearrange this as $\widetilde{R} = \frac{\pi}{B_1}$ and solve for the extended radius using the material buckling in place of the first-order geometric buckling (since they are equal):

$$\widetilde{R} = \frac{\pi}{B_1} = \frac{\pi}{B} = \frac{\pi}{7.74 \times 10^{-3}\ \text{cm}^{-1}} = 406\ \text{cm}$$

Then, we can find the critical radius:

$$R = \widetilde{R} - d = \widetilde{R} - 2.13D = 406\ \text{cm} - 2.13(3.95\ \text{cm}) = \boxed{398\ \text{cm}}$$

ii. As the end of Section 6.3 shows, the maximum flux in the bare spherical reactor is given by

$$\phi_{max} = \frac{\pi P}{4E_R \Sigma_f R^3},$$

where P is the thermal power level at which the reactor operates, E_R is the recoverable energy of the reactor in joules per fission, and Σ_f is the nominal one-group macroscopic fission cross-section of the moderating mixture. We first find the macroscopic fission cross-section using the results from Problem 6-1 and the values from Table 6.1:

$$\begin{aligned}\Sigma_f &= \Sigma_f^{Na} + \Sigma_f^{Pu} = N_{Na}\sigma_f^{Na} + N_{Pu}\sigma_f^{Pu} \\ &= \left(0.0254 \times 10^{24}\ \text{cm}^{-3}\right)\left(0 \times 10^{-24}\ \text{cm}^2\right) + \left(7.56 \times 10^{19}\ \text{cm}^{-3}\right)\left(1.85 \times 10^{-24}\ \text{cm}^2\right) = 1.40 \times 10^{-4}\ \text{cm}^{-1}.\end{aligned}$$

Assuming a recoverable energy of 200 MeV per fission, we get:

$$\phi_{max} = \frac{\pi \cdot 500\ \text{MW}}{4(200\ \text{MeV})\left(\frac{1.602 \times 10^{-19}\ \text{J}}{1\ \text{eV}}\right)(1.40 \times 10^{-4}\ \text{cm}^{-1})(398\ \text{cm})^3} = \boxed{1.39 \times 10^{15}\ \frac{\text{neutrons}}{\text{cm}^2 \cdot \text{sec}}}$$

iii. The probability that a fission neutron will escape or leak from the reactor is given by the inverse relation:

$$P_E = 1 - P_L = 1 - \frac{1}{k_\infty} = 1 - \frac{1}{2.31} = 0.568 = \boxed{56.8\%}$$

2. (a) *Problem 6-5.*

We consider the case of a critical bare cylindrical reactor of radius R and height H, as shown below. We denote by the term "radius of r" any position within the cylinder that has an $\mathcal{L}_2$-distance of r away from the central z-axis:

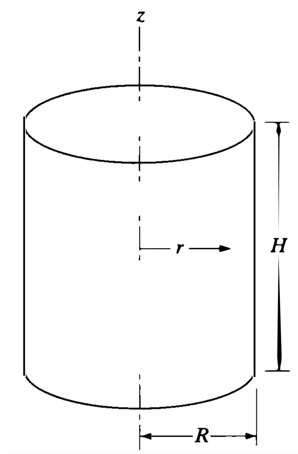

Figure 1: A critical bare cyndrical reactor (base at the origin) of radius R and height H at a central-axis $\mathcal{L}_2$-distance r.

We wish to derive the constant factor A that characterizes the flux of such a reactor at the specified radius r using the aformentioned Lebesgue measure as our metric of distance. We begin with the one-group reactor equation $\nabla^2\phi + B^2\phi = 0$, where B represents the buckling of the reactor. Since we are examining a cylindrical reactor structure, it would make the most sense to transform the one-group reactor equation into cylindrical coordinates (i.e. a transformation into circular polar coordinates in the xy-plane with the z-direction left untouched):

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0.$$

However, we recall that our boundary conditions remain intact, namely that the flux $\phi(r,z)$ must satisfy certain constraints with respect to its two parameters. One such condition is that the flux vanishes beyond the boundary of the moderator in the reactor. In other words,

$$\phi(\widetilde{R}, z) = 0$$

for any height z above the base of the cylinder. The other condition is that the net flux through any point in the circular surface through the center of the cylinder vanishes, as regardless of how the flux is distributed, the net flow through the center cancels out by symmetry. Since the center is at height $\widetilde{H}/2$, we have that the flux through any radius r away from the central axis in the $z = \widetilde{H}/2$ plane is zero. In other words,

$$\phi(r, \widetilde{H}/2) = 0$$

for any radius r away from the central axis. We can now propose using the method of separation of variables that ϕ is of the form $\phi(r,z) = \rho(r)Z(z)$, where ρ and Z are independent functions. Then, the differential equation is

$$\frac{d^2\rho}{dr^2}Z + \frac{Z}{r}\frac{d\rho}{dr} + \rho\frac{d^2 Z}{dz^2} + B^2\rho Z = 0.$$

Dividing through by ρZ, we have the equivalent differential equation:

$$\frac{1}{\rho}\frac{d^2\rho}{dr^2} + \frac{1}{\rho r}\frac{d\rho}{dr} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + B^2 = 0.$$

We now separate out the variables so that the equation reduces to

$$\frac{1}{\rho}\frac{d^2\rho}{dr^2} + \frac{1}{\rho r}\frac{d\rho}{dr} + B^2 = -\frac{1}{Z}\frac{d^2 Z}{dz^2}.$$

However, the LHS is purely a function of r, and the RHS is purely a function of z. Since we said that r and z are independent parameters, it must follow that both sides are equal to some constant, which we will call λ. Then, we end up getting two separate equations:

$$\frac{1}{\rho}\frac{d^2\rho}{dr^2} + \frac{1}{\rho r}\frac{d\rho}{dr} + B^2 = \lambda$$

$$-\frac{1}{Z}\frac{d^2 Z}{dz^2} = \lambda.$$

We will solve the second equation first. We first rewrite the second equation as the eigenvalue problem:

$$\frac{d^2 Z}{dz^2} + \lambda Z = 0.$$

We assume a solution of the form $Z = Z_0 e^{kz}$ for a constant k. Then, we can rewrite the differential equation as

$$(k^2 + \lambda) Z_0 e^{kz} = 0,$$

and since we don't want $Z = 0$ (for a non-trivial solution), we instead arrive at the characteristic $k^2 + \lambda = 0$ by the zero-factor property. We then break into a case-by-case analysis based on the value of λ:

- **Case 1.** $\lambda > 0$.
  In this case, to assert that $\lambda > 0$, we set $\lambda = \theta^2$ for some $\theta \neq 0$. Then, the solutions to the characteristic equation are $k = \pm\theta$. Thus, there are two solutions to the differential equation, namely $Z_0 e^{\theta z}$ and $Z_0 e^{-\theta z}$. However, by the superposition theorem, any linear combination of these two fundamental solutions is also a solution. Thus, we can say that the solution to the differential equation is

  $$Z = c_1 Z_0 e^{\theta z} + c_2 Z_0 e^{-\theta z} = w_1 e^{\theta z} + w_2 e^{-\theta z}$$

  for some other constants w_1 and w_2.
- **Case 2.** $\lambda = 0$.
  This is the simplest case. Here, the only solution to the characteristic equation is $k = 0$. Then, we have $Z = Z_0$.
- **Case 3.** $\lambda < 0$.
  In this case, to assert that $\lambda < 0$, we set $\lambda = -\theta^2$ for some $\theta \neq 0$. Then, the solutions to the characteristic equation are $k = \pm\theta i$. Thus, there are two solutions to the differential equation, namely $Z_0 \cos\theta z$ and $Z_0 \sin\theta z$. We again appeal to the superposition theorem to arrive at the following linear combination:

  $$Z = c_1 Z_0 \cos\theta z + c_2 Z_0 \sin\theta z = w_1 \cos\theta z + w_2 \sin\theta z.$$

Now, we apply our boundary conditions to all three cases. Since $\phi(r, \widetilde{H}/2)$ holds for *all* radii r, this implies that it is a condition on z alone, i.e. we see that $Z(\widetilde{H}/2) = 0$ is the true condition. Symmetry about the cylinder also tells us that $Z(0) = -Z(\widetilde{H}) \neq 0$. We now break into our case-by-case analysis:

- **Case 1.** $\lambda > 0$.
  In this case, the second boundary condition gives us that

  $$w_1 + w_2 = -w_1 e^{\theta\widetilde{H}} - w_2 e^{-\theta\widetilde{H}}.$$

  We can rearrange this and collect like terms to arrive at the following:

  $$w_1(1 + e^{\theta\widetilde{H}}) + w_2(1 + e^{-\theta\widetilde{H}}) = 0.$$

  Solving for w_2, we get the following:

  $$w_2 = -\frac{1 + e^{\theta\widetilde{H}}}{1 + e^{-\theta\widetilde{H}}} w_1.$$

  We rewrite the original solution now as the following:

  $$w_1 \left(e^{\theta z} - \frac{1 + e^{\theta\widetilde{H}}}{1 + e^{-\theta\widetilde{H}}} e^{-\theta z} \right).$$

  We now use the first boundary condition, which gives us that

  $$w_1 \left(e^{\theta\frac{\widetilde{H}}{2}} - \frac{1 + e^{\theta\widetilde{H}}}{1 + e^{-\theta\widetilde{H}}} e^{-\theta\frac{\widetilde{H}}{2}} \right) = 0.$$

  There are two possibilities: $w_1 = 0$ and the term in parentheses is 0. We first avoid the trivial solution:

$$e^{\theta\widetilde{H}} = \frac{1+e^{\theta\widetilde{H}}}{1+e^{-\theta\widetilde{H}}}.$$

This leads to $e^{\theta\widetilde{H}} + 1 = 1 + e^{\theta\widetilde{H}}$, which is true regardless of θ. Thus, it must be the case that $w_1 = 0$ for the boundary condition to hold. This tells us that $w_2 = 0$ as well, which leads to the trivial solution $Z = 0$, so we rule out this case.

- **Case 2.** $\lambda = 0$.
  The second boundary condition leads to $Z_0 = -Z_0 = 0$, and the first boundary condition directly gives us that $Z_0 = 0$. Thus, $Z = 0$, which is the trivial solution, so we rule out this case, too.
- **Case 3.** $\lambda < 0$.
  We apply the second boundary condition to get that

$$w_1 = -w_1 \cos\theta\widetilde{H} - w_2 \sin\theta\widetilde{H}.$$

Seting corresponding parts equal, we see that $\cos\theta\widetilde{H} = -1$ and $\sin\theta\widetilde{H} = 0$. Both conditions yield $\theta\widetilde{H} = \pi$. Thus, we see that the solution is of the form

$$w_1 \cos\frac{\pi z}{\widetilde{H}} + w_2 \sin\frac{\pi z}{\widetilde{H}}.$$

We apply the first boundary condition to get that

$$w_1 \cos\frac{\pi(\widetilde{H}/2)}{\widetilde{H}} + w_2 \sin\frac{\pi(\widetilde{H}/2)}{\widetilde{H}} = 0,$$

which reduces to $w_2 = 0$. It follows that our solutions are all of the form

$$w_1 \cos\frac{\pi z}{\widetilde{H}}.$$

Thus, the solution to this part of the differential equation is $Z(z) = w_1 \cos\frac{\pi z}{\widetilde{H}}$. We now solve the *other* equation...

$$\frac{1}{\rho}\frac{d^2\rho}{dr^2} + \frac{1}{\rho r}\frac{d\rho}{dr} + B^2 = \lambda = -\theta^2 = -\frac{\pi^2}{\widetilde{H}^2}.$$

Rearranging, we have that

$$\frac{d^2\rho}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} + \left(B^2 + \frac{\pi^2}{\widetilde{H}^2}\right)\rho = 0.$$

Let us denote by $B_*^2 = B^2 + \frac{\pi^2}{\widetilde{H}^2}$ the *effective buckling* of the reactor (i.e. $B_*^2 = B^2 + \theta^2$). Then, we have that

$$\frac{d^2\rho}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} + B_*^2\,\rho = 0.$$

We solve this differential equation by assuming a series solution form: $\rho(r) = \sum_{n=0}^{\infty} a_n r^n$. We proceed:

$$\sum_{n=0}^{\infty} n(n-1)a_n r^{n-2} + \sum_{n=0}^{\infty} n a_n r^{n-2} + B_*^2 \sum_{n=0}^{\infty} a_n r^n = 0.$$

We notice that some of these sum indices are redundant, so we can rewrite this compactly:

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}r^n + \frac{a_1}{r} + \sum_{n=0}^{\infty}(n+2)a_{n+2}r^n + B_*^2 \sum_{n=0}^{\infty} a_n r^n = 0.$$

Because we require that all exponents are to the positive power, we cannot have the a_1/r term, as it would not be canceled out by any term in the series (and the RHS is homogeneous). Thus, this term cannot exist, i.e. $a_1 = 0$:

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}r^n + \sum_{n=0}^{\infty}(n+2)a_{n+2}r^n + B_*^2 \sum_{n=0}^{\infty} a_n r^n = 0.$$

We end up with the following:

$$\sum_{n=0}^{\infty}\left((n+1+1)(n+2)a_{n+2}+B_*^2a_n\right)r^n=0.$$

Since the RHS is homogeneous, we need every term to cancel out, i.e. we need the following recurrence to hold:

$$a_{n+2}=-\frac{B_*^2}{(n+2)^2}a_n.$$

Since $a_1=0$, we see that $a_3=a_5=\cdots=0$ as well, so every odd term cancels out. We are left with only the even terms. Let us say that n is $2m$ for some $m\in\mathbb{Z}^{\geq 2}$. Then, $n+2$ is $2m+2$, and we have the following recurrence:

$$a_{2m+2}=-\frac{B_*^2}{(2m+2)^2}a_{2m},$$

which we can rewrite more neatly as $a_{2m}=-\frac{B_*^2}{(2m)^2}a_{2m-2}$. Thus, we can define the entire set of coefficients in terms of some base coefficient a_0 (indices are now over the m-range). That is:

$$a_{2m}=\frac{(-1)^mB_*^{2m}}{4^m(m!)^2}a_0.$$

Then, the entire solution is simply $\rho(r)=a_0+\sum_{m=1}^{\infty}a_{2m}r^{2m}$. We compute this below:

$$\rho(r)=a_0+a_0\sum_{m=1}^{\infty}\frac{(-1)^m(B_*r)^{2m}}{4^m(m!)^2}=a_0+a_0J_0(B_*r)\approx a_0J_0(B_*r),$$

where $J_0(\cdot)$ is Bessel's function of order zero. We leave out the first a_0 because its value is infinitesimal compared to the sum of infinite terms in the series. We now combine our solutions to the general form:

$$\phi(r,z)=\rho(r)Z(z)=a_0w_1J_0(B_*r)\cos\left(\frac{\pi z}{\widetilde{H}}\right)=AJ_0(B_*r)\cos\left(\frac{\pi z}{\widetilde{H}}\right)$$

for some other constant A. As the textbook gives, $B_*=\frac{2.405}{\widetilde{R}}$, so our solution is of the form

$$\phi(r,z)=\rho(r)Z(z)=AJ_0\left(\frac{2.405r}{\widetilde{R}}\right)\cos\left(\frac{\pi z}{\widetilde{H}}\right).$$

The only thing left to do is to determine the constant A. We find the power in the reactor to determine the constant:

$$\begin{aligned}
P&=E_R\Sigma_f\int_V\phi(r,z)dV\\
&=E_R\Sigma_f\int_0^{\widetilde{H}}\int_0^{2\pi}\int_0^{\widetilde{R}}\phi(r,z)rdrd\theta dz\\
&=AE_R\Sigma_f\int_0^{2\pi}d\theta\int_0^{\widetilde{H}}\cos\left(\frac{\pi z}{\widetilde{H}}\right)dz\int_0^{\widetilde{R}}rJ_0\left(\frac{2.405r}{\widetilde{R}}\right)dr\\
&=\frac{4\widetilde{R}^2\widetilde{H}AE_R\Sigma_f}{2.405^2}\int_0^{2.405}uJ_0(u)\,du\\
&=\frac{4\widetilde{R}^2\widetilde{H}AE_R\Sigma_f}{2.405^2}\left[uJ_1(u)\right]_0^{2.405}\\
&=\frac{4\widetilde{R}^2\widetilde{H}AE_R\Sigma_f}{2.405}\cdot J_1(2.405)=\frac{4\cdot 0.5191}{2.405}\widetilde{R}^2\widetilde{H}AE_R\Sigma_f=0.8634\widetilde{R}^2\widetilde{H}AE_R\Sigma_f
\end{aligned}$$

Then, we can rearrange to solve for A:

$$A=\frac{P}{0.8634\widetilde{H}\widetilde{R}^2E_R\Sigma_f}=\frac{(\pi/0.8634)P}{\left(\pi\widetilde{R}^2\widetilde{H}\right)E_R\Sigma_f}=\boxed{\frac{3.639P}{VE_R\Sigma_f}}$$

(b) *Problem 6-21.*
The result is the $\boxed{\text{same}}$ as in Problem 6-5, as we had assumed a power of P watts in the first place, as long as it is true that $R \approx \widetilde{R}$ and $H \approx \widetilde{H}$, i.e. $d \approx 0$.

3. *Problem 6-15.*

(a) We can find the atomic density of ${}^{235}_{92}$U using conversion factors:

$$N_U = 10 \ \frac{\text{g}}{\text{L}} \left(\frac{1 \text{ L}}{1000 \text{ cm}^3}\right) \left(\frac{1 \text{ mol}}{235 \text{ g}}\right) \left(\frac{0.6022137 \times 10^{24} \text{ atoms}}{1 \text{ mol}}\right) = \boxed{2.56 \times 10^{19} \ \frac{\text{atoms}}{\text{cm}^3}}$$

Similarly, we find the molecular density of H_2O using conversion factors. In the solution, we know that there is 1 L of water per 1 L of solution (i.e. the liquid solution is all water with U dissolved inside), so we get the following:

$$\begin{aligned} N_w &= 1 \ \frac{\text{L}}{\text{L}} \left(\frac{1000 \text{ cm}^3}{1 \text{ L}}\right) \left(\frac{1 \text{ L}}{1000 \text{ cm}^3}\right) \left(\frac{1 \text{ g}}{1 \text{ cm}^3}\right) \\ &= 1 \ \frac{\text{g}}{\text{cm}^3} \left(\frac{1 \text{ mol}}{18.015 \text{ g}}\right) \left(\frac{0.6022137 \times 10^{24} \text{ atoms}}{1 \text{ mol}}\right) = \boxed{3.34 \times 10^{22} \ \frac{\text{atoms}}{\text{cm}^3}} \end{aligned}$$

(b) We can then find the fuel utilization f using the method of Problem 6-1:

$$f = \frac{1}{1 + \frac{\Sigma_a^w}{\Sigma_a^U}} = \frac{1}{1 + \frac{N_w \sigma_a^w}{N_U g_a^{20^\circ} \sigma_a^U}} = \frac{1}{1 + \frac{(3.34 \times 10^{22} \text{ atoms/cm}^3)(0.664 \times 10^{-24} \text{ cm}^2)}{(2.56 \times 10^{19} \text{ atoms/cm}^3)(0.9780)(681 \times 10^{-24} \text{ cm}^2)}} = \boxed{0.435}$$

using the values for σ_a given in Example 6.10, part 2 and $g_a(20°\text{C}) = 0.9780$ from Table 3.2.

(c) The thermal diffusion area without a moderator L_{TM}^2 is given by

$$\begin{aligned} L_{TM}^2 &= \frac{D}{\Sigma_a} = \frac{2\overline{D_w}}{\Sigma_a^U + \Sigma_a^w} = \frac{2\overline{D_w}}{N_U g_a^{20^\circ} \sigma_a^U + N_w \sigma_a^w} \\ &= \frac{2(0.16 \text{ cm})}{(2.56 \times 10^{19} \text{ atoms/cm}^3)(0.9780)(681 \times 10^{-24} \text{ cm}^2) + (3.34 \times 10^{22} \text{ atoms/cm}^3)(0.664 \times 10^{-24} \text{ cm}^2)} \\ &= 8.15 \text{ cm}^2 \end{aligned}$$

where we used values from Table 3.2, Table 5.2, and Table 6.1. We can now find the true thermal diffusion area:

$$L_T^2 = (1 - f)(L_{TM}^2) = (1 - 0.435)(8.15 \text{ cm}^2) = \boxed{4.60 \text{ cm}^2}$$

Then, the thermal diffusion length is simply

$$L_T = \sqrt{L_T^2} = \boxed{2.14 \text{ cm}^2}$$

(d) We lastly find the infinite multiplication factor as follows:

$$k_\infty = \eta_U f = (2.065)(0.435) = \boxed{0.898}$$

where we used η_U at 20 °C from Table 6.3. Since $k_\infty < 1$, we see that the reactor is $\boxed{\text{subcritical}}$.

4. *Problem 6-19.*

We recall that in an infinite homogeneous mixture of fuel in moderator, the infinite multiplication factor is given by

$$k_\infty = \eta_T f,$$

where η_T is the nominal one-group constant for a fast reactor. We rearrange this as $f = \frac{k_\infty}{\eta_T}$. However, we know that $f = \frac{\Sigma_a^F}{\Sigma_a} = \frac{\Sigma_a^F}{\Sigma_a^F + \Sigma_a^S}$, which we can rewrite as $f = \frac{1}{1 + \frac{\Sigma_a^S}{\Sigma_a^F}} = \frac{1}{1 + 1/Z}$, where Z is the one-group ratio introduced in Section 6.6. Solving for Z:

$$\begin{aligned} f &= \frac{1}{1 + 1/Z} = \frac{k_\infty}{\eta_T} \\ Z &= \frac{k_\infty}{\eta_T - k_\infty} \end{aligned}$$

We recall from Equation 5.59 that $\Sigma_a^F = \frac{\sqrt{\pi}}{2} g_a(T_0)\Sigma_a^F(E_0)\sqrt{\frac{T_0}{T}}$ at some other temperature T, where $T_0 = 293.61$ K is room temperature and $E_0 = 0.0253$ eV. In our case, we want the value at $T = T_0$, so we expand everything:

$$\Sigma_a^F = \frac{\sqrt{\pi}}{2} g_a^F(T_0)\Sigma_a^F(E_0)\sqrt{\frac{T_0}{T}} = \frac{\sqrt{\pi}}{2} g_a^{20^\circ} N_F \sigma_a^F.$$

However, we know from the previous page that $Z = \Sigma_a^F/\Sigma_a^S$, so we can equally rewrite this as

$$\Sigma_a^F = \frac{\sqrt{\pi}}{2} g_a^{20^\circ} N_F \sigma_a^F = Z\Sigma_a^S.$$

Now, we can solve for N_F in terms of the other quantitites:

$$N_F = \frac{2}{\sqrt{\pi}}\left(\frac{Z\Sigma_a^S}{g_a^{20^\circ}\sigma_a^F}\right) = \frac{2}{\sqrt{\pi}}\left(\frac{k_\infty}{\eta_F - k_\infty}\right)\left(\frac{\Sigma_a^S}{g_a^{20^\circ}\sigma_a^F}\right),$$

where Σ_a^S is the macroscopic absorption cross-section of the moderator (the solution). We know that this reactor is critical, so $k_\infty = 1$. The other factors are moderator-dependent, so we do individual calculations.

(a) We find N_F for ^{235}U in various moderators. All values are from Table 3.2, Table 5.2, Table 6.3, and Tables II.2/3.

- *Moderator:* H_2O. Note the correction factor of $\sqrt{\pi}/2$ for Σ_a^w to offset the outer multiplication of $2/\sqrt{\pi}$.

$$\begin{aligned} N_U &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{\eta_U - 1}\right)\left(\frac{\Sigma_a^w}{g_a^{20^\circ}\sigma_a^U}\right) \\ &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2.065 - 1}\right)\left(\frac{(\sqrt{\pi}/2)(0.02220 \text{ cm}^{-1})}{(0.9780)(687.0 \times 10^{-24} \text{ cm}^2)}\right) \\ &= 3.10 \times 10^{19} \frac{\text{atoms}}{\text{cm}^3}\left(\frac{1 \text{ mol}}{0.6022137 \times 10^{24} \text{ atoms}}\right)\left(\frac{235 \text{ g}}{1 \text{ mol}}\right)\left(\frac{1000 \text{ cm}^3}{1 \text{ L}}\right) \\ &= \boxed{12.1 \frac{\text{g}}{\text{L}}} \end{aligned}$$

- *Moderator:* D_2O. We assume that it contains H_2O at 25 *w/o*.

$$\begin{aligned} N_U &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{\eta_U - 1}\right)\left(\frac{\Sigma_a^D}{g_a^{20^\circ}\sigma_a^U}\right) \\ &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2.065 - 1}\right)\left(\frac{9.3 \times 10^{-5} \text{ cm}^{-1}}{(0.9780)(687.0 \times 10^{-24} \text{ cm}^2)}\right) \\ &= 1.47 \times 10^{17} \frac{\text{atoms}}{\text{cm}^3}\left(\frac{1 \text{ mol}}{0.6022137 \times 10^{24} \text{ atoms}}\right)\left(\frac{235 \text{ g}}{1 \text{ mol}}\right)\left(\frac{1000 \text{ cm}^3}{1 \text{ L}}\right) \\ &= \boxed{0.0572 \frac{\text{g}}{\text{L}}} \end{aligned}$$

- *Moderator: graphite* (^{12}C).

$$\begin{aligned} N_U &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{\eta_U - 1}\right)\left(\frac{\Sigma_a^C}{g_a^{20^\circ}\sigma_a^U}\right) \\ &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2.065 - 1}\right)\left(\frac{2.4 \times 10^{-4} \text{ cm}^{-1}}{(0.9780)(687.0 \times 10^{-24} \text{ cm}^2)}\right) \\ &= 3.78 \times 10^{17} \frac{\text{atoms}}{\text{cm}^3}\left(\frac{1 \text{ mol}}{0.6022137 \times 10^{24} \text{ atoms}}\right)\left(\frac{235 \text{ g}}{1 \text{ mol}}\right)\left(\frac{1000 \text{ cm}^3}{1 \text{ L}}\right) \\ &= \boxed{0.148 \frac{\text{g}}{\text{L}}} \end{aligned}$$

(b) We find N_F for ^{239}Pu in various moderators. All values are from Table 3.2, Table 5.2, Table 6.3, and Tables II.2/3.

- *Moderator:* H_2O. Note the correction factor of $\sqrt{\pi}/2$ for Σ_a^w to offset the outer multiplication of $2/\sqrt{\pi}$.

$$\begin{aligned}
N_{Pu} &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{\eta_{Pu}-1}\right)\left(\frac{\Sigma_a^w}{g_a^{20^\circ}\sigma_a^{Pu}}\right) \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2.035-1}\right)\left(\frac{(\sqrt{\pi}/2)(0.02220\ \text{cm}^{-1})}{(1.0723)(1020\times 10^{-24}\ \text{cm}^2)}\right) \\
&= 1.96\times 10^{19}\ \frac{\text{atoms}}{\text{cm}^3}\left(\frac{1\ \text{mol}}{0.6022137\times 10^{24}\ \text{atoms}}\right)\left(\frac{239\ \text{g}}{1\ \text{mol}}\right)\left(\frac{1000\ \text{cm}^3}{1\ \text{L}}\right) \\
&= \boxed{7.78\ \frac{\text{g}}{\text{L}}}
\end{aligned}$$

- *Moderator:* D_2O. We assume that it contains H_2O at 25 w/o.

$$\begin{aligned}
N_{Pu} &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{\eta_{Pu}-1}\right)\left(\frac{\Sigma_a^D}{g_a^{20^\circ}\sigma_a^{Pu}}\right) \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2.035-1}\right)\left(\frac{9.3\times 10^{-5}\ \text{cm}^{-1}}{(1.0723)(1020\times 10^{-24}\ \text{cm}^2)}\right) \\
&= 9.27\times 10^{16}\ \frac{\text{atoms}}{\text{cm}^3}\left(\frac{1\ \text{mol}}{0.6022137\times 10^{24}\ \text{atoms}}\right)\left(\frac{239\ \text{g}}{1\ \text{mol}}\right)\left(\frac{1000\ \text{cm}^3}{1\ \text{L}}\right) \\
&= \boxed{0.0368\ \frac{\text{g}}{\text{L}}}
\end{aligned}$$

- *Moderator: graphite* (^{12}C).

$$\begin{aligned}
N_{Pu} &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{\eta_{Pu}-1}\right)\left(\frac{\Sigma_a^C}{g_a^{20^\circ}\sigma_a^{Pu}}\right) \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2.035-1}\right)\left(\frac{2.4\times 10^{-4}\ \text{cm}^{-1}}{(1.0723)(1020\times 10^{-24}\ \text{cm}^2)}\right) \\
&= 2.39\times 10^{17}\ \frac{\text{atoms}}{\text{cm}^3}\left(\frac{1\ \text{mol}}{0.6022137\times 10^{24}\ \text{atoms}}\right)\left(\frac{239\ \text{g}}{1\ \text{mol}}\right)\left(\frac{1000\ \text{cm}^3}{1\ \text{L}}\right) \\
&= \boxed{0.0949\ \frac{\text{g}}{\text{L}}}
\end{aligned}$$

5. (a) *Problem 6-23.*
We consider the case of a critical bare cubical reactor of side length a, as shown below.

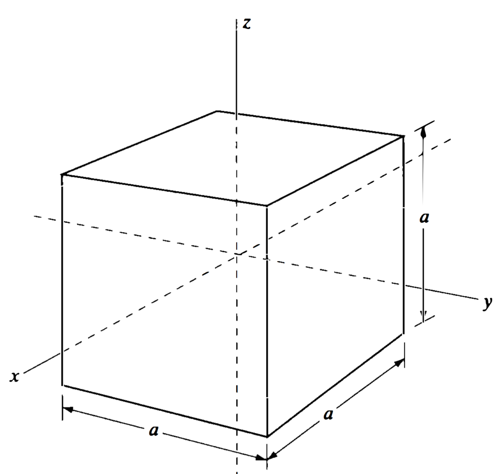

Figure 2: A critical bare cubical reactor (bottom-left corner at the origin) of side length a.

We begin with the one-group reactor equation $\nabla^2\phi + B^2\phi = 0$, where B represents the buckling of the reactor. Since we are working with a Cartesian coordinate system, it makes most sense to write the equation in them:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + B^2 \phi = 0.$$

We now consider boundary conditions on the flux in this reactor. Just as in Problem 6-5, the flux at the midpoint of each face of the cube must vanish. That is, we have the following:

$$\phi\left(\frac{\widetilde{a}}{2}, y, z\right) = \phi\left(x, \frac{\widetilde{a}}{2}, z\right) = \phi\left(x, y, \frac{\widetilde{a}}{2}\right) = 0.$$

Similarly, symmetry about the cube gives us the following three constraints:

$$\phi(0, y, z) = -\phi(\widetilde{a}, y, z) \neq 0$$
$$\phi(x, 0, z) = -\phi(x, \widetilde{a}, z) \neq 0$$
$$\phi(x, y, 0) = -\phi(x, y, \widetilde{a}) \neq 0.$$

We can now propose using the method of separation of variables that ϕ is of the form $\phi(x, y, z) = R(x, y)Z(z)$, where R and Z are independent functions. We now break apart B into its x-, y-, and z-components, i.e. $B^2 = B_x^2 + B_y^2 + B_z^2$ for the buckling x-component B_x, y-component B_y, and z-component B_z. Then, we get:

$$\frac{\partial^2 R}{\partial x^2} Z + \frac{\partial^2 R}{\partial y^2} Z + R \frac{d^2 Z}{dz^2} + B^2 R Z = 0.$$

Dividing through by RZ and rearranging, we get the following:

$$\frac{1}{R}\frac{\partial^2 R}{\partial x^2} + \frac{1}{R}\frac{\partial^2 R}{\partial y^2} + B_x^2 + B_y^2 = -\frac{1}{Z}\frac{d^2 Z}{dz^2} - B_z^2.$$

Since R and Z are both independent functions, the only way the LHS and RHS are equal is if they are both equal to some other constant, which we we will call m. Then, we end up getting two separate equations:

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2}\right) + B_x^2 + B_y^2 = m$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} + B_z^2 = -m.$$

We first rewrite the second equation as $\frac{d^2 Z}{dz^2} + (B_z^2 + m)Z = 0$ or $\frac{d^2 Z}{dz^2} + tZ = 0$ for some other constant t. We then break up the first equation, which can be rewritten as follows:

$$\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + (B_x^2 + B_y^2 - m)R = 0.$$

We again use separation of variables, assuming that $R(x, y)$ is of the form $R(x, y) = X(x)Y(y)$, where X and Y are again independent functions. Then, the differential equation becomes

$$\frac{d^2 X}{dx^2} Y + X \frac{d^2 Y}{dy^2} + (B_x^2 + B_y^2 - m)XY = 0.$$

Dividing through by XY and rearranging, we get the following:

$$\frac{1}{X}\frac{d^2 X}{dx^2} + B_x^2 = m - \frac{1}{Y}\frac{d^2 Y}{dy^2} - B_y^2.$$

Again, both sides are equal to some other constant k. Then, we end up getting two more equations:

$$\frac{d^2 X}{dx^2} + (B_x^2 - k)X = 0$$

$$\frac{d^2 Y}{dy^2} + (B_y^2 + k - m)Y = 0.$$

We can rewrite these as

$$\frac{d^2X}{dx^2} + rX = 0$$

$$\frac{d^2Y}{dy^2} + sY = 0$$

for some other constants r and s. Note that $r + s + t = B_x^2 + B_y^2 + B_z^2 = B^2$. We now convert all of our boundary conditions into equivalent conditions for X, Y, and Z:

$$X\left(\frac{\widetilde{a}}{2}\right) = Y\left(\frac{\widetilde{a}}{2}\right) = Z\left(\frac{\widetilde{a}}{2}\right) = 0$$

$$X(0) = -X(\widetilde{a}) \neq 0$$

$$Y(0) = -Y(\widetilde{a}) \neq 0$$

$$Z(0) = -Z(\widetilde{a}) \neq 0.$$

These differential equations with these boundary conditions are exactly identical to those that we explored in Problem 6-5 earlier. Thus, we already have our general solution (there is no point repeating the same analysis):

$$X(x) = w_1 \cos\frac{\pi x}{\widetilde{a}}$$

$$Y(y) = w_2 \cos\frac{\pi y}{\widetilde{a}}$$

$$Z(z) = w_3 \cos\frac{\pi z}{\widetilde{a}}.$$

We thus know our overall solution:

$$\phi(x,y,z) = X(x)Y(y)Z(z) = \left(w_1 \cos\frac{\pi x}{\widetilde{a}}\right)\left(w_2 \cos\frac{\pi y}{\widetilde{a}}\right)\left(w_3 \cos\frac{\pi z}{\widetilde{a}}\right) = w_1 w_2 w_3 \cos\frac{\pi x}{\widetilde{a}} \cos\frac{\pi y}{\widetilde{a}} \cos\frac{\pi z}{\widetilde{a}},$$

which we can rewrite as

$$\boxed{\phi(x,y,z) = A\cos\frac{\pi x}{\widetilde{a}} \cos\frac{\pi y}{\widetilde{a}} \cos\frac{\pi z}{\widetilde{a}}}$$

for some other constant A. We notice that the flux is symmetric in all directions, as expected.

(b) *Problem 6-24.*

We use the method of Problem 6-5 again, noting that if the reactor is operating at a power of P watts, this power is related to the flux ϕ via the full real-volume integral:

$$\begin{aligned}
P &= E_R \Sigma_f \int_V \phi(x,y,z) dV \\
&= A E_R \Sigma_f \int_0^a \cos\frac{\pi x}{\widetilde{a}} dx \int_0^a \cos\frac{\pi z}{\widetilde{a}} dy \int_0^a \cos\frac{\pi z}{\widetilde{a}} dz \\
&= A E_R \Sigma_f \left[\int_0^a \cos\frac{\pi p}{\widetilde{a}} dp\right]^3 \\
&= A E_R \Sigma_f \left[\frac{\widetilde{a}}{\pi} \int_0^{a\pi/\widetilde{a}} \cos u \, du\right]^3 \\
&= A E_R \Sigma_f \left[\frac{2\widetilde{a}}{\pi} \int_0^{a\pi/2\widetilde{a}} \cos u \, du\right]^3 \\
&= A E_R \Sigma_f \frac{8\widetilde{a}^3}{\pi^3} \left[\sin u\Big|_0^{a\pi/2\widetilde{a}}\right]^3 = A E_R \Sigma_f \frac{8\widetilde{a}^3}{\pi^3} \sin^3\left(\frac{\pi a}{2\widetilde{a}}\right)
\end{aligned}$$

Thus, we can solve for the constant A, which gives us $\boxed{A = \frac{\pi^3 P}{8 E_R \Sigma_f \widetilde{a}^3 \sin^3(\pi a / 2\widetilde{a})}}$

(c) *Problem 6-25.*

i. In this case, we wish to solve for the critical boundaries, given the composition. In particular, we are given that the ratio $\beta = n_F/n_S$ is 1.0×10^{-5}, where n_F is the atomic composition of the uranium-235 fuel and n_S is the atomic composition of the moderator in the solution/mixture, which is graphite. Now, we turn to Section 6.6 and examine Case 2, which describes the case where the atomic density composition is specified:

$$B^2 = \frac{k_\infty - 1}{M_T^2},$$

where $M_T^2 = L_T^2 + \tau_T$ and τ_T is the neutron age. Also, for a cubical reactor of side length a, we know that $\widetilde{a} = \dfrac{\pi\sqrt{3}}{B}$ as described later in Section 6.6. Thus, we can rewrite this as

$$\widetilde{a} = \pi\sqrt{\frac{3(L_T^2 + \tau_T)}{k_\infty - 1}} = \pi\sqrt{\frac{3\left(L_{TM}^2(1-f) + \tau_T\right)}{k_\infty - 1}}.$$

As we saw in Problem 6-19, we can write k_∞ as $k_\infty = \eta_T f$, where $f = \dfrac{1}{1+1/Z}$ and $Z = \dfrac{\Sigma_a^F}{\Sigma_a^S}$. Before doing any computation, we realize that we can rewrite $1 - f$ as

$$1 - f = 1 - \frac{1}{1+1/Z} = \frac{1}{Z+1},$$

which makes our work slightly easier, as now $\widetilde{a} = \pi\sqrt{\dfrac{3\left(\dfrac{L_{TM}^2}{Z+1} + \tau_T\right)}{k_\infty - 1}}$. However, since $k_\infty = \eta_T f$, we see that

$$k_\infty - 1 = \eta_T f - 1 = \frac{\eta_T Z}{Z+1} - 1 = (\eta_T - 1)\left(\frac{Z-1}{Z+1}\right).$$

Thus, we can rewrite $\widetilde{a}$ as

$$\widetilde{a} = \pi\sqrt{\frac{3\left(L_{TM}^2 + \tau_T(Z+1)\right)}{(\eta_T - 1)(Z-1)}}.$$

Now, we write an equation for Z:

$$Z = \frac{\Sigma_a^F}{\Sigma_a^S} = \frac{\frac{\sqrt{\pi}}{2} g_a^F(T)\Sigma_a^F(E_0)\sqrt{\frac{T_0}{T}}}{\frac{\sqrt{\pi}}{2}\Sigma_a^S(E_0)\sqrt{\frac{T_0}{T}}} = \frac{g_a^F(T)\Sigma_a^F(E_0)}{\Sigma_a^S(E_0)} = \frac{g_a^F(T) N_F \sigma_a^F(E_0)}{N_S \Sigma_a^S(E_0)} = \frac{N_F}{N_S}\frac{g_a^F(T)\sigma_a^F(E_0)}{\sigma_a^S(E_0)} = \beta g_a^F(T)\frac{\sigma_a^F(E_0)}{\sigma_a^S(E_0)}.$$

Using $T = 250\ ^\circ\text{C} = 523.15$ K and $E_0 = 0.0253$ eV, we get:

$$Z = (1.0 \times 10^{-5})(0.9402)\left(\frac{687.0 \times 10^{-24}\ \text{cm}^2}{0.0034 \times 10^{-24}\ \text{cm}^2}\right) = 1.90,$$

where we used values from Table 3.2[1] and Table II.3.

We now find L_{TM} at T (instead of T_0) using the adjustment formula from Chapter 5:

$$L_T^2(\rho, T) = L_T^2(\rho_0, T_0)\left(\frac{\rho_0}{\rho}\right)^2\sqrt{\frac{T}{T_0}}.$$

However, since $\beta \ll 1$, we see that $n_F \ll n_S$, i.e. the mixture is almost purely graphite. Thus, the density of the mixture is approximately that of pure graphite, or $\rho \approx \rho_0$. Then:

$$L_{TM}^2(\rho, T) \approx L_{TM}^2(\rho_0, T_0)\sqrt{\frac{T}{T_0}} = (3500\ \text{cm}^2)\sqrt{\frac{523.15\ \text{K}}{293.61\ \text{K}}} = 4672\ \text{cm}^2,$$

where we used values from Table 5.2. Then, we can finally solve for $\widetilde{a}$:

[1] Using the extended table on Page 21 of "Effective Cross-Section Values for Well-Moderated Thennal Reactor Spectra", C.H. Westcott.

$$\widetilde{a} = \pi\sqrt{\frac{3\left(4672\text{ cm}^2 + (368\text{ cm}^2)(1.90+1)\right)}{(2.065-1)(1.90-1)}} = \boxed{410\text{ cm}}$$

where we used values from Table 5.3 and Table 6.3.

ii. The critical mass is based on the full real-volume, so we must first compute a from $\widetilde{a}$:

$$a = \widetilde{a} - 2d = \widetilde{a} - 2(2.13\overline{D}) = 411\text{ cm} - 4.26(0.84\text{ cm}) = 406\text{ cm},$$

where we used $\overline{D}$ from Table 5.2. We can now find the atomic density of U as follows:

$$N_U = N_F = \beta N_S = \beta N_C = (1.0\times10^{-5})\left(0.08023\times10^{24}\ \frac{\text{atoms}}{\text{cm}^3}\right) = 0.08023\times10^{19}\ \frac{\text{atoms}}{\text{cm}^3},$$

where we used values from Table II.3. We can represent this quantity as mass density instead:

$$0.08023\times10^{19}\ \frac{\text{atoms}}{\text{cm}^3}\left(\frac{1\text{ mol}}{0.6022137\times10^{24}\text{ atoms}}\right)\left(\frac{235\text{ g}}{1\text{ mol}}\right) = 3.13\times10^{-4}\ \frac{\text{g}}{\text{cm}^3}.$$

Finally, we find the critical mass:

$$m_U = \rho V = \rho a^3 = \left(3.13\times10^{-4}\ \frac{\text{g}}{\text{cm}^3}\right)(406\text{ cm})^3\left(\frac{1\text{ kg}}{1000\text{ g}}\right) = \boxed{20.9\text{ kg}}$$

iii. We recall from Problems 6-23 and 6-24 that the flux is given by

$$\phi(x,y,z) = \frac{\pi^3 P}{8E_R\Sigma_f\widetilde{a}^3\sin^3(\pi a/2\widetilde{a})}\cos\frac{\pi x}{\widetilde{a}}\cos\frac{\pi y}{\widetilde{a}}\cos\frac{\pi z}{\widetilde{a}}.$$

We can maximize this by choosing x, y, and z so that each of the cosine terms is maximized. This occurs principally when $x = y = z = 0$, so each cosine term is unity. Then, we can simply compute the rest. We first find Σ_f for the uranium-235:

$$\Sigma_f^U(T) = \frac{\sqrt{\pi}}{2}g_f^U(T)N_U\sigma_f^U(T_0)\sqrt{\frac{T_0}{T}},$$

where we have $T = 250°\text{C} = 523.15$ K and $T_0 = 293.61$ K. Then:

$$\Sigma_f^U = \frac{\sqrt{\pi}}{2}(0.9346)\left(0.08023\times10^{19}\ \frac{\text{atoms}}{\text{cm}^3}\right)(587\times10^{-24}\text{ cm}^2)\sqrt{\frac{293.61\text{ K}}{523.15\text{ K}}} = 2.92\times10^{-4}\text{ cm}^{-1}.$$

Finally, we find the maximum flux assuming that a recoverable energy of 200 MeV:

$$\phi_{max} = \phi(0,0,0) = \frac{\pi^3(1\text{ kW})}{8(200\text{ MeV})\left(\frac{1.602\times10^{-19}\text{ J}}{1\text{ eV}}\right)(2.92\times10^{-4}\text{ cm}^{-1})(410\text{ cm})^3\sin^3\left(\frac{\pi(406\text{ cm})}{2(410\text{ cm})}\right)}(1)(1)(1)$$

which evaluates to

$$\phi_{max} = \boxed{6.01\times10^9\ \frac{\text{neutrons}}{\text{cm}^2\cdot\text{sec}}}$$