COS 423: Homework Assignment

due Wed 04/4/2012

Problem 1

Consider an arrangement of n lines in the plane. (You may assume that no line is vertical and no three of them intersect in a point.) Assume that each cell (ie, 2D face) is labeled 1 through N.

1. What is the value of N as a function of n? (Is it always the same for a given n?)
2. Some cells are unbounded (ie, they go to infinity). How many of those must there be?
3. Prove that the number of vertices of the arrangement that lie on any of the unbounded cells is $O(n)$?
4. Our goal is to preprocess the arrangement in $O(n^2)$ time and space, so that, given a query point, we can find which cell contains it in time $O(\log n)$. Hint: First find an $O(\log^2 n)$ time solution based on a judicious partition of the cells into classes, and then apply fractional cascading to it.

Problem 2

Suppose that someone shows you three matrices, A, B, C with elements in GF(2) and claims that $AB = C$. You can check the truth of that claim in $O(n^3)$ time by multiplying the two matrices. (You can actually do a little better but never mind.) Show how to do the checking with a randomized algorithm that takes $O(n^2)$ time and makes an error with probability less than one over the number of particles in the universe.

Problem 3

Prove that it is possible to place n points in the unit square so that none of the $\binom{n}{3}$ triangles they form has an area smaller than c/n^2 for a suitable small constant $c > 0$. To prove this, throw $2n$ points at random and clean up by removing all points involved in a triangle of area smaller than c/n^2. The key is to prove that the cleanup does not remove too many points.

Problem 4

Recall the randomized median-finding algorithm we discussed in class. Roughly, we pick a subset of $\lceil n^{3/4} \rceil$ elements from S and sort it. We identify the two elements ranked (roughly) $\frac{1}{2}n^{3/4} - \sqrt{n}$ and $\frac{1}{2}n^{3/4} + \sqrt{n}$ and locate them in S. Finally we extract the elements from S between these two elements. If there are more than $cn^{3/4}$ of them, for a large enough constant c, stop. If the median of S is not among them, stop. Otherwise, sort that set and find the median. Prove, using a second-moment method, that this algorithm produces the median element of S in $O(n)$ time with probability at least $1 - O(n^{-1/4})$.

Problem 5

Given an undirected graph G with n nodes, we denote by d_i the degree of the i-th one. The independence number $I(G)$ is defined as the maximum number of nodes with no edges joining them. We want to prove that

$$I(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

To do that, consider the following randomized algorithm: order the n nodes at random and, for each one in that order, keep it only if none of its neighbors has been visited. Prove that this produces an independent set and bound its expected size.