

Discrepancy Bounds for Geometric Set Systems with Square Incidence Matrices

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Abstract

Alexander has proven the existence of a set of n points in the plane such that, given any two-coloring of the points, there exists a halfplane within which one color outnumbers the other by $\Omega(n^{1/4})$. We strengthen this result by showing that the halfplane can be chosen among n fixed ones. In other words, we build a point/halfplane set system of discrepancy $\Omega(n^{1/4})$, whose incidence matrix is $n \times n$. By a result of Matoušek, this lower bound is tight.

The second result is an $n \times n$ variant of a classical lower bound of Roth on the discrepancy of arithmetic progressions. Stated in dual form, our result asserts the existence of a set system of discrepancy $\Omega(n^{1/4})$, whose $n \times n$ incidence matrix $(a_{i,j})$ is formed as follows: each row (resp. column) corresponds to a line segment (resp. horizontal line) in the plane, and $a_{i,j} = 1$ if segment i and line j intersect in an integer point. Matoušek and Spencer have shown the lower bound to be tight.

1 Introduction

Schmidt [8] has shown the existence of n points in the plane such that, given any two-coloring of the points, there is always an axis-parallel box within which one color outnumbers the other by $\Omega(\log n)$. It is natural to ask the question: can the box be chosen among a small set of prespecified boxes? Given an incidence matrix A of a set system, we define the discrepancy of A ,

$$D(A) = \left\{ \min \|Ax\|_\infty : x \in \{-1, 1\} \right\}.$$

Is there an $n \times n$ incidence matrix for boxes, $A = (a_{i,j})$, of discrepancy $\Omega(\log n)$, such that $a_{i,j} = 1$ if the box associated with row i contains the point associated with column j ? As it happens, the answer is trivially affirmative. Here is why: the set of candidate boxes for Schmidt's bound can obviously be restricted to $O(n^4)$. So, by applying the bound for $n' = O(n^{1/4})$ points, we can make the number of candidates equal to n . Next, we form an $n \times n$ incidence matrix A by filling in $n - n'$ columns of zeroes derived from the addition of $n - n'$ dummy points. The discrepancy of the resulting set system is $\Omega(\log n') = \Omega(\log n)$. Of course,

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this trick does not work if the discrepancy is of the form n^α , for some constant α . By using a more subtle argument, we prove the following:

Theorem 1.1 *There exist n points and n halfplanes in \mathbf{R}^2 , such that the $n \times n$ incidence matrix $A = (a_{i,j})$ has discrepancy $D(A) = \Omega(n^{1/4})$, where $a_{i,j} = 1$ if and only if the i -th halfplane contains the j -th point.*

Theorem 1.2 *There exist n horizontal lines and n segments in \mathbf{R}^2 , such that the $n \times n$ incidence matrix $A = (a_{i,j})$ has discrepancy $D(A) = \Omega(n^{1/4})$, where each row (resp. column) corresponds to a segment (resp. horizontal line), and $a_{i,j} = 1$ if and only if segment i and line j intersect in an integer point.*

Both theorems are optimal. This follows from results by Matoušek [6] and Matoušek and Spencer [7], respectively. It remains an open question whether the same bounds hold for $m \times n$ matrices, where m is asymptotically smaller than n .

2 The Proof of Theorem 1.1

Let P be the set of n integer points in $[1, \sqrt{n}]^2$ (assume that n is a large square), and let x be a vector in \mathbf{R}^n whose i -th coordinate x_i is associated with $p_i \in P$. Given a closed halfplane h bounded above by a nonvertical line, we define $f(h) = \sum_{p_i \in h} x_i$. Let ω be the motion-invariant measure for lines, normalized so as to provide a probability measure for the lines crossing the square $[1, \sqrt{n}]^2$. Alexander [1] has proven that if $x_1 + \dots + x_n = 0$, then

$$\int f^2(h) d\omega(h) = \Omega(1/\sqrt{n}) \|x\|_2^2.$$

For completeness, we repeat (with only a few modifications) the argument we used in [4] to discretize Alexander's result and derive Lemma 2.1 below. We subdivide the space of lines crossing $[1, \sqrt{n}]^2$ into $N + O(n^2)$ regions within which $f(h)$ remains invariant. By choosing n and N large enough, say, $N = 2^n$, we can easily ensure that the ω -area σ of N of these regions is exactly the same, ie, about $1/N$, while the other $O(n^2)$ regions have smaller areas. Thus, the error produced in computing $\int f^2(h) d\omega(h)$ by integrating f^2 only over the equal-area regions is $O(n^2/N) \sup f^2$. Because $|f|$ cannot exceed

$$|x_1| + \dots + |x_n| \leq \sqrt{n} \|x\|_2,$$

this error is bounded by $O(n^3 \|x\|_2^2 / N)$. Let B be the $N \times n$ matrix whose rows are indexed by the N equal-area regions $\hat{\sigma}$ and are the characteristic vectors of the set of x_i 's appearing in (the unique form) $f(h)$, for $h \in \hat{\sigma}$. We have

$$\left| \|Bx\|_2^2 - \frac{1}{\sigma} \int f^2(h) d\omega(h) \right| = O(n^3) \frac{\|x\|_2^2}{N\sigma},$$

and because $\sigma = 1/N \pm O(n^2/N^2)$,

$$\left| \|Bx\|_2^2 - N \int f^2(h) d\omega(h) \right| = O(n^3 \|x\|_2^2).$$

Lemma 2.1 [4]

$$\det B^T B = \Omega\left(N/\sqrt{n}\right)^{n-1}.$$

Proof: Let $\mu_1 \geq \dots \geq \mu_n \geq 0$ be the eigenvalues of $B^T B$ and let $\{v_i\}$ be an orthonormal eigenbasis, with μ_i corresponding to v_i . Let (ξ_1, \dots, ξ_n) be the coordinates of x in the basis $\{v_i\}$. The solution space of the system of equations, $x_1 + \dots + x_n = 0$ and $\xi_j = 0$ ($j < n-1$), is of dimension at least 1. It lies in the (ξ_{n-1}, ξ_n) -plane, so it intersects the cylinder $\xi_{n-1}^2 + \xi_n^2 = 1$. For any point x of the intersection,

$$\|Bx\|_2^2 = \sum_{i=1}^n \mu_i \xi_i^2 = \mu_{n-1} \xi_{n-1}^2 + \mu_n \xi_n^2 \leq \mu_{n-1}.$$

This implies that for the unit vector x ,

$$\mu_{n-1} \geq N \int f^2(h) d\omega(h) - O(n^3 \|x\|_2^2) \geq \Omega(N/\sqrt{n}) - O(n^3),$$

and hence,

$$\mu_{n-1} \geq \Omega(N/\sqrt{n}). \quad (1)$$

We need a lower bound on the smallest eigenvalue. With N being large enough, we can always assume that, for each point p_i , there exist two lines (adding them on, if necessary, and updating N accordingly), each represented by a distinct row of B , that pass right above and below p_i . The contribution of these two rows to $\|Bx\|_2^2$ is of the form $\Phi^2 + (\Phi + x_i)^2$, which is always at least $x_i^2/2$. It follows that $\|Bx\|_2^2 \geq \frac{1}{2} \|x\|_2^2$, and hence, $\mu_n \geq 1/2$. The lemma follows from (1) and the fact that $\det B^T B$ is the product of the eigenvalues. \square

By the Binet-Cauchy formula,¹

$$\det B^T B = \sum_{1 \leq j_1 < \dots < j_n \leq N} \left| \det B \begin{pmatrix} j_1 & j_2 & \dots & j_n \\ 1 & 2 & \dots & n \end{pmatrix} \right|^2.$$

Therefore, there exists an $n \times n$ submatrix A of B such that

$$\begin{aligned} (\det A)^2 &= \left| \det B \begin{pmatrix} j_1 & j_2 & \dots & j_n \\ 1 & 2 & \dots & n \end{pmatrix} \right|^2 \\ &\geq \binom{N}{n}^{-1} \det B^T B = \Omega(1)^n \left(\frac{n}{eN}\right)^n \left(\frac{N}{\sqrt{n}}\right)^{n-1} \\ &\geq (cn)^{n/2}, \end{aligned}$$

for some fixed $c > 0$. Lovász et al. [5] define the hereditary discrepancy, $D^H(A)$, of the incidence matrix A to be the maximum value of $D(A')$ over all matrices A' formed by subsets of the columns of A . They prove that

$$D^H(A) = \Omega(|\det A|^{1/n}).$$

In our case, this implies that

$$D^H(A) = \Omega(n^{1/4}).$$

Let A' be the (or any) submatrix of A that achieves the hereditary discrepancy, and let M be the matrix derived from A by zeroing out the columns not in the submatrix A' . By introducing

¹The notation following $\det B$ refers to the matrix obtained by picking the rows indexed j_1, \dots, j_n in B .

artificial points if necessary, we can make M the incidence matrix of a point/halfplane set system, whose discrepancy is thus $\Omega(n^{1/4})$. This completes the proof of Theorem 1.1. \square

For reference it might be useful to make a general lemma out of the technique we just used.

Lemma 2.2 *If B is an $N \times n$ incidence matrix of a set system, then there exists an $n \times n$ matrix A formed by n rows of B , such that*

$$D^H(A) \geq c \sqrt{\frac{n}{N}} \left(\det B^T B \right)^{1/2n},$$

for some constant $c > 0$.

3 The Proof of Theorem 1.2

Let B be the $N \times n$ incidence matrix of the following set system: each set is an arithmetic progression modulo m , of length k and difference at most $6k$, where $k = \lfloor \sqrt{n/6} \rfloor$; note that $N = O(n\sqrt{n})$. By adapting an argument of Roth [9], Beck and Sós [3] have shown that the matrix $B^T B$ has all its eigenvalues in $\Omega(n)$. This implies that

$$\det B^T B = \Omega(n)^n.$$

By Lemma 2.2, we derive the existence of an $n \times n$ submatrix A , such that

$$D^H(A) = \Omega(n^{1/4}).$$

This result can be interpreted in terms of arithmetic progressions, but it is perhaps better grasped in dual space. Since arithmetic progressions are considered modulo n , a row of A might consist of two distinct progressions. By doubling the number of rows if necessary we can make them into regular arithmetic progressions. Let n denote the new number of rows. If $b, a + b, \dots, ka + b$ is the progression associated with row i , let us now associate with that row the segment on line $Y = aX + b$ running from $X = 0$ to $X = k$. Column j is associated with line $Y = j$. The lower bound on the hereditary discrepancy implies that the restriction of the set system to a certain subset of lines has discrepancy $\Omega(n^{1/4})$. By zeroing out the leftover columns and adding dummy column lines, we thus create an $n \times n$ incidence matrix of discrepancy $\Omega(n^{1/4})$, where element $a_{i,j}$ is 1 if and only if segment i and line j intersect in an integer point. This proves Theorem 1.2. \square

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