A Lower Bound on the Complexity of Approximate Nearest-Neighbor Searching on the Hamming Cube*

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Abstract

We consider the nearest-neighbor problem over the d-cube: given a collection of points in $\{0,1\}^d$, find the one nearest to a query point (in the L^1 sense). We establish a lower bound of $\Omega(\log\log d/\log\log\log d)$ on the worst-case query time. This result holds in the cell probe model with (any amount of) polynomial storage and word-size $d^{O(1)}$. The same lower bound holds for the approximate version of the problem, where the answer may be any point further than the nearest neighbor by a factor as large as $2^{\lfloor(\log d)^{1-\varepsilon}\rfloor}$, for any fixed $\varepsilon > 0$.

1 Introduction

For a variety of practical reasons ranging from molecular biology to web searching, nearest-neighbor searching has been a focus of attention lately [2]–[9], [11]–[21], [26]. In the applications considered, the dimension of the ambient space is usually high, and predictably, classical lines of attack based on space partitioning fail. To overcome the well-known "curse of dimensionality," it is typical to relax the search by seeking only approximate answers. Curiously, no lower bound has been established — to our knowledge — on the complexity of the approximate problem in its canonical setting, i.e., points on the hypercube. (A very recent result due to Borodin et.al. does

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give a good lower bound on the *exact* version of the problem.) Our work is an attempt to remedy this.

Given a *database* $S \subseteq \{0,1\}^d$, a δ-*approximate nearest neighbor* (δ-*ANN*) of a *query point* $x \in \{0,1\}^d$ is any $y \in S$ such that $\|x-y\|_1 \leq \delta \|x-z\|_1$, for any $z \in S$. The parameter $\delta \geq 1$ is called the *approximation factor* of the problem. Given some δ, the problem is to preprocess S so as to be able to find a δ-*ANN* of any query point efficiently. The data structure consists of a table T whose entries hold $d^{O(1)}$ bits each. This means that a point can be read in constant time. This assumption might be unrealistically generous when d is large, but note that this only strengthens our lower bound result.

Theorem 1.1 *Suppose the table T, constructed from preprocessing a database S of n points in $\{0,1\}^d$, is of size polynomial in n and d and holds $d^{O(1)}$-bit entries. Then, for any algorithm using T for δ-ANN searching, there exists some S such that the query time is $\Omega(\log\log d / \log\log\log d)$. This holds for any approximation factor $\delta \leq 2^{\lfloor (\log d)^{1-\varepsilon} \rfloor}$, for any fixed $\varepsilon > 0$.*

How good is the lower bound? First, note that the problem can be trivially solved *exactly* in constant time, by using a table of size 2^d. Moreover, recent results of [20, 16], when adapted to our model of computation, show that for *constant* $\delta > 1$ there is a polynomial sized table with d-bit entries and a *randomized* algorithm that enables us to answer δ-*ANN* queries using $O(\log\log d)$ probes to the table. Although there seems to be only a small gap between this upper bound and our lower bound, the two bounds are in fact incomparable because of the randomization. An important open theoretical question regarding *ANN* searching is to extend our lower bound to allow randomization.

In this context it is worth mentioning that a stronger lower bound for *exact* nearest neighbor search is now known. Very recent results of Borodin et. al. [7] show that even randomized algorithms for this problem require $\Omega(\log d)$ query time in our model.

2 The Cell Probe Model

Yao's *cell probe model* [27] provides a framework for measuring the number of memory accesses required by a search algorithm. Because of its generality, any lower bound in that model can be trusted to apply to virtually any conceivable sequential algorithm. In his seminal paper [1], Ajtai established a nontrivial lower bound for predecessor queries in a discrete universe (for recent improvements, see [5, 22, 25]). Our proof begins with a similar adversarial scenario. Given a key-set of n points in $\{0,1\}^d$, a table T is built in preprocessing: its size is $(dn)^c$, for fixed (but arbitrary) $c > 0$ and each entry holds $d^{O(1)}$ bits. (For simplicity, we assume that an entry consists of exactly d bits; the proof is very easily generalized if this is number d is changed to $d^{O(1)}$.) To answer queries, the algorithm has at its disposal an infinite supply of functions f_1, f_2, etc. Given a query x, the algorithm evaluates the index $f_1(x)$ and looks up the table entry $T[f_1(x)]$. If $T[f_1(x)]$ is a δ-*ANN* of x, it can stop after this single round. Otherwise, it evaluates $f_2(x, T[f_1(x)])$ and looks up the entry $T[f_2(x, T[f_1(x)])]$. Again it stops if it this entry is the desired answer (at a cost of two rounds), else it goes on in this fashion. The query time of the algorithm is defined to be the maximum number of rounds, over all queries $x \in \{0,1\}^d$, required to find a δ-*ANN* of x in the table. Note that we do not charge the algorithm for the time it takes to compute the functions f_r. Note also that we require the last entry of T fetched by the algorithm to be the answer that it will give (this adds at most one to the query time).

Following [22, 23], we couch our cell-probe arguments in a communication-complexity setting as we model the algorithm as a game between Bob and Alice [10, 19]. The algorithm is modeled by a set of functions $f_1, f_2, \ldots$. Alice starts out with a set $P_1 \subseteq \{0,1\}^d$ of candidate queries and Bob holds a collection $K_1 \subseteq 2^{\{0,1\}^d}$ of candidate key-sets; each set in K_1 being of size n. The goal of Alice and Bob is to force as many communication rounds as possible.

The possible values of $f_1(x)$ (provided by Alice for every x) partition P_1 into equivalence classes. Bob chooses one such class and the corresponding value of $f_1(x)$, thus restricting the set of possible queries to $P_2 \subseteq P_1$. Given this fixed value of $f_1(x)$, the entry $T[f_1(x)]$ depends only on Bob's choice of key-set. All the possible values of that entry partition K_1 into equivalence classes. Bob picks one of them and communicates the corresponding value of $T[f_1(x)]$ to Alice, thus restricting the collection of possible key-sets to $K_2 \subseteq K_1$. Alice and Bob can then iterate on this process. This produces two nested sequences of *admissible* query sets,

$$P_1 \supseteq P_2 \supseteq \cdots \supseteq P_t\,,$$

and *admissible* key-set collections,

$$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_t\,.$$

An element of $P_r \times K_r$ specifies a problem instance. The set $P_r \times K_r$ is called *nontrivial* if it contains at least two problem instances with distinct answers, meaning that no point can serve as a suitable ANN in both instances. If $P_r \times K_r$ is nontrivial, then obviously round r is needed, and possibly others as well.

We show that for some appropriate value of $n = n(d)$, there exists an admissible starting $P_1 \times K_1$, together with a strategy for Alice and Bob, that leads to a nontrivial $P_t \times K_t$, for $t = \Theta(\log\log d / \log\log\log d)$. What makes the problem combinatorially challenging is that a greedy strategy can be shown to fail. Certainly Alice and Bob must ensure that P_r and K_r do not shrink too fast; but just as important, they must choose a low-discrepancy strategy that keeps query sets and key-sets "entangled" together. To achieve this, we adapt to our needs a technique used by Ajtai [1] for probability amplification in product spaces.[1]

3 The Lower Bound

Throughout the proof, we assume that d is large enough and that logarithms are to the base 2. The term "distance" refers to the Hamming distance between two points in $\{0,1\}^d$. A "ball of radius r centered at x" denotes the set $\{y \in \{0,1\}^d : \mathrm{dist}(x,y) \le r\}$. To begin with, we specify the size n of the admissible key-sets and the number t of rounds, and also define two auxiliary numbers h and β:

$$h \stackrel{\text{def}}{=} 6ct \tag{1}$$

$$\beta \stackrel{\text{def}}{=} 16 \cdot 2^{\lfloor (\log d)^{1-\varepsilon} \rfloor} \tag{2}$$

$$t = \left\lceil \frac{\varepsilon \log\log d}{2 \log\log\log d} \right\rceil \tag{3}$$

$$n = (h-1)^{t-1} d^{5t} \tag{4}$$

The significance of these formulae will become clear in the proofs of Lemmas 3.4–3.7.

[1] This simple but powerful technique, which is described in §3.3, has been used elsewhere in communication complexity, for example by Karchmer and Wigderson [17].

3.1 Admissible Queries

Before we get down to defining the sets P_r we shall need to prove a geometric fact about the hypercube.

Definition 3.1 *A family of balls is said to be γ-separated if the distance between any two points belonging to distinct balls in the family is more than γ times the distance between any two points belonging to any one ball in the family. Here γ is any positive real quantity.*

Lemma 3.2 *Let $B \subseteq \{0,1\}^d$ be a ball of radius $k \leq d$ large enough. For any $\gamma \geq 16$ there exists a $\gamma/16$-separated family of balls within B, such that the size of the family is at least $2^{k/13}$ and the radius of each ball in the family is k/γ.*

Proof: We use an argument similar to the proof of Shannon's theorem. Let V_r be the volume of (i.e. the number of points in) a ball in $\{0,1\}^d$ of radius r, centered at a point in $\{0,1\}^d$. (Notice that this number does not depend on the center). Clearly

$$V_r = \sum_{i=0}^{\lfloor r \rfloor} \binom{d}{i}.$$

Consider the ball B', concentric with B and of radius $k/3$, and call its points initially *unmarked.* We proceed to mark the points of B' as follows: while there is an unmarked point left in B', pick one and mark all the points at distance at most $k/4$ from that point. The number N of points we pick in B' satisfies

$$N \geq \frac{V_{k/3}}{V_{k/4}}.$$

We can estimate N from below.

$$N \geq \frac{V_{k/3}}{V_{k/4}} = \frac{\sum_{i=0}^{\lfloor k/3 \rfloor} \binom{d}{i}}{\sum_{i=0}^{\lfloor k/4 \rfloor} \binom{d}{i}} \geq \frac{\binom{d}{\lfloor k/3 \rfloor}}{\sum_{i=0}^{\lfloor k/4 \rfloor} \binom{d}{i}}.$$

Note that in each term of the sum in the denominator i is at most $d/4$. For such i,

$$\frac{\binom{d}{i}}{\binom{d}{i-1}} = \frac{d-i+1}{i} \geq 3, \qquad \text{so} \qquad \sum_{i=0}^{\lfloor k/4 \rfloor} \binom{d}{i} \leq \frac{3}{2} \binom{d}{\lfloor k/4 \rfloor}.$$

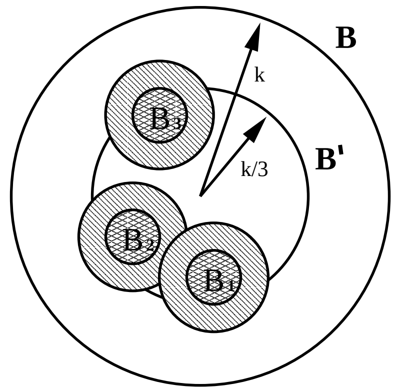

Figure 1: Picking balls $B_1, B_2, \ldots$. Light shading: marked points. Heavy shading: picked balls.

$$\begin{aligned} N &\geq \frac{2}{3}\frac{\binom{d}{\lfloor k/3\rfloor}}{\binom{d}{\lfloor k/4\rfloor}} = \frac{2}{3}\cdot\prod_{i=\lfloor k/4\rfloor}^{\lfloor k/3\rfloor}\frac{d-i+1}{i} \\ &\geq \frac{2}{3}(\underbrace{2\cdot\ldots\cdot 2}_{k/12-2}) \geq 2^{k/12-3}, \end{aligned}$$

and for large enough k, this implies $N \geq 2^{k/13}$.

Now pick balls of radius k/γ centered at the N picked points; their centers are in B' and their common radius is at most $k/16$, so these balls lie within B. Moreover, it is easy to see that they form a $\gamma/16$-separated family. To see why, suppose on the contrary that two points p and q in balls centered at distinct points p_0 and q_0 lie within $k/8$ of each other. Then,

$$\begin{aligned} \mathrm{dist}(p_0,q_0) &\leq \mathrm{dist}(p_0,p)+\mathrm{dist}(p,q)+\mathrm{dist}(q,q_0) \\ &\leq k/\gamma + k/8 + k/\gamma \leq k/4, \end{aligned}$$

since $\gamma \geq 16$. But this is a contradiction since by construction $\mathrm{dist}(p_0,q_0) > k/4$. □

Corollary 3.3 *For k divisible by β, there exists a $\beta/16$-separated family of radius-(k/β) balls within B, of size $2^{k/\beta}$.*

The partitioning of P_r into equivalence classes can be unwieldy, so we restrict the admissible query sets to be part of another, better-structured, nested sequence

$$P_1^* \supseteq P_2^* \supseteq \cdots \supseteq P_t^*,$$

where each $P_r^* \supseteq P_r$.

To define P_r^*, we build a hierarchy of balls: Let $\mathcal{H}$ be the tree whose root is associated with the ball of $\{0,1\}^d$ of radius d centered at $(0,\ldots,0)$. The children of the root are each associated with one of the $2^{d/\beta}$ balls specified by the above corollary.[2] Their children, grand-children, etc., are defined similarly. In general, a node of depth k (root being of depth 0) is associated with a ball of radius d/β^k and its number of children is $2^{d/\beta^{k+1}}$. We iterate this recursive construction until the leaves of $\mathcal{H}$ are of depth h^{t-1}. Note that the balls associated with the leaves of $\mathcal{H}$ are of radius at least $d/\beta^{h^{t-1}}$, and thus, by our choice of t, large enough for the application of Lemma 3.2; specifically, its corollary.

The tree $\mathcal{H}$ is used to build other trees, each one associated with a separate round. We begin with the round-one tree $\mathcal{H}_1$. Given $v \in \mathcal{H}$, let $\mathcal{H}_1^*(v)$ denote the subtree of depth h^{t-2} rooted at v. For each node v of $\mathcal{H}$ whose depth is divisible by h^{t-2}, remove from $\mathcal{H}$ all the nodes of $\mathcal{H}_1^*(v)$, except for its leaves, which we keep in $\mathcal{H}$ and make into the children of v: these operations transform $\mathcal{H}$ into a tree $\mathcal{H}_1$ of depth h. In this way, each node v of $\mathcal{H}_1$ (together with its children) forms a contraction of the tree $\mathcal{H}_1^*(v)$. We can easily check that a node of $\mathcal{H}_1$ of depth $k < h$ has exactly

$$2^{\nu d/\beta^{kh^{t-2}+1}}$$

children, where $\nu = (1 - 1/\beta^{h^{t-2}})/(1 - 1/\beta)$.

For $1 < r < t$, we define $\mathcal{H}_r$ by induction. We pick some internal node v of $\mathcal{H}_{r-1}$ and consider the tree $\mathcal{H}_{r-1}^*(v)$ of which it is the contraction. This tree now plays the role of $\mathcal{H}$ earlier: For $z \in \mathcal{H}_{r-1}^*(v)$, we let $\mathcal{H}_r^*(z)$ denote the subtree of $\mathcal{H}_{r-1}^*(v)$ of depth h^{t-r-1} rooted at z. If the depth of z in $\mathcal{H}_{r-1}^*(v)$ is divisible by h^{t-r-1}, we turn the leaves of $\mathcal{H}_r^*(z)$ into the children of z, which transforms $\mathcal{H}_{r-1}^*(v)$ into a tree of depth h that is the desired $\mathcal{H}_r$.

For $r = t$, we define $\mathcal{H}_r$ (with respect to an internal node $v \in \mathcal{H}_{r-1}$) as simply the tree formed by v and its children. Observe that, for any $r > 1$, the definition of $\mathcal{H}_r$ is not deterministic, since the initial choice of v is left unspecified.

[2] To simplify the notation, we shall assume that d is a large enough power of 2. Note that β is already a power of 2.

Lemma 3.4 *Any internal node v of any $\mathcal{H}_r$ satisfies $2^{\sqrt{d}} < \kappa(\mathcal{H}_r, v) < 2^{2d/\beta}$, where $\kappa(\mathcal{T}, v)$ denotes the number of children of node v in tree $\mathcal{T}$.*

Proof: Observe that $\kappa(\mathcal{H}_t, v) = \kappa(\mathcal{H}_{t-1}, v)$. So, it suffices to prove the lemma for $1 \leq r \leq t-1$. Pick any such r and consider any internal node v of $\mathcal{H}_r$: $\kappa(\mathcal{H}_r, v)$ is the number of leaves of $\mathcal{H}_r^*(v)$, which itself is a subtree of $\mathcal{H}$ of depth h^{t-r-1}. So, if k is the depth of v in $\mathcal{H}$, then

$$\kappa(\mathcal{H}_r, v) = \prod_{i=1}^{h^{t-r-1}} 2^{d/\beta^{k+i}}.$$

It follows that the number $\kappa(\mathcal{H}_r, v)$ is largest when $r = 1,\ k = 0$, and smallest when $r = t-1,\ k = h^{t-1} - 1$. Thus

$$\kappa(\mathcal{H}_r, v) \leq \prod_{i=1}^{h^{t-2}} 2^{d/\beta^i} = 2^{\nu d/\beta} < 2^{2d/\beta}.$$

On the other hand, $\kappa(\mathcal{H}_r, v) \geq 2^{d/\beta^{h^{t-1}}}$, so it suffices to prove that

$$h^{t-1} \log \beta < \frac{1}{2} \log d. \tag{5}$$

But this follows after some routine algebra from (1), (3), and the fact that d is large enough. □

The association between balls and nodes of $\mathcal{H}_r$ is inherited from $\mathcal{H}$ in the obvious manner. The centers of the balls at the leaves of $\mathcal{H}$ constitute the set P_1^*. For $r > 1$, we define P_r^* as the intersection of P_{r-1}^* with the balls at the leaves of $\mathcal{H}_r$. We define $P_1 = P_1^*$. For $r > 1$, Alice chooses the set P_r to be a certain subset of P_r^* according to a strategy to be specified in Section 3.4. Recall that to be admissible a query must be consistent with all of the information exchanged between Bob and Alice so far. For $r > 1$, we keep the set P_r of admissible queries from being too small by requiring the following:

- QUERY INVARIANT: The fraction of the leaves in $\mathcal{H}_r$ whose associated balls intersect P_r is at least $1/d$.

Note that the size of the initial collection P_1 of admissible queries is not quite as large as 2^d, although it is still a fractional power of it. Indeed,

$$|P_1| = (2^d)^{\frac{1-1/\beta^{h^{t-1}}}{\beta-1}}.$$

By our assumption on table size, the index $f_1(x)$ that Alice gives Bob during the first round can take on at most $(dn)^c$ distinct values. This subdivides P_1 into as many equivalence classes. The same is true at any around $r < t$, and so P_r is partitioned into the classes $P_{r,1}, \ldots, P_{r,(dn)^c}$. An internal node v of $\mathcal{H}_r$ is called *dense for* $P_{r,i}$ if the fraction of its children whose associated balls intersect $P_{r,i}$ is at least $1/d$. The node v is said to be dense if it is dense for at least one $P_{r,i}$.

Lemma 3.5 *The union of the balls associated with the dense non-root nodes of $\mathcal{H}_r$ contains at least a fraction $1/2d$ of the balls at the leaves.*

Proof: Consider one of the partitions $P_{r,i}$. Color the nodes of $\mathcal{H}_r$ whose associated balls intersect $P_{r,i}$. Further, mark every colored non-root node that is dense for $P_{r,i}$. Finally, mark every descendant in $\mathcal{H}_r$ of a marked node. For $1 \le k \le h$, let L_k be the number of leaves of $\mathcal{H}_r$ whose depth-k ancestor in $\mathcal{H}_r$ is colored and unmarked. (We include v as one of v's ancestors.) Let L be the number of leaves of $\mathcal{H}_r$. Clearly $L_1 \le L$. For $k > 1$, an unmarked colored depth-k node is the child of a colored depth-$(k-1)$ node that is *not* dense for $P_{r,i}$. It follows that $L_k < L_{k-1}/d$ and so, for any $k \ge 1$, $L_k \le L/d^{k-1}$.

Repeating this argument for all the $P_{r,i}$'s in the partition, we find that all the unmarked, colored nodes, at a fixed depth $k \ge 1$, are ancestors of at most $(dn)^c L/d^{k-1}$ leaves. In particular, the number of unmarked, colored leaves is at most

$$(dn)^c L/d^{h-1} < L/2d. \tag{6}$$

This last inequality follows from (1) and (4). Incidentally, the quantity h is defined the way it is precisely to make this inequality hold.

The query invariant ensures at least L/d colored leaves, so there are at least $L/2d$ colored, marked leaves. Moving up the tree $\mathcal{H}_r$, we find that the marked nodes whose parents are unmarked are ancestors of at least $L/2d$ leaves. All such nodes are dense, which completes the proof. □

3.2 Admissible Key-Sets

The collections K_r of admissible key-sets need not be specified explicitly. Instead, we define a probability distribution $\mathcal{D}_r$ over the set of all $\binom{2^d}{n}$ key-sets of size n and indicate a lower bound on the probability that a random key-set drawn from $\mathcal{D}_r$ is admissible, i.e., belongs to K_r. Beginning with the case $r = 1$, we define a random key-set S_1 recursively in terms of a random variable S_2, which itself depends on $S_3, \ldots, S_t$. To treat all these cases at once, we define S_r, for $1 \le 1 \le t$:

- For $r < t$, we define a *random* S_r *within* $\mathcal{H}_r$ in two stages:

  [1] For each $k = 1, 2, \ldots, h-1$, choose d^5 nodes of $\mathcal{H}_r$ of depth k at random, uniformly without replacement among the nodes of depth k that are not descendants of chosen nodes of smaller depth. The $(h-1)d^5$ nodes chosen in this way are said to be *picked by* S_r.

  [2] For each node v picked by S_r, recursively choose a random S_{r+1} within the corresponding tree $\mathcal{H}_{r+1}$ (i.e., defined with respect to node v). Such a S_{r+1} is called the *canonical projection* of S_r on v. The union of these $(h-1)d^5$ projections S_{r+1} defines a *random* S_r *within* $\mathcal{H}_r$.

- For $r = t$, a random S_t within (some) $\mathcal{H}_t$ is obtained by selecting d^5 nodes at random, uniformly without replacement, among the leaves of the depth-one tree $\mathcal{H}_t$: S_t consists of the d^5 centers of the balls associated with these leaves.

Note that a random S_r consists of exactly $(h-1)^{t-r}d^{5(t-r+1)}$ points, thus satisfying the definition of n in (4) for the case of S_1. A random S_1 is admissible with probability one (since no information has been exchanged yet), and so the set of all S_1's constitutes K_1. Obviously, this cannot be true for $r > 1$, since for one thing S_r does not even have the right size, i.e., n.

Suppose we have defined the distribution $\mathcal{D}_{r-1}$, for some $r > 1$. As we shall see from Bob's strategy, this implies the choice of a specific $\mathcal{H}_{r-1}$. To define $\mathcal{D}_r$, we choose some node v in $\mathcal{H}_{r-1}$ (which immediately implies the choice of $\mathcal{H}_r$ for the next round). Any key-set S_1 whose construction involves choosing an S_r within the tree $\mathcal{H}_r$ associated with node v is called v*-based* and its subset formed by the corresponding S_r is called its v*-projection*.

By abuse of terminology, we say that S_r is admissible if it is a v-projection of at least one key-set of K_{r-1}: for each admissible S_r, choose one such key-set arbitrarily and call it the v*-extension* of S_r; for any other S_r, choose as its (unique) v-extension any v-based key-set whose v-projection is S_r (such a key-set is non-admissible). To define the distribution $\mathcal{D}_r$, we assign probability zero to any key-set S_1 that is not a v-extension; if it is, we assign it the probability of its v-projection with respect to the distribution of a random S_r. During round $r-1$, Bob gets to choose K_r among the key-sets with nonzero probability in $\mathcal{D}_r$.

We set a lower bound on the number of admissible key-sets by requiring Bob's strategy to enforce the following

- KEY-SET INVARIANT: A random S_r is admissible with probability at least 2^{-d^2}.

The underlying distribution is the one derived from the construction of S_r, which is also equivalent to $\mathcal{D}_r$.

In what follows, we shall need a tail estimate for the hypergeometric distribution. The next lemma provides it:

Lemma 3.6 *Consider a set of N of objects, a fraction $1/T$ of which are "good". Let us pick a subset of size $m \leq N$ of these objects, uniformly at random, and let the random variable X denote the fraction of elements of this subset that are good. Then for any real $t > 0$ we have* $\mathrm{Prob}[\frac{1}{T} - X \geq t] \leq e^{-2mt^2}$.

Proof: See [15].

Lemma 3.7 *Fix an arbitrary $\mathcal{H}_r$ $(r < t)$. There exists some k_0 $(1 \leq k_0 < h)$ such that, with probability at least 2^{-d^2-1}, a random S_r within $\mathcal{H}_r$ is admissible and picks at least d^3 dense nodes of $\mathcal{H}_r$ of depth k_0.*

Proof: By Lemma 3.5, the dense non-root nodes of $\mathcal{H}_r$ are ancestors of at least a fraction $1/2d$ of the leaves. By the pigeonhole principle, for some k_0 with $1 \leq k_0 < h$, at least a fraction $1/2dh$ of the nodes of depth k_0 are dense. Of course, not all these nodes can be picked by S_r: only those that do not have ancestors that have been picked further up the tree are candidates. But this rules out fewer than hd^5 nodes, which by Lemma 3.4, represents a fraction at most $hd^5 2^{-\sqrt{d}}$ of all the nodes of depth k_0. This means that from among the set of depth-k_0 nodes that can be picked by S_r, the fraction $1/T_0$ that is dense satisfies

$$\frac{1}{T_0} \geq \frac{\frac{1}{2dh} - \frac{hd^5}{2^{\sqrt{d}}}}{1 - \frac{hd^5}{2^{\sqrt{d}}}} > \frac{1}{3dh}.$$

Among the d^5 nodes we pick at depth k_0, we expect at least $d^5/3dh$ of them to be dense, and thus we should exceed the lemma's target of d^3 with overwhelming probability, say, $1 - 2^{-d^2-1}$. Using Lemma 3.6 we see that this is indeed the case: choose the set of objects in the lemma to be the set of depth-k_0 nodes that are available for picking by S_r and let the "good" objects among these nodes be the dense nodes. Choose $m = d^5$, $T = T_0$ and $t = 1/T_0 - 1/d^2 > 0$. The lemma now says that the number R of dense nodes we pick satisfies

$$\text{Prob}[R \leq d^3] = \text{Prob}[R/d^5 \leq 1/d^2] \leq e^{-2d^5(\frac{1}{T_0} - \frac{1}{d^2})^2}$$

But, as observed above, $T_0 \leq 3dh$ and so, after some routine algebra, we obtain $\text{Prob}[R \leq d^3] \leq 2^{-d^2-1}$.

The key-set invariant completes the proof. □

3.3 Probability Amplification

In the r^{th} round, the table entry $T[f_r(x, T[f_1(x)], ...)]$ that Bob returns to Alice can take on at most 2^d distinct values, and so the collection of admissible key-sets is partitioned into equivalence classes $K_{r,1}, \ldots, K_{r,2^d}$. Bob has to choose one of these classes to form the new collection K_{r+1} of admissible key-sets. Unfortunately, such a large number of classes is likely to cause a violation of the key-set invariant. To amplify the probability that a random key-set is admissible back to 2^{-d^2}, we exploit the fact that the distribution is defined over a product space, and borrowing an idea from Ajtai [1], we project the distribution on its "highest-density" coordinate.

Lemma 3.8 *For $r < t$, there exists a dense node v of $\mathcal{H}_r$ such that the conditional probability that the canonical projection on v of a random S_r is admissible, given that it picks v, is at least* $1/2$.

Proof: Let D be a subset of dense nodes of depth k_0 (referred to in Lemma 3.7). We define $\mathcal{E}_D$ to be the event that the set of dense nodes of depth k_0 picked by S_r is exactly D. Let p_D be the probability that S_r is admissible and that $\mathcal{E}_D$ occurs, and let c_D be the conditional probability that S_r is admissible, given $\mathcal{E}_D$. By Lemma 3.7, summing over all subsets D of dense depth-k_0 nodes of size at least d^3, we find that

$$\sum_D c_D \cdot \text{Prob}[\mathcal{E}_D] = \sum_D p_D \geq 2^{-d^2-1},$$

and therefore $c_{D_0} \geq 2^{-d^2-1}$, for some D_0 of size at least d^3.

Now we derive a key fact from the product construction of the probability spaces for key-sets. Consider the $|D_0|$-dimensional space, where each $v \in D_0$ defines a coordinate. Each point in this space represents an S_r and is characterized by a vector $\langle n_1, ..., n_{|D_0|}\rangle$, where n_i is the canonical projection of S_r onto the i^{th} node of D_0. By the definition of admissibility for S_r's, if $\langle n_1, ..., n_{|D_0|}\rangle$ is an admissible S_r, then all the n_i's in its vector representation are admissible S_{r+1}'s. Let A_{n_i} be the set of *all* admissible S_{r+1}'s within the $\mathcal{H}_{r+1}$ corresponding to the i^{th} node of D_0. Clearly the admissible S_r's that belong to the $|D_0|$-dimensional space are all contained in

$$A_{n_1} \times ... \times A_{n_{|D_0|}}$$

the size of which is a fraction $\prod_{v \in D_0} c_v$ of the S_r's for which D_0 is exactly the set of dense nodes of depth k_0 picked by S_r, where c_v is the probability that a random S_{r+1} within the $\mathcal{H}_{r+1}$ corresponding to v is admissible. Because within S_r the random construction of any S_{r+1} is independent of $S_r \setminus S_{r+1}$, c_v is also the conditional probability that the canonical projection on v of a random S_r is admissible, given that it picks v. Thus we see that

$$c_{D_0} \leq \prod_{v \in D_0} c_v.$$

Since $|D_0| \geq d^3$, it follows that

$$c_v \geq \left(2^{-d^2-1}\right)^{1/|D_0|} \geq \frac{1}{2},$$

for some $v \in D_0$. □

3.4 Maintaining the Invariants

We summarize the strategies of Alice and Bob and discuss the enforcement of the two invariants. Skipping the trivial case $r = 1$, we show that if the invariants hold at the beginning of round $r < t$, they also hold at the beginning of round $r + 1$. Prior to round r, consider the node v from $\mathcal{H}_r$ described in Lemma 3.8. Since v is dense there is some $P_{r,i}$ such that the fraction of v's children whose associated balls intersect $P_{r,i}$ is at least $1/d$. Alice chooses such a $P_{r,i}$ and defines $P_{r,i} \cap P^*_{r+1}$ to be P_{r+1}, the set of admissible queries prior to round $r + 1$. The tree $\mathcal{H}_{r+1}$ is then rooted at v, and its leaves coincide with the children of v in $\mathcal{H}_r$. Thus, the fraction of the leaves of $\mathcal{H}_{r+1}$ whose associated balls intersect P_{r+1} is at least $1/d$, and the query invariant holds.

Turning now to the key-set invariant, recall that during round r, Bob is presented with a table entry, which holds one of 2^d distinct values. By the choice of v in Lemma 3.8, the probability that a random S_{r+1} at v is admissible is at least a half. A key observation is that this is the same probability that a random key-set from $\mathcal{D}_{r+1}$ is in K_r. By the pigeonhole principle, there is a value of the table entry for which, with probability at least $(1/2)2^{-d}$, a random key-set from $\mathcal{D}_{r+1}$ is in K_r and produces a table with that specific entry value. Since $2^{-d-1} > 2^{-d^2}$, the key-set invariant holds after round r.

3.5 Forcing t Rounds

To complete the proof of Theorem 1.1, we must show that the invariants on query-sets and key-sets are strong enough to guarantee that $P_t \times K_t$ is nontrivial, i.e. that after $t-1$ rounds, we still have at least two admissible problem instances which produce different answers. We shall soon prove that there exists at least one key-set $S \in K_t$ which picks two distinct leaves v_1 and v_2 of the tree $\mathcal{H}_t$ whose associated balls contain queries q_1 and q_2, respectively, in P_t. Notice that by construction, the family of balls associated with the leaves of $\mathcal{H}_t$ is a $\beta/16$-separated family. Since any key must lie within some ball in this family, no key can be a $\beta/16$-*ANN* for both q_1 and q_2. But (2) says that $\beta/16 = 2^{\lfloor (\log d)^{1-\varepsilon} \rfloor}$ which concludes the argument.

We prove the existence of such an S by contradiction. For any S_t let $\nu(S_t)$ denote the number of queries in P_t that it picks (which is shorthand for "the number of nodes it picks each of whose balls contains at least one query in P_t"). Suppose that no admissible S_t picks more than one query. Then the probability p that a random S_t is admissible satisfies

$$p \leq \text{Prob}[\nu(S_t) = 0] + \text{Prob}[\nu(S_t) = 1].$$

To form a random S_t we pick d^5 leaves of $\mathcal{H}_t$ at random, uniformly. By the query invariant, at least $1/d$ of them belong to P_t. So,

$$p < \Big(1 - \frac{1}{d}\Big)^{d^5} + 2^d \Big(1 - \frac{1}{d}\Big)^{d^5 - 1} < e^{-d^4} + 2^{d+1} e^{-d^4} < e^{-d^3}.$$

By the key-set invariant, we must have $p > 2^{-d^2}$, hence a contradiction. This concludes the proof of Theorem 1.1. □

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