Points and Triangles in the Plane and Halving Planes in Space*

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Abstract. We prove that for any set $S$ of $n$ points in the plane and $n^{3-\epsilon}$ triangles spanned by the points in $S$ there exists a point (not necessarily in $S$) contained in at least $n^{3-3\epsilon} / (c \log^2 n)$ of the triangles. This implies that any set of $n$ points in three-dimensional space defines at most $\sqrt[3]{(c/2)n^{2/3}} \log^{2/3} n$ halving planes.

1. Introduction

Let $S$ be a set of $n$ points in the plane and let $\mathcal{T}$ be the set of $\binom{n}{3}$ triangles whose vertices are in $S$. Boros and Füredi [5] show that there is a point $x$, not necessarily in $S$, contained in at least $n^{3/7}$ triangles and this bound is tight up to an additive

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quadratic term. The main result of this paper is a generalization of this result to the case where \( \mathcal{F} \) is only a subset of the \( \binom{n}{3} \) triangles. Specifically, we show that if \( |\mathcal{F}| = n^3 - 3 \), then there exists a point \( x \) contained in at least \( n^3 - 3n/(2^2 \log^2 n) \) of the triangles.\(^1\) An application of this result is that it allows three-dimensional space to define at most \( 2^{5/3/3} \log^2 n \) halving planes, where a halving plane of a three-dimensional set of \( n \) points in general position is a plane passing through three of the points that has an equal number of points on each of its sides (assuming \( n \) is odd). This improves the recent \( O(n^{3-1/34}) \) upper bound of Bárány et al. [4].

The organization of this paper is as follows. Section 2 introduces some one-dimensional selection lemmas proved by application of the pigeonhole principle. These form the primary combinatorial tools used to prove the main result in Section 3. Section 4 discusses the application to halving planes of a finite point set in space.

2. One-Dimensional Selection Lemmas

The primary combinatorial tools used to prove the main result of this paper are two variants of a result for points and intervals on a line. They are formulated as parts (i) and (ii) of Lemma 2.1. We refer to (i) as the "unweighted selection lemma" and to (ii) as the "weighted selection lemma." The unweighted selection lemma has also been used in a companion paper [6]; for the sake of completeness we repeat its proof which is not difficult. For two points \( p \) and \( q \) on the real line we call \( I_{pq} = \{ x|p < x < q \text{ or } q < x < p \} \) the interval of the edge \( \{p, q\} \).

**Lemma 2.1.** Let \( V \) be a set of \( n \geq 2 \) points on the real line and let \( E \) be a set or multiset of \( m \geq 2n \) edges, that is, unordered point pairs. For a point \( x \notin V \), let \( E(x) \) denote any subset or submultiset of edges in \( E \) whose intervals contains \( x \) (\( E(x) \) does not necessarily contain all such edges). Define \( m(x) = |E(x)| \) and let \( n(x) \) be the number of points of \( V \) incident to edges in \( E(x) \).

(i) If \( E \) is a set, then there is a point \( x \) and a set \( E(x) \) with \( m(x) \geq m^2/(4n^2) \).
(ii) If \( E \) is a multiset, then there is a point \( z \) and a multiset \( E(z) \) with \( m(z)/n(z) \geq m/(2n \log n) \).

**Proof.** In order to show (i) choose \( k - 1 \) points, none in \( V \), cutting the line into \( k \) intervals so that each contains no more than \( [n/k] \) points of \( V \). The number of edges (including those not in \( E \)) whose intervals contain none of the \( k - 1 \) delimiters is therefore at most

\[
k([n/k]) \leq \frac{n^2 + nk}{2k}.
\]

\(^1\) All logarithms in this paper are to the base 2.

The remaining intervals contain at least \( n - m - (n^2 + nk)/2k \) such intervals, \( k = \lceil n^2/(m - n) \rceil \). By the pigeonhole principle, \( m \geq n^2/(m - n) \).

To prove (ii) replace the \( k - 1 \) distinct points \( V \) by the \( n - 1 \) chosen points, so that the order of the points is \( z \) define \( E(z) \) as the set of edges in \( E \) incident to \( z \). In this way each edge is counted at most \( m(z) \) times. Because each point of \( V \) is on at most one edge per level we also have \( m(z)/n(z) < m/(n(1 + \log n)) \) for each \( z \), a contradiction. This implies that the ratio \( m(z)/n(z) \geq m/(2n \log n) \).

Remark. It is easy to see that (i) is tight in the worst case. In [6] we also show that this result is best possible up to a multiplicative constant in the worst case.

3. Selecting a Point in a Set of Triangles

This section studies the problem of selecting a point in a given collection of \( t \) open triangles in \( \mathbb{R}^2 \), in general because all triangles can intersect, we must have \( t \geq n^3/27 \) with \( n \) the total number of distinct vertices. Our result is a generalization of a result that the.set of all \( \binom{n}{3} \) triangles defines a subset of size at least \( n^3/27 \) with respect to others. A generalization of the unweighted version of this result can be found in the remainder of this section.

**Theorem 3.1.** Let \( S \) be a set of \( n \) points in \( \mathbb{R}^2 \). The set of all open triangles spanned by the points has at least \( t^3/(2^n \log^2 n) \) such that \( t \geq n^3/27 \).

The proof of this theorem is based on the unweighted selection lemma and the pigeonhole principle.
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The remaining intervals contain at least one of the \( k - 1 \) delimiters each and there are at least \( m - (n^2 + nk)/2k \) such intervals, which is at least \( m/2 \) if we choose \( k = [n^2/(m - n)] \). By the pigeonhole principle one of the chosen points is contained in at least \( m/(2(k - 1)) \geq (m^2 - mn)/(2n^2) \geq m^2/(4n^2) \) intervals.

To prove (ii) replace the \( k - 1 \) delimiters by \( n - 1 \) points, one between any two adjacent points of \( V \), and build a minimum height binary tree whose nodes are the \( n - 1 \) chosen points, so that the inorder sequence coincides with the left-to-right order of the points. The height of the tree is \( \lceil \log(n - 1) \rceil < \log n \). For a node \( z \) define \( E(z) \) as the set of edges in \( E \) whose intervals contain \( z \) but no ancestors of \( z \). In this way each edge is counted at exactly one node which implies

\[
\sum_{z} m(z) = m.
\]

Because each point of \( V \) can be incident to edges associated with at most one node per level we also have \( \sum_{z} n(z) < n(1 + \log n) \). Now assume that \( m(z)/n(z) < m/(n(1 + \log n)) \) for each node \( z \). But then

\[
\sum_{z} m(z) < \frac{m}{n(1 + \log n)} \sum_{z} n(z) < m,
\]

a contradiction. This implies that there is a point \( z \) with

\[
\frac{m(z)}{n(z)} \geq \frac{m}{n(1 + \log n)} \geq \frac{m}{2n \log n}.
\]

\[ \square \]

**Remark.** It is easy to see that (i) is tight up to a multiplicative constant in the worst case. In [6] we also show that a set variant of (ii) is tight (up to a multiplicative constant) in the worst case.

3. Selecting a Point in a Set of Triangles

This section studies the problem of finding a point that lies in many members of a given collection of \( t \) (open) triangles in the plane. This problem is not interesting in general because all triangles can be disjoint. However, if \( t \) is much larger than \( n \), the total number of distinct vertices of the triangles, then we can prove that there must be a point (not necessarily a vertex) contained in many of the triangles. Our result is a generalization of a theorem of Boros and Füredi [5] who prove that the set of all \( \binom{n}{3} \) triangles defined by any set of \( n \) points in the plane has a subset of size at least \( n^3/27 \) with nonempty common intersection. Our result is also a generalization of the unweighted selection lemma to two dimensions; other such generalizations can be found in [6]. The theorem and its proof constitute the remainder of this section.

**Theorem 3.1.** Let \( S \) be a set of \( n \) points in the plane and let \( \mathcal{T} \) be a set of \( t = n^3 - a \) open triangles spanned by the points of \( S \). Then there exists a point contained in at least \( t^2/(2^8 n^6 \log^5 n) = n^{3 - 3a}/(2^8 \log^5 n) \) of the triangles.
Remark. In the case \( x = 0 \) (or, more precisely, when \( t = \Omega(n^3) \)) our techniques can be slightly modified to prove a lower bound of \( cn^3 \) for the number of triangles with nonempty common intersection, thus reproducing (albeit with a smaller constant of proportionality) and extending the result of [5].

All steps of the proof of Theorem 3.1 are elementary and make use of the two selection lemmas presented in the preceding section.

Proof. The first step associates each triangle in \( \mathcal{T} \) with one of its edges as follows. Choose a direction in the plane with the property that no line in this direction passes through more than one point of \( S \). Without loss of generality assume this to be the \( x \)-direction. For each triangle in \( \mathcal{T} \) define its base to be the edge with the longest orthogonal \( x \)-projection. Each triangle in \( \mathcal{T} \) is associated with its base. For each segment \( ab \) connecting a pair of points in \( S \) define its multiplicity \( \mu_{ab} \) to be the number of triangles that have \( ab \) as a base. Clearly, \( \sum_{a} \mu_{ab} = t \).

We next choose an integer \( \xi \geq 2 \) with the property that

\[
\sum_{\xi \leq n/2} \mu_{ab} \geq \frac{t}{\log n}.
\]

The existence of such a \( \xi \) can be established if we assume that \( t > n^2 \) which is no loss of generality as Theorem 3.1 is void otherwise. Cover the interval \([2, n - 2]\) of integers with \([\log(n - 2) - 1 \leq \log n\) subintervals of the form \([2^j, 2^{j+1}]\), for \( j = 1, \ldots, \log(n - 2) - 1 \), and consider for each subinterval the subset of edges whose multiplicities it contains. By the pigeonhole principle, there must be an \( \xi \) subinterval \([2^j, 2^{j+1}]\) so that \( \xi = 2^j \) satisfies the above condition.

Let \( \mathcal{T}_\xi \) denote the collection of triangles in \( \mathcal{T} \) whose bases have multiplicity in \([\xi, 2\xi]\). The number of such triangles is \( t_\xi \geq t/\log n \). Since \( t_\xi \) is also bounded from above by \( 2^\xi \binom{n}{2} \), it follows that

\[
\xi \geq \frac{t}{n^2 \log n}.
\]

From now on we consider only the triangles in \( \mathcal{T}_\xi \).

In the second step of the proof we orthogonally project all points of \( S \) onto the \( x \)-axis, and define a multiset of intervals delimited by the projected points as follows. For each base \( ab \) whose multiplicity is in \([\xi, 2\xi]\) we take every pair of distinct triangles \( abc, abd \) in \( \mathcal{T}_\xi \) associated with \( ab \) and obtain the corresponding interval \( c'd' \), where \( c' \) and \( d' \) are the projections of the "inner" points \( c \) and \( d \). See Fig. 3.1 for an illustration. The number of resulting intervals, counted with multiplicity, is \( m_0 \geq t_\xi (\xi - 1)/2 \geq t_\xi/4 \), since each triangle of \( \mathcal{T}_\xi \) is paired with at least \( \xi - 1 \) triangles, and each such pair arises twice.

We can thus apply the weighted selection lemma (Lemma 2.1(iii)) to deduce that there exists a point \( z_0 \) on the \( x \)-axis contained in \( m_1 \) intervals (counted with multiplicity), which are delimited by a set of intervals \( m_1 \) as in Fig. 3.1. Let \( l \) denote the vertical intervals delimited by \( a \) and \( c \), and that the interval \( l \) is the intersection of the two corresponding triangles.

In the third step of the proof we construct a set \( S \) defined. We have \( m_2 = m_1 \) such in \( \mathcal{T}_\xi \), and \( m_2 \) is defined by some base while \( m_1 \) is obtained in the previous step. It follows that \( m_2 \) is at most \( m_1 \).

We now apply the unweighted selection lemma (Lemma 2.1) to the set \( S \) of intervals and points. Note that no interval \( l \) is of the same interval (assuming \( l \) is chosen so that \( S \) meet \( l \) at the same point), because each triangle \( S \) is determined at least \( 4m_2 \) triangles, and each such pair arises twice.

We can thus apply the weighted selection lemma (Lemma 2.1(iii)) to deduce that there exists a point \( z_1 \) on the \( x \)-axis contained in \( m_2 \) intervals (counted with multiplicity), which are delimited by a set of intervals \( m_2 \) as in Fig. 3.1. Let \( p \) denote the vertical segments \( ab \) and \( cd \), and that the interval \( p \) is the intersection of the two corresponding triangles.

In the fourth step of the proof we construct a set \( S \) defined. We have \( m_3 = m_2 \) such in \( \mathcal{T}_\xi \), and \( m_3 \) is obtained in the previous step. It follows that \( m_3 \) is at most \( m_2 \).

We now apply the unweighted selection lemma (Lemma 2.1) to the set \( S \) of intervals and points. Note that no interval \( l \) is of the same interval (assuming \( l \) is chosen so that \( S \) meet \( l \) at the same point), because each triangle \( S \) is determined at least \( 4m_3 \) triangles, and each such pair arises twice.
If we assume that \( t > n^2 \) which is no otherwise. Cover the interval \([2, n - 2]\) of intervals of the form \([2^i, 2^{i+1}]\), for each subinterval the subset of edges on whole principle, there must be one above condition.

In \( \mathcal{F} \) whose bases have multiplicity \( m_{1} \geq t/(\log n) \). Since \( t_{0} \) is also bounded.

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Fig. 3.1. Pairing two triangles with a common base.

multiplicity), which are delimited by at most \( n_{1} \) distinct endpoints, such that

\[
\frac{m_{1}}{n_{1}} \geq \frac{m_{0}}{2n \log n} \geq \frac{t_{0} \xi}{8n \log n}.
\]

Each of the \( m_{1} \) resulting intervals \( c'd' \) corresponds to a quadruplet \( Q = \{a, b, c, d\} \), as in Fig. 3.1. Let \( l \) denote the vertical line \( x = z_{0} \). Note that \( l \) intersects both segments \( ab \) and \( cd \), and that the interval \( l_{0} = (l \cap ad, l \cap bc) \) is contained in the union of the two corresponding triangles \( abc, abd \) of \( \mathcal{F}_{0} \).

In the third step of the proof we consider the system of intervals along \( l \) as just defined. We have \( m_{2} = m_{1} \) such intervals. By how many endpoints are they delimited? Each such endpoint is the intersection of \( l \) with a segment \( ad \) so that \( a \) is an endpoint of some base while \( d \) is one of the at most \( n_{1} \) “inner” points obtained in the previous step. It follows that the number \( n_{2} \) of endpoints along \( l \) is at most \( m_{1} \).

We now apply the unweighted selection lemma (Lemma 2.1(ii)) to this system of intervals and points. Note that no two distinct quadruplets can give rise to the same interval (assuming \( l \) is chosen so that no two segments connecting points in \( S \) meet \( l \) at the same point), because the endpoints of the interval uniquely determine the four points \( a, b, c, d \) forming the quadruplet. Thus the unweighted selection lemma is indeed applicable, and it yields a point \( x_{0} \in l \) that is contained in at least

\[
\frac{m_{2}}{4n_{2}} \geq \frac{m_{1}}{4n^{2}n_{1}} \geq \frac{1}{4n^{2}} \left( \frac{t_{0} \xi}{8n \log n} \right)^{2} = \frac{t_{0} \xi^{2}}{28n^{4} \log^{2} n}.
\]

intervals. Since each interval \( I_{0} \) is contained in the union of the two triangles forming \( Q \), it follows that \( x_{0} \) must lie in at least this many triangles of \( \mathcal{F} \), where each triangle is now counted with multiplicity, which fortunately is at most \( 2 \xi \) since recall that a triangle \( abc \) can be coupled with at most \( 2 \xi \) points \( d \) to form a
quadruple of the type we consider). It follows that \( x_0 \) lies in at least

\[
\frac{t_0^3}{2^n n^4 \log^2 n} \geq \frac{t^3}{2^n n^6 \log^5 n} = \frac{n^{3-3x}}{2^x \log n}
\]

distinct triangles, where the last expression is obtained by substituting the lower bounds on \( t_0, \xi \) that were noted above.

**Remark.** Notice that Theorem 3.1 is reasonably strong only as long as \( x \) is small; in particular, it is void if \( x \geq 1 \). We can prove that there is a point contained in at least \( t/2n = n^{3-x} \) triangles, which improves Theorem 3.1 if \( x \geq 1 \). For each point \( p \in S \) take the sum of the angles at \( p \) covered by the incident triangles. Since the sum of angles of any triangle is \( \pi \), there must be a point \( p \) whose sum of angles is at least \( t \pi /n \). The full angle at \( p \) is \( 2\pi \) which implies that there is a point near \( p \) covered by at least \( t/2n \) of the angles. This point lies in at least \( t/2n = \frac{n^{3-x}}{2} \) triangles.

### 4. An Application to Halving Planes in Space

Let \( S \) be a set of \( n \) points in \( d \)-dimensional space. A \( k \)-set of \( S \) is a subset \( T \subseteq S \) with \( |T| = k \) and \( T = S \cap h \) for some half-space \( h \). We define \( f_d,k(n) \) as the maximum number of \( k \)-sets realized by any set of \( n \) points in \( d \) dimensions. Obtaining sharp bounds on \( f_d,k(n) \) has remained an elusive task for almost 20 years. In the plane the known bounds are \( f_{2,k}(n) = \Omega(n \log k) \) [9], [10] and \( f_{2,k}(n) = O(n \sqrt{k} \log^* k) \) [12]. The problem of counting the total number of \( j \)-sets for \( 1 \leq j \leq k \) is much better understood. The maximum number of such \( j \)-sets for \( n \) points in \( d \) dimensions is \( \Theta(n^{j/d} k^{(d+1)/d}) \) [1], [7], [11].

The most difficult case in the analysis of \( f_d,k(n) \) seems to be when \( k \) is about half of \( n \). In three dimensions call a triangle \( abc \), \( a, b, c \in S \), a halving triangle if the plane through these points has at most \( (n - 3)/2 \) points of \( S \) on each side; call the plane through \( a, b, \) and \( c \) a halving plane. A recent result of Bárány et al. [4] shows that the number of halving planes of \( n \) points in three dimensions is \( O(n^{3-1/343}) \); this is the first nontrivial upper bound on this quantity. The best lower bound is \( \Omega(n \log n) \) [8].

Using Theorem 3.1 we can improve the upper bound on the number of halving planes.

**Theorem 4.1.** The number of halving planes of a set of \( n \) points in three dimensions is less than \( 2^{8/3} n^{8/3} \log^{5/3} n \).

**Proof.** We make use of the fact, established in [4], that no line in space can intersect more than \( \binom{n}{2} \) halving triangles of \( S \). Project the points and all halving triangles onto a plane so that no two points have coinciding projections. Theorem

3.1 implies that if \( S \) has more than \( n \) points in this plane contained in the projective image of the \( k \)-triangles. This implies that the line of intersection of many halving triangles must be smaller than \( n^{1/2} \), which is infeasible. A similar argument holds for \( k \)-planes.

**Remark.** The above bound also shows that \( k \)-planes spanned by a set \( S \) of \( n \) points where \( k \) is the number of facets bounded to \( S \) and the plane contains exactly one of its open half-spaces. In other words, the bound on the number of \( k \)-sets of a set of \( n \) points.

The maximum number of halving planes is proportional to the maximum number of \( k \)-sets in an arrangement of \( n \) planes in space, and level vertically onto the \( xy \)-plane we have a higher-order power diagram of the set of \( n \) circles in \( d \), so that the number of facets of the median cut is less than \( 1 \).

### 5. Discussion

This paper proves that for any set of \( n \) points in \( d \)-dimensional space there exists a point containing \( \Theta(n \log k) \) \( k \)-sets. A corollary of this result is that any set of \( n \) points in \( d \)-dimensional space contains a \( 2 \)-set \( \Theta(n \log n) \) with an \( \Omega(n \log n) \) \( 2 \)-set. The extension of these results to higher dimensions always contains a corollary: if \( d \geq 3 \), it seems likely that \( n \) halving planes in \( d \)-dimensional space have \( \Omega(n \log n) \) halving planes.

Another promising line of research involves the use of hyperrectangular boxes and spheres.

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that \( x_0 \) lies in at least
\[
\frac{n^3}{3^x} = \frac{n^3}{2^9 \log^5 n}
\]
which is obtained by substituting the lower bound on \( x \).

Remark. The above bound also serves as an upper bound for the number of \( k \)-triangles spanned by a set \( S \) of \( n \) points, where a \( k \)-triangle is a triangle whose vertices belong to \( S \) and the plane containing it contains exactly \( k \) points of \( S \) in one of its open half-spaces. In other words, the above bound is also an upper bound on the number of \( k \)-sets of a set of \( n \) points in three-dimensional space, for any \( k \). Of course, if \( k \) is very small, then better bounds are known (see [7]).

The maximum number of halving planes of a set of \( n \) points in space is proportional to the maximum number of facets of the so-called median level in an arrangement of \( n \) planes in space, see [8]. If we project the facets of a median level vertically onto the \( xy \)-plane we get a subdivision known as a higher-order power diagram of a set of \( n \) circles in the plane, see [2]. Thus, Theorem 4.1 implies that the number of facets of the median level in space and the number of regions in a higher-order power diagram is \( O(n^{8.3} \log^{5.3} n) \).

5. Discussion

This paper proves that for any set of \( n^{5.3} \) triangles with a total of \( n \) vertices in the plane there exists a point contained in at least \( n^{5.3}/(2^9 \log^5 n) \) of the triangles. A corollary of this result is that any set of \( n \) points in three-dimensional space defines at most \( 2^{9.3} n^{8.3} \log^{5.3} n \) halving triangles. An interesting open problem is the extension of these results to higher dimensions. In view of a result of Bárány [3], who shows that the set of \( \binom{n}{d+1} \) simplices defined by \( n \) points in \( d \) dimensions always contains a constant fraction of simplices with nonempty common intersection, it seems likely that the result on triangles can be generalized to dimensions \( d \geq 3 \). Such a generalization would then imply new upper bounds on the number of halving hyperplanes in \( d+1 \) dimensions.

Another promising line of research is to find other applications of the combinatorial selection lemmas of Section 2. Indeed, these lemmas can be generalized to hyperrectangular boxes and spheres in higher dimensions [6].

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References


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Some Provably Hard Crossing Problems

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*Abstract.* This paper presents a conjecture that a graph with the minimum number of edge pseudolines, a topic well-studied by any given arrangement can be forced to be a plane graph of an appropriate graph. Using some of Sturmfels, this yields that there exist a special straight-line drawing of a graph, with no pseudoline crossings. While this conjecture is fairly clear, we have used it to prove that a straight-line drawing of a graph with polygonal edges must have at least one crossing. Here we obtain a lower bound for the number of breakpoints which are required in any straight-line drawing of a graph.

1. Introduction

A drawing of a simple graph G is a representation of a different point, and line, and an ordering of the closed unit interval I with respect to the plane, where every point is either common vertex, where their dual edges may intersect. The problem (the term 2-color number problem consists of producing an edge-coloring with the minimum number of colors (this problem is denoted by c(G)). This problem is a graph layout problem (see [17]), and it has

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