# RAMANUJAN GRAPHS AND THEIR 2-FOLD COVERS

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NSERC USRA SUMMER RESEARCH PROJECT
20 August 2004
University of British Columbia

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## ABSTRACT

In this paper, a graph is discussed as being Ramanujan if its second largest eigenvalue, calculated from its adjacency matrix, follows the relation $\lambda_2 \leq 2\sqrt{(d-1)}$. Several tests were performed regarding the percentage of Ramanujan graphs and covers, but the main focus was on the relationship between the signed cover and the base eigenvalues. Numerical tests done on d-regular Ramanujan graphs show that there is a threshold base eigenvalue where a good base expander has a greater chance of doing worse as a cover (i.e. the signed cover highest eigenvalue is greater than the base graph $\lambda_2$) then turns to a decent base expander that has a greater chance of doing no worse as a cover (i.e. the signed cover highest eigenvalue is smaller than the base graph $\lambda_2$). This threshold depends on both the degree of the vertices and the number of vertices, in that the threshold increases as these two quantities do.

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# LIST OF COMPUTER PROGRAMS USED

NOTE:

- These programs were written by the authors of this paper using MATLAB.
- The program arguments use the following notation:

    n = number of vertices
    k = degree of vertices in k-regular graph
    A = base graph
    B = signed cover
    C = cover

*makeGraph2(n, k)*

- generates a random k-regular graph in short time (for example, an adjacency matrix for a graph with n = 1000000, k = 4 takes between 15 – 17 seconds to create)

*makeBpGraph(k, n)*

- generates only bipartite graphs

*makeSignedCover(A)*

- generates a signed cover for the base graph A

*makeCover(B)*

- generates a cover for base graph A, but requires a signed cover B to do this
- takes a significant amount of time

*findEVal_1(A, k)*

- used for finding the second eigenvalue of original base matrix A and cover C

*findEVal_2(lambda2_A, B, k)*

- used for finding the second eigenvalue of cover using the base graph and signed cover (instead of generating the cover C and directly finding its second eigenvalue)

*compareTimes(k, n)*

- a program that creates base A, signed cover B, cover C, finds the second eigenvalue of A and of the cover using the two methods mentioned above (compare eigenvalues of A and B and chose the second highest or directly calculate the second highest of the cover), and compares the times of the two methods

- results from using this program indicate that for small n, the faster way to create a cover is to make C (after making A and B) and find the second eigenvalue of the cover directly, whereas for n ~> 1000, it is faster to make just A and B then find the highest eigenvalue of B and compare it with the second highest of A. Although this is faster, it puts us at a disadvantage if we wish to make a tower since we would need to generate a cover C anyway.

*manySignedCovers(n, k)*

- a program that more compactly generates a user-defined number of signed covers for the same start graph

## KEYWORDS

***Adjacency Matrix:***

- Each graph has an associated adjacency matrix, A.
- To construct this matrix, label the graph's vertices from 1 to n. If vertices i and j are connected, then entry $a_{ij}$ of matrix A is 1. If the graph is not directed, A will be symmetric and entry $a_{ji}$ will also be 1.
- A in not unique since it depends on the labeling of the vertices. But different adjacency matrices can represent the same graph, and will give the same eigenvalues for the same graph.

***Bipartite Graph:***

- A bipartite graph, also known as a bigraph, is comprised of two distinct sets of vertices where vertices are connected if and only if they lie in different sets.
- If the graph is d-regular, a bipartite graph produces $\lambda_1 = d$ and $\lambda_n = -d$.

***Connected Graph:***

- There is a path joining every pair of distinct vertices.

***Degree of a Vertex:***

- The degree of a vertex of a graph is the number of edges connected to the vertex. The degree may not necessarily be the same for each vertex in the graph.

***Directed Graph:***

- Edges in the graph have a distinct direction from one vertex to another. If an edge points from vertex i to vertex j then only entry $a_{ij}$ of the adjacency matrix is 1.

***Disconnected Graph:***

- A graph that is not connected.
- A disconnected d-regular graph produces $\lambda_1 = d$ with an algebraic multiplicity > 1.

***d-Regular Graph:***

- A graph is regular if each vertex has the same degree. A graph is referred to as a d-regular graph when each vertex has degree d.

***Graph:***

- A graph, G, is a set of vertices connected by edges and is represented by G(V, E) where V is the set of vertices and E is the set of edges.

***Loop:***

- An edge that begins and ends at the same vertex.

***Multigraph:***

- A multigraph is not a simple graph. There can be more than one edge connecting any two vertices.

***n-Fold Cover[1]:***

- Each vertex of a base graph is lifted to n vertices in the cover and they are connected randomly, such that there is an n:1 covering map onto the base graph and the adjacency of corresponding vertices in the base graph and cover are preserved.

***Signed Cover:***

- Associated with a 2-fold cover.
- Generated by taking each element $a_{ij}$ in the base graph matrix and changing it to either $+a_{ij}$ or $-a_{ij}$.

***Simple Graph:***

- A simple graph is a graph for which at most one edge connects any two vertices.

[1] This definition and information about covers is based on two papers: Yonatan Bilu and Nati Linial, "Lifts, Discrepancy and Nearly Optimal Spectral Gaps", December 2003; and Alon Amit and Nathan Linial, "Random Graph Coverings I: General Theory and Graph Connectivity", revised November 16, 2000.

# 1.0 INTRODUCTION

A graph is a set of vertices connected by edges. Denote n as the number of vertices and d as the degree of each vertex. In application, a vertex can be regarded as a node, location, or an entity, while the edge would be the path between each node. An example would be a country in which there are many cities (vertices) and various roads (edges) connecting cities to each other.

A graph has certain properties including whether or not it is Ramanujan. To decided this, one must look at the eigenvalues of the graph's adjacency matrix. Though there are variations of the Ramanujan condition, in this paper, the property that a graph is Ramanujan is when the second highest arithmetic eigenvalue $\lambda_2 \leq 2\sqrt{(d-1)}$.

This research project also looks at whether a cover of a graph is Ramanujan or not. There are two ways to generate a cover (this applies only to 2-fold covers). First generate a base graph $G_A$ with adjacency matrix A. To generate the graph's signed cover, each entry of A is changed to the negative of itself or left the same, in a manner that keeps the resulting matrix symmetric. These changes are done at random, so that a random graph is generated. Geometrically, this plus/minus change is represented on the 2-fold cover. Suppose vertices i and j are connected for $G_A$. The 2-fold cover now has two vertices where i used to be and two where j used to be. If the top vertex of i connects to the top of j (and so the bottom of i to the bottom of j) then the signed cover matrix, B, has a 1 in position $a_{ij}$ and $a_{ji}$. But if the top vertex of i connects to the bottom of j, then the entries of B are -1 instead. The signed cover matrix is then n x n like A, but the cover matrix, C, is 2n x 2n, and is obtained as one would an adjacency matrix from any other graph.

There are two ways to find the eigenvalues of the cover. Let $\lambda_{2A}$ be the second highest eigenvalue of the base graph $G_A$. Let $\lambda_{1B}$ be the highest eigenvalue of the signed cover B. Let $\lambda_{2C}$ be the second highest eigenvalue of the cover C. Let $\lambda_2$ be the second highest eigenvalue of the cover. The first way to find $\lambda_2$ is to find the old eigenvalues and the new eigenvalues[2]. The union of these eigenvalue sets give the eigenvalues of the cover. Thus, one need only compare $\lambda_{1B}$ and $\lambda_{2A}$ to find $\lambda_2 = \max(\{\lambda_{1B}, \lambda_{2A}\})$. The second way is to look directly at the cover adjacency matrix to get $\lambda_2 = \lambda_{2C}$.

[2] The word "new" refers to the eigenvalues of the signed cover, based on the nomenclature of Dr. Joel Friedman, "Relative Expanders or Weakly Relatively Ramanujan Graphs ", last modified April 8, 2002, Duke Mathematical Journal.

# 2.0 TESTS PERFORMED

Four tests were performed using the programming resources mentioned in "List of Computer Programs Used".

NOTE: Only d-regular graphs and 2-fold covers were used in this research.

## 2.1 Test 1: Random Graphs

Check if graphs of different n are Ramanujan.

Ranges Tested:
n: $0 < n \leq 500\,000$
d: $d = 4$

## 2.2 Test 2: Random Graphs with Random 2-Fold Covers

Check if 2-fold covers of graphs of different n are Ramanujan.

Ranges Tested:
n: $0 < n \leq 500\,000$
d: $d = 4$

## 2.3 Test 3: Random Graph with Many Random Signed Covers

For a random graph, generate many signed covers and investigate if there is a pattern in $\lambda_{1B}$ and $\lambda_{2C}$ in relation to the base eigenvalue $\lambda_{2A}$. The procedure is repeated with many start graphs of the same n and d.

Ranges Tested:
n: $n = 1000, 2000, 5000, 10000$
d: $d = 4, 6$

To estimate N, the number of total signed covers that must be made for each old eigenvalue, a 90% confidence interval with a margin of error of ± 5% in the new eigenvalues was used. The average probability that a non-Ramanujan graph is generated was estimated as p = 25%. The $z_c = 1.65$ for the 90% confidence interval. The margin of error E = 5%.

$$\begin{aligned} N &= p\,(1- p\,)\,(z_c\,/\,E\,)^2 \\ &= (0.25)(0.75)[(1.65)/(0.05)]^2 \\ &= 204 \end{aligned}$$

The number of graphs needed for each old eigenvalue was also estimated. For this, a 90% confidence interval with a ± 10% margin of error was used. From the same formula, N = 50.

Figure 1 below breaks down the actual number of graphs generated.

**Figure 1 :** Total Number of Graphs Generated for Test 3

*d* ---------------------------- 4 ---------------------------------- 6 -------------------------

*n* -------------- 1000, 2000, 5000, 10000 ------- 1000, 2000, 5000, 10000------------

*old* $\lambda_2$ --------------- 50 for each n ---------------------50 for each n -------------------

*new* $\lambda_2$ ----------- 200 for each old $\lambda_2$ ------------- 200 for each old $\lambda_2$ ---------------

*Total graphs generated: 80000*

## 2.4 Test 4: Towers

Start with a random graph of different n. Generate a cover and use this as a base graph, for which another cover is made. Repeat this procedure until the graph becomes non-Ramanujan (i.e. make a cover of a cover of a cover…) Check how ‘big’ the towers are – how many vertices the iterations reach until the graph becomes non-Ramanujan.

Ranges Tested[3]:
n: n = 500, 1000, 10000
d: d = 4

## 2.5 Tests Discussion

Investigations for various number of vertices were done. If the results from Test 1 indicate that the percentage of Ramanujan graphs are approximately the same among different vertex ranges then the results and patterns of Test 2 and 3 can be extended to other (higher) number of nodes.

Also, there occurs a local phenomenon where the eigenvalues go awry and become particularly high, especially for d = 4. To ensure consistency in the results, d = 6 was also investigated.

[3] The number of vertices, n, in Ranges Tested indicates the starting value of the towers.

## 3.0 OBSERVATIONS AND RESULTS

### 3.1 Test 1: Random Graphs

Table 1 shows the percentage of Ramanujan graphs for different ranges of n and how many graphs were generated in each range. Though a more rigorous selection of graphs is needed to draw definite conclusions, the pattern seems to hover at around 84% of graphs being Ramanujan and 16% being non-Ramanujan. The numbers fluctuate throughout the ranges of n tested so it is hard to tell whether the percentages are increasing, decreasing or remaining approximately the same as n increases. We can cautiously say that the percentage of Ramanujan graphs rests around the same number and so assume that the patterns of Test 3 can be extended to other numbers of vertices.

**Table 1 :** Test 1 – Percentage of Ramanujan Graphs for Various Vertex Ranges

| | Ramanujan Graph? | | | Ramanujan Graph? | |
|---|---|---|---|---|---|
| n | # of yes | # of no | total # | % of yes | % of no |
| 0 < n ≤ 10000 | 135 | 28 | 163 | 82.8 | 17.2 |
| 10000 | 67 | 8 | 75 | 89.3 | 10.7 |
| 10000 < n ≤ 100000 | 145 | 23 | 168 | 86.3 | 13.7 |
| 100000 | 60 | 14 | 74 | 81.1 | 18.9 |
| 100000 < n ≤ 200000 | 229 | 50 | 279 | 82.1 | 17.9 |
| 200000 | 84 | 16 | 100 | 84.0 | 16.0 |
| 200000 < n ≤ 500000 | 10 | 2 | 12 | 83.3 | 16.7 |
| 500000 | 10 | 2 | 12 | 83.3 | 16.7 |
| 0 < n ≤ 500000 | 462 | 87 | 549 | 84.2 | 15.8 |

### 3.2 Test 2: Random Graphs with Random 2-Fold Covers

Table 2 shows the percentage of covers that are Ramanujan for various ranges of n. As with Test 1, a greater number of graphs must be investigated to have more concrete results. However, we may again cautiously assume that the percentage of covers generated that are no worse that the base graph (the number of scenarios called "yellow" in the table) lies anywhere from 25% to 34% at around a 32% average. (Only the percentages for ranges of n were used and not percentages for single n values.) This scenario is the best expected from a cover, but since it only appears 32% of the time, it will arise with less probability than the scenario that the cover is worse than the base graph though still Ramanujan.

The percentage of non-Ramanujan covers (scenarios called "green" in the table) lies from 15% to 22%. Further tests may show that these percentage ranges are smaller, and that all intervals of n lie at approximately the same number. If this is indeed the case, the percentage of non-Ramanujan covers would lie at around 19%, judging from the current results. This is echoed, as may be expected, in the 16% of non-Ramanujan graphs from Test 1.

| **Table 2 :** Test 2 – Percentages for Cover Situations for Various Vertex Ranges | | | | | |
|---|---|---|---|---|---|
| | # green | # yellows | | | |
| n | $\lambda_{1B} > 2\sqrt{(d-1)}$ | $\lambda_{1B} < \lambda_{2A}$ | # covers | % yellow | % green |
| 0 < n ≤ 10000 | 36 | 51 | 161 | 31.7 | 22.4 |
| 10000 | 20 | 30 | 73 | 41.1 | 27.4 |
| 10000 < n ≤ 100000 | 34 | 51 | 158 | 32.3 | 21.5 |
| 100000 | 9 | 28 | 71 | 39.4 | 12.7 |
| 100000 < n ≤ 200000 | 27 | 60 | 177 | 33.9 | 15.3 |
| 200000 | 3 | 15 | 50 | 30.0 | 6.0 |
| 200000 < n ≤ 500000 | 3 | 3 | 12 | 25.0 | 25.0 |
| 500000 | 3 | 3 | 12 | 25.0 | 25.0 |

### 3.3 Test 3: Random Graph with Many Random Signed Covers

Graphs 1 and 2 have on the x-axis the base $\lambda_{2A}$ values and on the y-axis the percentage of how many yellow, green, and white scenarios occur with the condition that the base graph is Ramanujan. These scenarios show the three possible relationships between the new and old eigenvalues and are described below.

The procedure of this Test is as follows: A random graph is made and its $\lambda_{2A}$ value found. A random signed cover is then made and its $\lambda_{1B}$ is found. Many other signed covers are made for this same base graph.

- If $\lambda_{1B} < \lambda_{2A}$ then $\lambda_2 = \lambda_{2A}$. This situation is represented by the yellow scenario. This means that the cover is no worse than the base graph, since the cover has the same eigenvalue as the base. This situation is the best one expected, assuming we want to generate graphs that are Ramanujan. So, the smaller the new eigenvalues are, the better.
- If $\lambda_{2A} < \lambda_{1B} \leq 2\sqrt{(d-1)}$ then $\lambda_2 = \lambda_{1B}$. This situation is represented by the white scenario. In this case the cover is still Ramanujan but worse than the base graph. This is the next best case.
- If $\lambda_{1B} > 2\sqrt{(d-1)}$ then $\lambda_2 = \lambda_{2B}$ and the cover is not Ramanujan. This is the worst case and is represented by the green scenario.

The following general observations are common to all the graphs:

- The green set of points are arranged in an approximately linear, horizontal regression. This implies that the percentage of non-Ramanujan covers per graph is approximately the same for each base graph. Among all n and d pairs, this percentage is the same at ~24% (using a 90% confidence interval, this number is correct to within ± 10% at the current count of base graphs tested). Therefore there is a ~76% chance of obtaining a Ramanujan graph (cover) from another Ramanujan graph. That the percentage of non-Ramanujan covers is constant for various n is not too surprising since Test 1 results show that the percentage of non-Ramanujan graphs remains approximately constant over various intervals of n.

- The yellow curve diverges from the white curve at the minimum and maximum base eigenvalues with asymptotes characteristic of a logistic curve. The two curves cross somewhere in the $\lambda_{2A}$ range. That is, when $\lambda_{2A}$ is small, there are many white scenarios expected. This means that:
  a.) there is little chance of obtaining a cover that is no worse than the base, and
  b.) < 76% of the covers generated will be Ramanujan but a bit worse than the base (with a larger $\lambda_2$).

  Consequently, when $\lambda_{2A}$ is large but still under $2\sqrt{(d-1)}$, there are many yellow scenarios expected. This means that:
  a.) there is great chance of obtaining a cover that is no worse than the base, and
  b.) < 76% of the covers will be Ramanujan and have the same eigenvalue as the base graph – an optimal scenario.
- Since the yellow and white curves behave in reverse ways, they cross somewhere in between the smallest $\lambda_{2A}$ and the largest base $\lambda_{2A}$ possible. We will call this intersection a threshold and name it $\lambda_{threshold}$. The value represented by this threshold is where a white and yellow scenario have equal chance of occurring. Beyond this threshold, for higher $\lambda_{2A}$, the covers have a dominating chance of being no worse than the base graph and still Ramanujan – the best scenario. It is interesting to calculate how many base graphs for a certain n and d pair are above the threshold. Table 3 shows this. Averaging the percentages for d = 4 and d = 6, we see that ~ 49% of the base graphs have $\lambda_{2A} > \lambda_{threshold}$.

**Table 3 :** Test 3 – Percentages of Base Eigenvalues Greater than Threshold Value

| | d = 4 | | d = 6 | |
|---|---|---|---|---|
| n | $\lambda_{threshold}$ | % $\lambda_{2A}$ above $\lambda_{threshold}$ | $\lambda_{threshold}$ | % $\lambda_{2A}$ above $\lambda_{threshold}$ |
| 1000 | 3.453 | 50 | 4.453 | 46 |
| 2000 | 3.456 | 68 | 4.461 | 44 |
| 5000 | 3.46 | 52 | 4.465 | 44 |
| 10000 | 3.462 | 44 | 4.468 | 44 |
| | | average % 53.5 | | average % 44.5 |

**For n=1000 d=4:**
- The threshold value is at ~ 3.453
- Non-Ramanujan covers are generated 23.0 % of the time

**For n=2000 d=4:**
- The threshold value is at ~ 3.456
- Non-Ramanujan covers are generated 23.0 % of the time

**For n=5000 d=4:**
- The threshold value is at ~ 3.460
- Non-Ramanujan covers are generated 22.3 % of the time

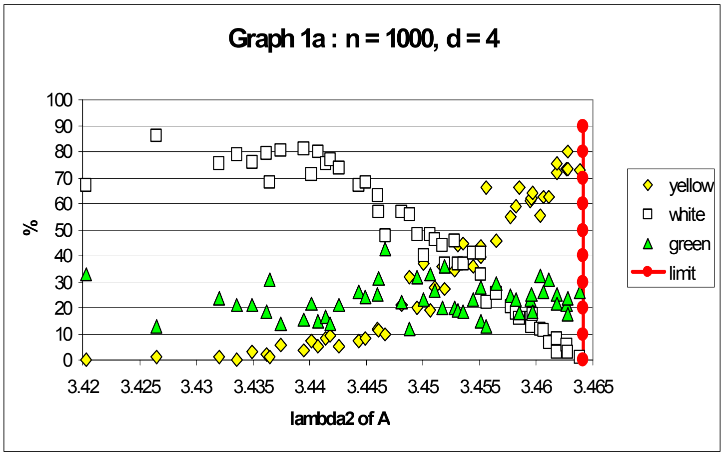
**For n=10000 d=4:**
- The threshold value is at ~ 3.462
- Non-Ramanujan covers are generated 19.4 % of the time

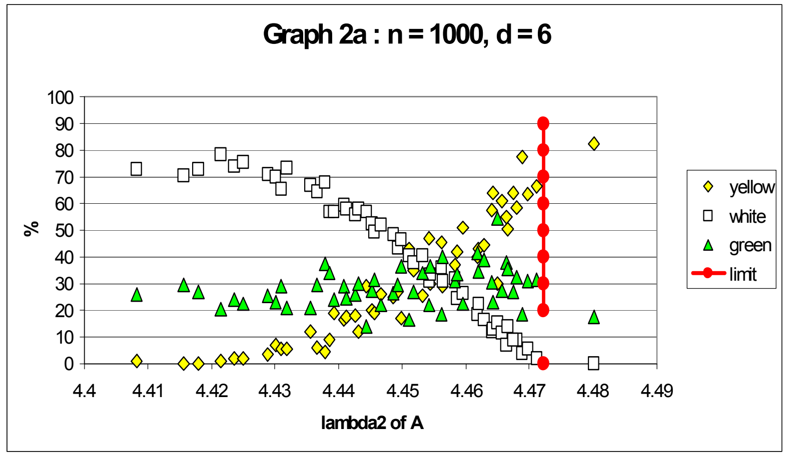
**For n=1000 d=6:**
- The threshold value is at ~ 4.453
- Non-Ramanujan covers are generated 28.5 % of the time

**For n=2000 d=6:**
- The threshold value is at ~ 4.461
- Non-Ramanujan covers are generated 26.0 % of the time

**For n=5000 d=6:**
- The threshold value is at ~ 4.465
- Non-Ramanujan covers are generated 23.7% of the time

**For n=10000 d=6:**
- The threshold value is at ~ 4.468
- Non-Ramanujan covers are generated 21.9 % of the time


Graph 1a : n = 1000, d = 4
%
lambda2 of A
yellow
white
green
limit


Graph 2a : n = 1000, d = 6
%
lambda2 of A
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white
green
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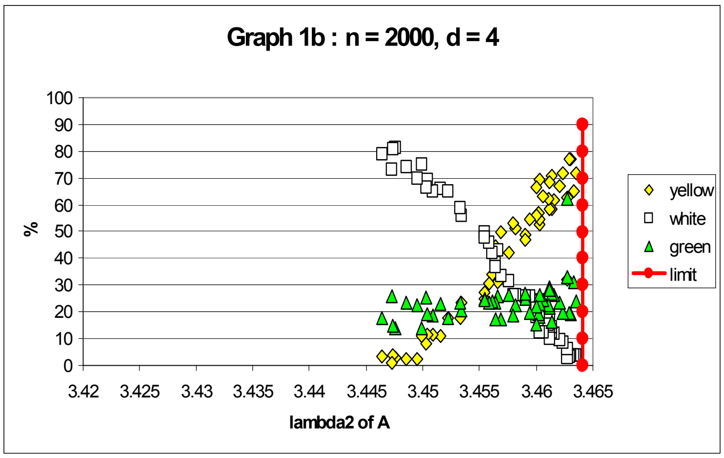
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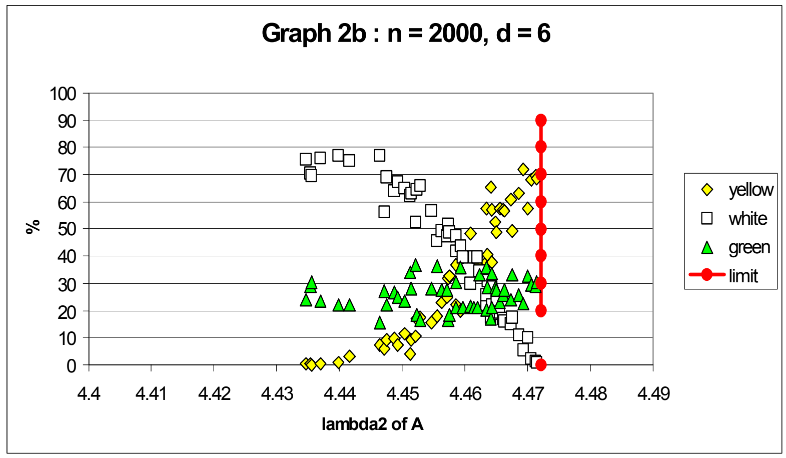
%
lambda2 of A
yellow
white
green
limit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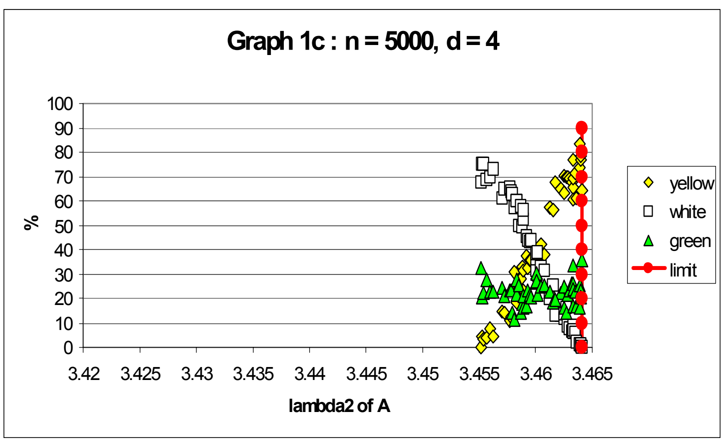
green
limit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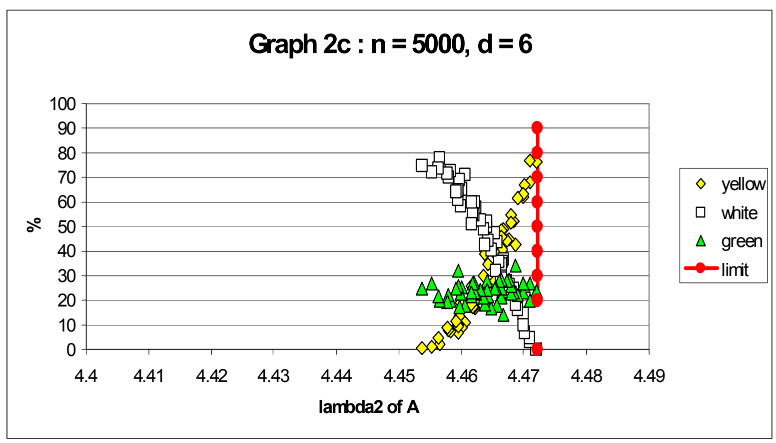
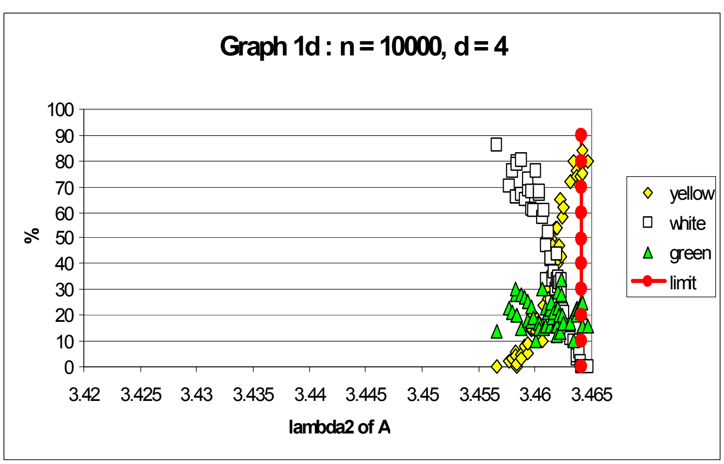
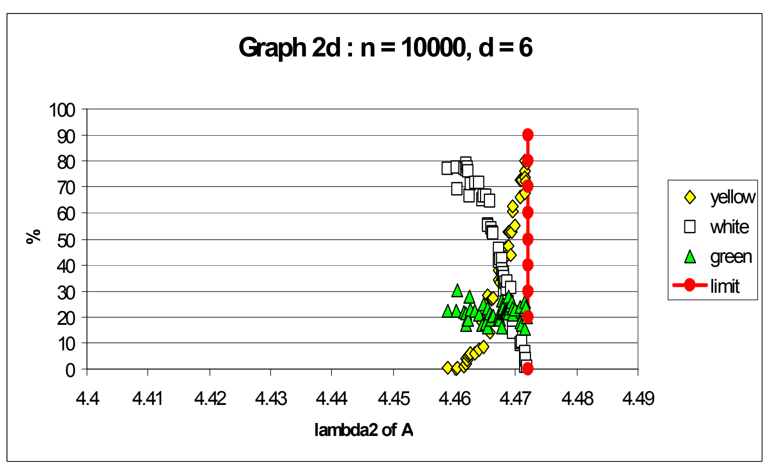
green
limit


Graph 2c : n = 5000, d = 6
%
lambda2 of A
yellow
white
green
limit


Graph 1d : n = 10000, d = 4
%
lambda2 of A
yellow
white
green
limit


Graph 2d : n = 10000, d = 6
%
lambda2 of A
yellow
white
green
limit

### 3.3.1 Observation 1

As suggested by the similarities between the graphs, we pose the following conjecture:

**Conjecture 1:** The threshold value depends on the value of n, increasing as n does.

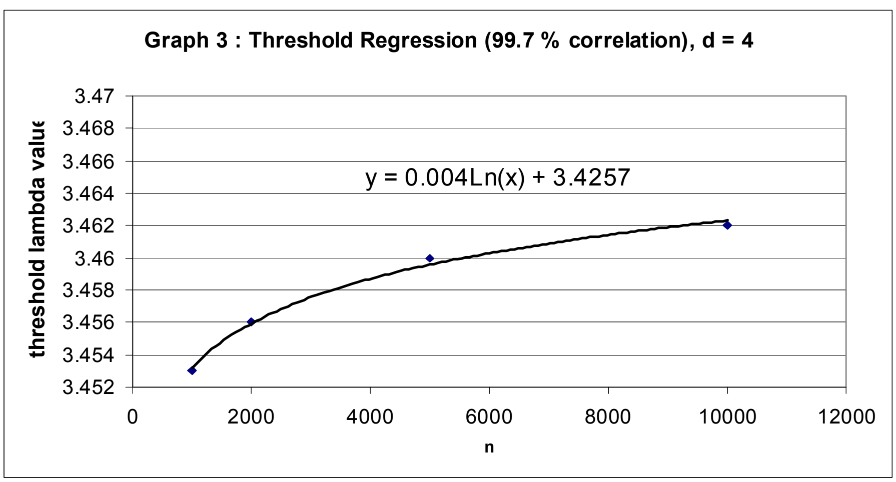
We know that since $\lambda_2 \rightarrow 2\sqrt{(d-1)}$ as $n \rightarrow \infty$, the average $\lambda_2$ value increases as n increases, and so the threshold value is expected to also increase with increasing n. The graphs above also support Conjecture 1.

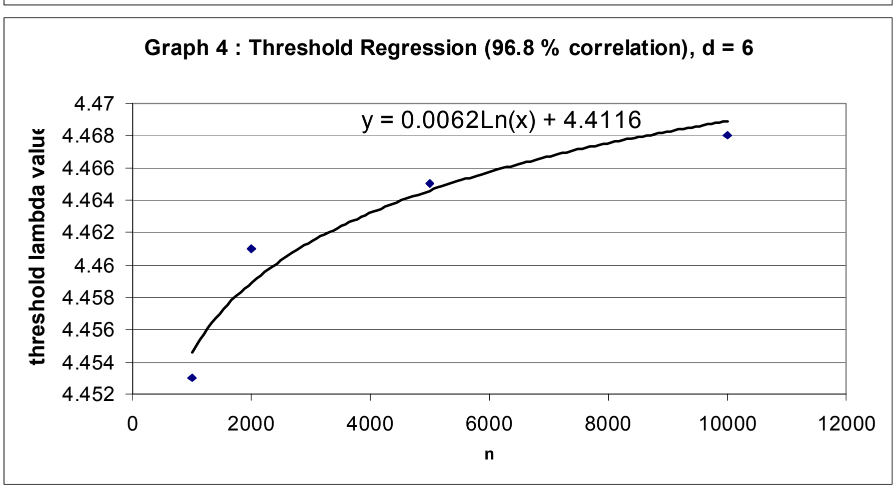
How does this value relate to n and is it always less than $2\sqrt{(d-1)}$? The threshold values have been plotted in Graph 3 (d = 4) and Graph 4 (d = 6) and connected with a logarithmic curve. The correlation factor for Graph 3 is 99.7% and for Graph 4 is 96.8%. The logarithmic curve is a sensible approximation, but the correct equation may be $O(\log^2 n)$ or a higher power, or perhaps a curve with a horizontal asymptote at $\lambda_{threshold} = 2\sqrt{(d-1)}$ if the threshold value is always less that $2\sqrt{(d-1)}$, as may be expected.

The present equations for the threshold value, $\lambda_{threshold}$, function of n are:

For d = 4: $\lambda_{threshold}(n) = 0.004 \ln(n) + 3.4257$

For d = 6: $\lambda_{threshold}(n) = 0.0062 \ln(n) + 4.4116$



### 3.3.2 Observation 2

The differences between the graphs are also of interest. The higher n graphs are shifted horizontally to the right and compressed horizontally. This means that for the same base $\lambda_{2A}$ value, n = 10000 gives a low chance of getting a cover no worse than the base and n = 1000 gives a high chance.

**Conjecture 2:**

For the same base $\lambda_{2A}$ value, there is greater probability of obtaining a cover no worse than the base when n is small. That is, the percentage of yellow scenarios decreases with increasing n, for each particular $\lambda_{2A}$ value.

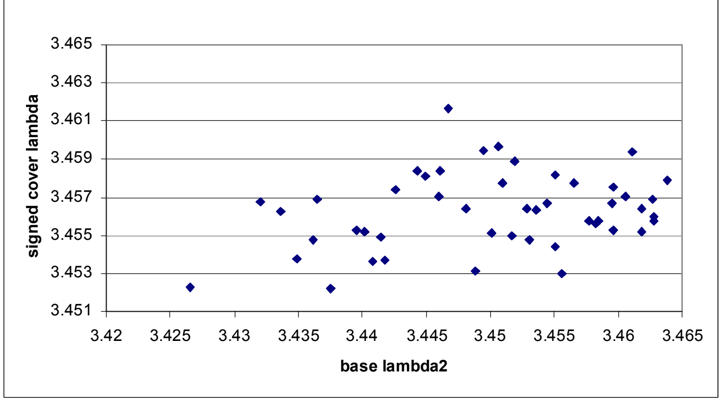
To show this, we use the shapes of the curves discussed above. Suppose n1 < n2. Since $\lambda_2 \rightarrow 2\sqrt{(d-1)}$ as $n \rightarrow \infty$ then for larger n, the lowest base $\lambda_{2A}$ is greater for n2 than for n1. We assume that the shape of the yellow and white curves from Graphs 1 and 2 are accurate. For the lowest base $\lambda_{2A}$ there is close to 0% yellow and 100% white. The lowest $\lambda_{2A}$ for n2 also has it's yellow at 0% but at an $\lambda_{2A}$ value higher than for n1. Therefore the yellow graph is shifted to the right horizontally for n2, meaning that for the same $\lambda_{2A}$, there is a greater chance of obtaining yellow for low n than for high n. The same argument can be used for the white graph, which is only an inverted version of the yellow graph.

Also, by observation, the smaller n values give a larger range of $\lambda_{2A}$ eigenvalues, which is why the graph for n = 1000 is so stretched out compared to larger n. Because of this, there is a greater area available for getting a cover no worse than the base. Consequently, on average, $\lambda_{2A} > \lambda_{1B}$ (more yellow) for smaller n and $\lambda_{2A} < \lambda_{1B}$ (less yellow) for larger n.

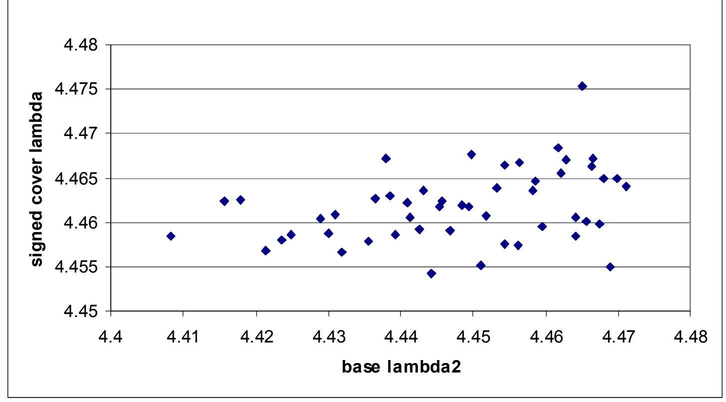
### 3.3.3 Observation 3

The correlation between $\lambda_{2A}$ and the average $\lambda_{1B}$ for each base eigenvalue is depicted in Graphs 5 and 6. (Of the 50 $\lambda_{1B}$ signed cover eigenvalues generated for each base graph, their average is calculated and this average is depicted in the graphs). The graph points are scattered with no apparent correlation, indicating there is no formulaic relationship between $\lambda_{2A}$ and $\lambda_{1B}$. This observation may suggest that the backtracking walks along a graph are more important to the graph than other types of walks.

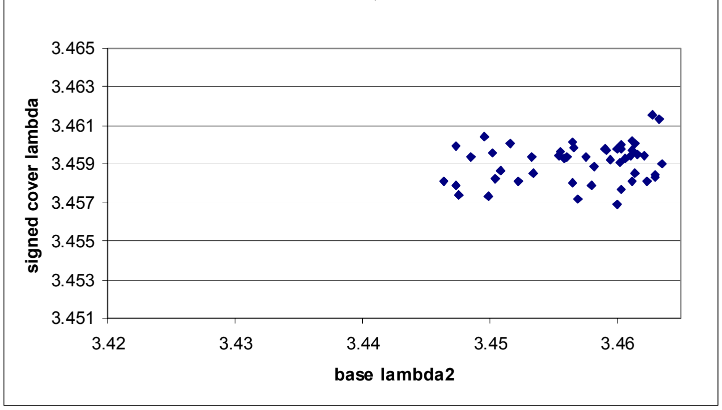
We note also that the cluster of old vs. new eigenvalues appears to be contracting, with the $\lambda_{1B}$ average for each graph converging towards a point with coordinates $( 2\sqrt{(d-1)}, 2\sqrt{(d-1)} )$. This indicates that as n increases, the new eigenvalues form less of a spread and, like $\lambda_{2A}$, their average approaches $2\sqrt{(d-1)}$ as $n \rightarrow \infty$.



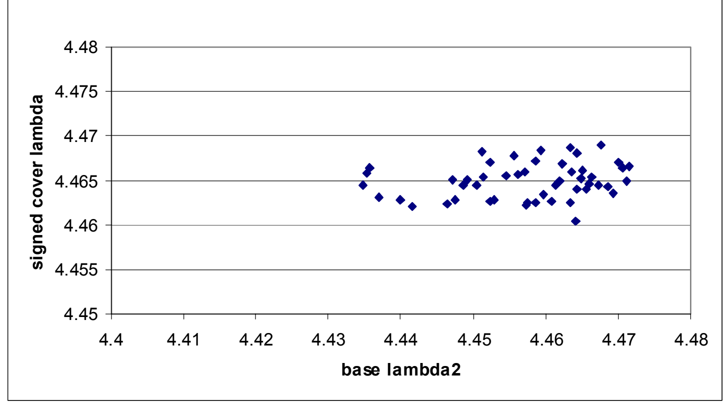
Graph 5a : Base Lambda vs. New Lambda for n = 1000, d = 4



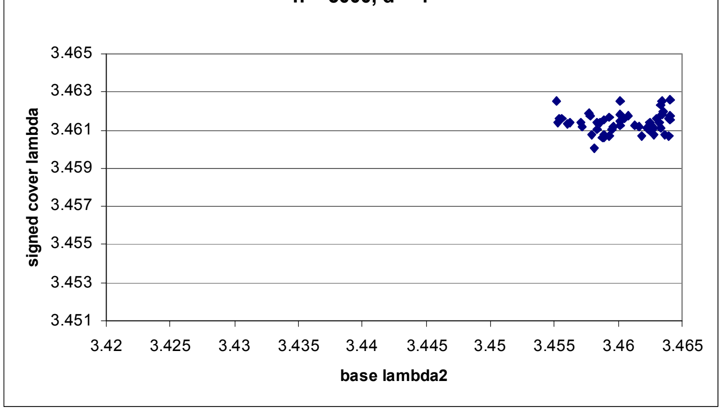
Graph 6a : Base Lambda vs. New Lambda for n = 1000, d = 6



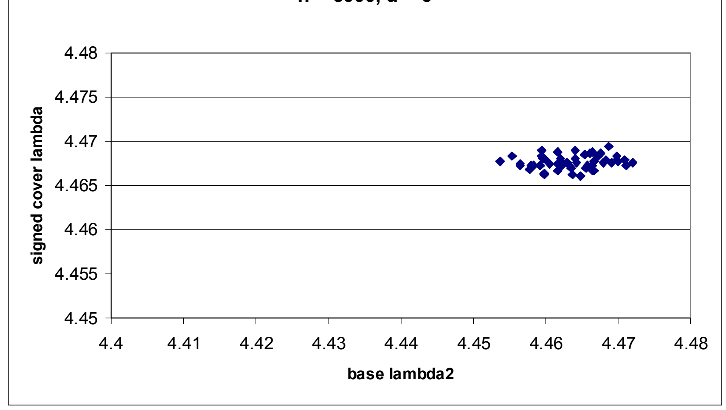
Graph 5b : Base Lambda vs. New Lambda for n = 2000, d = 4



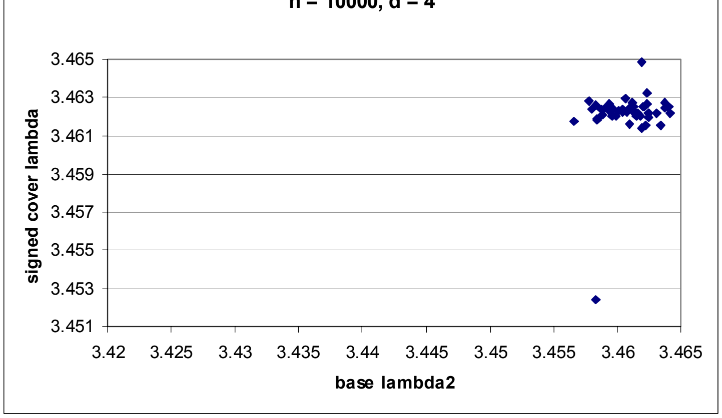
Graph 6b : Base Lambda vs. New Lambda for n = 2000, d = 6



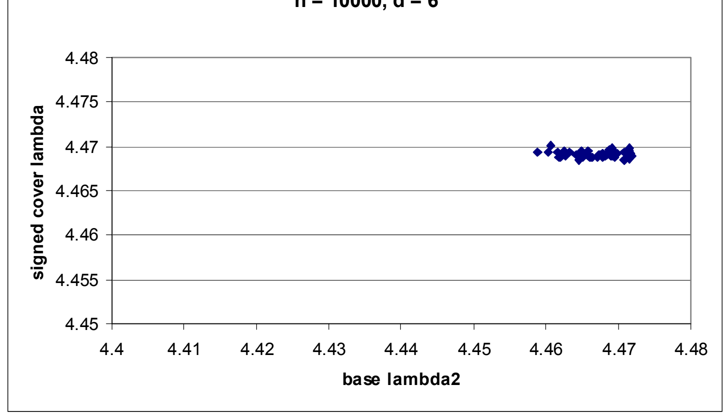
Graph 5c : Base Lambda vs. New Lambda for n = 5000, d = 4



Graph 6c : Base Lambda vs. New Lambda for n = 5000, d = 6



Graph 5d : Base Lambda vs. New Lambda for n = 10000, d = 4



Graph 6d : Base Lambda vs. New Lambda for n = 10000, d = 6

## 3.4 Test 4: Towers

The towers created in this Test had a variety of heights, the largest rising from n = 500 to n = 16000. Inasmuch as only a few towers were created, the results are not conclusive; there are a couple of possible conjectures that may be correct:

1. Towers are big when the initial $\lambda_{2A}$ is small. This generally occurs when the number of initial nodes are small also.

2. Towers are big when the initial $\lambda_{2A}$ is greater than the threshold value for the specific n and d as discussed in Test 3 Observations. The threshold value for the starting n was calculated from the logarithmic formula and so is a rough approximation.

# 4.0 CONCLUSION

If $\lambda_{2A}$ is high for the adjacency matrix A of a Ramanujan graph, then it is more likely to have the $\lambda_2$ for the cover come from the base graph rather than from the signed cover. Further, for the same base $\lambda_{2A}$ value for different values of n, a cover no worse than the base is more probable when n is small. There does exist a threshold base eigenvalue for every n and d pair, such that $\lambda_{2A}$ values above $\lambda_{threshold}$ will yield a higher probability that the cover in no worse than the base. These threshold values were plotted and their resulting regression appears to be O(log(n)) to a good correlation factor. However, a better fit may be a graph where $\lambda_{threshold} \rightarrow 2\sqrt{(d-1)}$ as $n \rightarrow \infty$.

No concrete association was observed between the old eigenvalue and the new average eigenvalue. But it was noticed that for larger n the cluster constricted and seemed to be converging to one point. We speculate that its coordinates are ( $2\sqrt{(d-1)}$, $2\sqrt{(d-1)}$ ).

The steps below show how to best obtain a Ramanujan graph:

1. Obtain a base graph with $\lambda_{2A} \geq \lambda_{threshold}$.
   - The probability that a Ramanujan graph is generated as a base graph is ~ 84%.
   - The probability that a graph with $\lambda_{2A} > \lambda_{threshold}$ is generated is 54% for d = 4 and 45% for d = 6.
   - $\lambda_{threshold}$ can be approximately deduced for any n from the formulas:

     For d = 4: $\lambda_{threshold}(n) = 0.004 \ln(n) + 3.4257$

     For d = 6: $\lambda_{threshold}(n) = 0.0062 \ln(n) + 4.4116$

2. Obtain a signed cover.
   - The probability of obtaining a Ramanujan cover per base graph is ~76%.
   - If the graph was chosen as stated in Step 1, the majority of the signed cover $\lambda_{1B}$ values should be less than $\lambda_{2A}$.
   - Depending on how high above the threshold $\lambda_{2A}$ was chosen, the cover will have ~ 40% to ~ 80% chance of having the cover $\lambda_{2C}$ the same as $\lambda_{2A}$.

3. Convert the signed cover into a cover.
   - This is useful if the cover will be used as another graph, like as a base graph in a tower construction.

# 5.0 FUTURE PROJECTS

As was inferred in “Observations and Results”, it would be interesting and beneficial to generate more graphs for Test 1 and 2 and observe whether the conjecture regarding the percentage of Ramanujan graphs and Ramanujan covers are correct.

This project, however, was more focused on the concept of covers, and seems a likely topic for future projects also. The potential that covers have for giving a method of generating Ramanujan graphs can play a large part in determining how good expanders can be obtained for use in efficiency of networks and other such systems. In relation to this project, further investigation must be done into how useful the threshold value for the base second eigenvalue is in generating covers that are good expanders. Though it is certainly optimal to obtain a cover no worse than the base, it remains to be investigated whether this is necessary for having a high tower construction.

As well, the percentages and values in this project must be supported by further mathematical proof, even though, at present, the graphs, tables, and results do suggest that the conjectures presented in this paper are correct. An interesting branch from this research would be to detail what our conclusions mean in terms of the geometry of the graph, as in Observation 3.