Synthesizing Bijective Lenses

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Bidirectional transformations between different data representations occur frequently in modern software systems. They appear as serializers and deserializers, as parsers and pretty printers, as database views and view updaters, and as a multitude of different kinds of ad hoc data converters. Manually building bidirectional transformations—by writing two separate functions that are intended to be inverses—is tedious and error prone. A better approach is to use a domain-specific language in which both directions can be written as a single expression. However, these domain-specific languages can be difficult to program in, requiring programmers to manage fiddly details while working in a complex type system.

We present an alternative approach. Instead of coding transformations manually, we synthesize them from declarative format descriptions and examples. Specifically, we present Optician, a tool for type-directed synthesis of bijective string transformers. The inputs to Optician are a pair of ordinary regular expressions representing two data formats and a few concrete examples for disambiguation. The output is a well-typed program in Boomerang (a bidirectional language based on the theory of lenses). The main technical challenge involves navigating the vast program search space efficiently enough. In particular, and unlike most prior work on type-directed synthesis, our system operates in the context of a language with a rich equivalence relation on types (the theory of regular expressions). Consequently, program synthesis requires search in two dimensions: First, our synthesis algorithm must find a pair of "syntactically compatible types," and second, using the structure of those types, it must find a type- and example-compliant term. Our key insight is that it is possible to reduce the size of this search space without losing any computational power by defining a new language of lenses designed specifically for synthesis. The new language is free from arbitrary function composition and operates only over types and terms in a new disjunctive normal form. We prove (1) our new language is just as powerful as a more natural, compositional, and declarative language and (2) our synthesis algorithm is sound and complete with respect to the new language. We also demonstrate empirically that our new language changes the synthesis problem from one that admits intractable solutions to one that admits highly efficient solutions, able to synthesize lenses between complex file formats with great variation in seconds. We evaluate Optician on a benchmark suite of 39 examples that includes both microbenchmarks and realistic examples derived from other data management systems including Flash Fill, a tool for synthesizing string transformations in spreadsheets, and Augeas, a tool for bidirectional processing of Linux system configuration files.

1 INTRODUCTION

Programs that analyze consumer information, performance statistics, transaction logs, scientific records, and many other kinds of data are essential components in many software systems. Often times, the data analyzed comes in ad hoc formats, making tools for reliably parsing, printing, cleaning, and transforming data increasingly important. Programmers often need to reliably transform back-and-forth between formats, not only transforming source data into a target format but also safely transforming target data back into the source format. Lenses [13] are back-and-forth transformations that provide strong guarantees about their round-trip behavior, guarding against data corruption while reading, editing, and writing data sources.
A lens comprises two functions, \textit{get} and \textit{put}. The \textit{get} function translates source data into the target format. If the target data is updated, the \textit{put} function translates this edited data back into the source format. A benefit of lens-based languages is that they use a single term to express both \textit{get} and \textit{put}. Furthermore, well-typed lenses give rise to \textit{get} and \textit{put} functions guaranteed to satisfy desirable invertibility properties.

Lens-based languages are present in variety of tools and have found mainstream industrial use. Boomerang [3, 5] lenses provide guarantees on transformations between \textit{ad hoc} string document formats. Augeas [25], a popular tool that reads Linux system configuration files, uses the \textit{get} part of a lens to transform configuration files into a canonical tree representation that users can edit either manually or programmatically. It uses the lens’s \textit{put} to merge the edited results back into the original string format. Other lens-based languages and tools include GRoundTram [19], BiFluX [29], BiYacc [40], Brul [39], BiGUL [20], bidirectional variants of relational algebra [4], spreadsheet formulas [26], graph query languages [18], and XML transformation languages [24].

Unfortunately, these languages impose fiddly constraints on lenses, making lens programming slow and tedious. For example, Boomerang programmers often must rearrange the order of data items by recursively using operators that swap adjacent fields. Furthermore, the Boomerang type checker is very strict, disallowing many programs because they contain ambiguity about how certain data is transformed. In short, lens languages provide strong bidirectional guarantees at the cost of forcing programmers to satisfy finicky type systems.

To make programming with lenses faster and easier, we have developed \textit{Optician}, a tool for synthesizing lenses from simple, high-level specifications. This work continues a recent trend toward streamlining programming tasks by synthesizing programs in a variety of domain-specific languages [10, 15, 23, 30], many guided by types [10, 11, 14, 28, 31]. Specifically, Optician supports the synthesis of \textit{bijective lenses}, a useful subset of Boomerang. As inputs, Optician takes specifications of the source and target formats, plus a collection of concrete examples of the desired transformation. Format specifications are supplied as ordinary regular expressions. Because regular expressions are so widely understood, we anticipate such inputs will be substantially easier for everyday programmers to work with than the unfamiliar syntax of lenses. Moreover, including these format descriptions communicates a great deal of information to the synthesis system. Thus, requiring user input of regular expressions makes synthesis robust, helps the system scale to large and complex data sources, and constrains the search space sufficiently that the user typically needs to give very few, if any, examples.

Despite the benefits of Boomerang’s informative types, Boomerang is not well-suited to support synthesis directly. Specifically, Boomerang’s types are regular expression pairs, and each regular expression is equivalent to an infinite number of other regular expressions. To synthesize all Boomerang terms, a type-directed synthesizer must sometimes be able to find, amongst all possible equivalent regular expressions, the one with the right syntactic structure to guide the subsequent search for a well-typed, example-compatible Boomerang term.

To resolve these issues, we introduce a new language of \textit{Disjunctive Normal Form (DNF) lenses}. Just as string lenses have pairs of regular expressions as types, DNF lenses have pairs of \textit{DNF regular expressions} as types. The typing judgements for DNF lenses limit how equivalences can be used, greatly reducing the size of the search space. Despite the restrictive syntax and type system of DNF lenses, we prove our new language is equivalent to a natural, declarative specification of the bijective fragment of Boomerang.

Figure 1 shows a high-level, schematic diagram for Optician. First, Optician uses the function \(\downarrow\) to convert the input regular expressions into DNF regular expressions. Next, \textit{SYNTHDNFLENS} performs type-directed synthesis on these DNF regular expressions and the input examples to
synthesize a DNF lens. Finally, this DNF lens is converted back into a regular lens with the function $\uparrow$, and returned to the user.

**Contributions.** Optician makes bidirectional programming more accessible by obviating the need for programmers to write lenses by hand. We begin by briefly reviewing some background on regular expressions and core lens combinators (§2). After we motivate our problem with an extended, real-world example (§3), we offer the following technical contributions:

- We introduce a new lens language (DNF Lenses) that is suitable for synthesis (§4 and §5). We show how to convert ordinary regular expressions and lenses into the corresponding DNF forms, and we prove that DNF lenses are sound and complete with respect to the high-level bijective lens syntax.
- We present an efficient, type-directed synthesis algorithm for synthesizing lenses (§6). We prove that if there is a lens that satisfies the input specification, this algorithm will return such a lens.
- We evaluate Optician, its optimizations, and existing synthesis tools on 39 benchmarks, including examples derived from Flash Fill [15] and the Augeas [25] system (§7). We show that our optimizations are critical for synthesizing many of the complex lenses in our benchmark suite and that our full algorithm succeeds on all benchmarks in under 5 seconds.
- While we are not aware of any other systems for automatically synthesizing bijective transformations, we establish a baseline for our the effectiveness of our techniques by comparing our synthesis algorithm with the one used in Flash Fill [15], a well-known and influential synthesis system deployed in Microsoft Excel. Flash Fill only synthesizes transformations in one direction, but it was only able to complete synthesis of 3 out of 39 of our benchmarks. We conjecture that the extra information we supply the synthesis system via our regular format descriptions, allows it to scale to significantly more complex and varied formats than is possible in current string synthesis systems that do not use this information.

We close with related work (§8) and conclusions (§9).

## 2 PRELIMINARIES

**Technical Report.** Throughout the paper, we will state a number of theorems. We have omitted these theorems for space, and have included these details in the auxiliary technical report.
Regular Expressions. We use Σ to denote the alphabet of individual characters c; strings s and t are elements of Σ*. Regular expressions, abbreviated REs, are used to express languages, which are subsets of Σ*. REs over Σ are:

\[ S, T ::= s \mid \emptyset \mid S^* \mid S \cdot T \mid S \mid T \]

\[ \mathcal{L}(S) \subseteq \Sigma^* \], the language of S, is defined as usual.

Unambiguity. The typing derivations of lenses require regular expressions to be written in a way that parses text unambiguously. S and T are unambiguously concatenable, written \( S \cdot T \), if, for all strings \( s_1, s_2 \in \mathcal{L}(S) \) and \( t_1, t_2 \in \mathcal{L}(T) \), whenever \( s_1 \cdot t_1 = s_2 \cdot t_2 \) it is the case that \( s_1 = s_2 \) and \( t_1 = t_2 \). Similarly, S is unambiguously iterable, written \( S^\ast \), if, for all \( n, m \in \mathbb{N} \) and for all strings \( s_1, \ldots, s_n, t_1, \ldots, t_m \in \mathcal{L}(S) \), whenever \( s_1 \cdot \ldots \cdot s_n = t_1 \cdot \ldots \cdot t_m \) it is the case that \( n = m \) and \( s_i = t_i \) for all i.

A regular expression S is strongly unambiguous if one of the following holds: (a) \( S = s \), or (b) \( \mathcal{L}(S) = \{ \} \), or (c) \( S = S_1 \cdot S_2 \) with \( S_1^\ast \cdot S_2 \), or (d) \( S = S_1 | S_2 \) with \( S_1 \cap S_2 = \emptyset \), or (e) \( S = (S')^* \) with \( (S')^\ast \). In the recursive cases, \( S_1, S_2, \) and \( S' \) must also be strongly unambiguous.

Equivalences. S and T are equivalent, written \( S \equiv T \), if \( \mathcal{L}(S) = \mathcal{L}(T) \). There exists an equational theory for determining whether two regular expressions are equivalent, presented by Conway [7], and proven complete by Krob [22]. Conway’s axioms consist of the semiring axioms (associativity, commutativity, identities, and distributivity for | and \cdot \) plus the following rules for equivalences involving the Kleene star:

\[
\begin{align*}
(S \mid T)^* & \equiv (S^* \cdot T)^* \cdot S^* & \text{Sumstar} \\
(S \cdot T)^* & \equiv e \mid (S \cdot (T \cdot S)^*) \cdot T & \text{Prodstar} \\
(S^*)^* & \equiv S^* & \text{Starstar} \\
(S \mid T)^* & \equiv ((S \mid T) \cdot T \mid (S \cdot T)^* \cdot S^*) \cdot (e \mid (S \mid T) \cdot ((S \cdot T)^* \mid \ldots \mid (S \cdot T)^n)) & \text{Dicyc}
\end{align*}
\]

While this equational theory is complete, naïvely using it in the context of lens synthesis presents several problems. In the context of lens synthesis, we instead use the equational theory corresponding to the axioms of a star semiring [9]. If two regular expressions are equivalent within this equational theory, they are star semiring equivalent, written \( S \equiv^s T \). The star semiring axioms consist of the semiring axioms plus the following rules for equivalences involving the Kleene star:

\[
\begin{align*}
S^* & \equiv^s e \mid (S \cdot S^*) & \text{Unrollstar}_L \\
S^* & \equiv^s e \mid (S^* \cdot S) & \text{Unrollstar}_R
\end{align*}
\]

In §3, we provide intuition for why synthesis with full regular expression equivalence is problematic and justify our choice of using star semiring equivalence instead.

Bijective Lenses. All bijections between languages are lenses. We define bijective lenses to be bijections created from the following Boomerang lens combinators, \( l \):

\[
l ::= \text{const}(s_1 \in \Sigma^*, s_2 \in \Sigma^*) \\
| \text{iterate}(l) \\
| \text{concat}(l_1, l_2) \\
| \text{swap}(l_1, l_2) \\
| \text{or}(l_1, l_2) \\
| l_1 ; l_2 \\
| \text{id}_S
\]

The denotation of a lens \( l \) is \( \llbracket l \rrbracket \subseteq \text{String} \times \text{String} \). If \( (s_1, s_2) \in \llbracket l \rrbracket \), then \( l \) maps between \( s_1 \) and \( s_2 \).

\[\text{There are other complete axiomatizations for regular expression equivalence, such as Kozen’s[21] and Salomaa’s[32]. We focus on Conway’s for the sake of specificity, but discuss alternative theories in §8.}\]
where \( s \) operates by repeatedly applying \( l \) as a function when applied from left to right, as the language create a bijective function. For example, if \( l \) requires side conditions about unambiguity. These side conditions guarantee that the semantics of \( L(l) \) operate in both directions by applying the identity function to strings in \( L(l) \) operated right to left. Replacing string \( s \) to subparts of a string. Figure 2 gives the typing relation. Many of the typing derivations simply lens in the combinator language is the constant lens between strings \( s \) and \( t \), \( \text{const}(s, t) \). The lens \( \text{const}(s, t) \), when operated left to right, replaces the string \( s \) with \( t \), and when operated right to left, replaces string \( t \) with \( s \). The identity lens on a regular expression, \( \text{id}_S \), operates in both directions by applying the identity function to strings in \( L(S) \). The composition combinator, \( l_1 \circ l_2 \), operates by applying \( l_1 \) then \( l_2 \) when operating left to right, and applying \( l_2 \) then \( l_1 \) when operating right to left.

Each of the other lenses manipulates structured data. For instance, \( \text{concat}(l_1, l_2) \) operates by applying \( l_1 \) to the left portion of a string, and \( l_2 \) to the right, and concatenating the results. The combinator \( \text{swap}(l_1, l_2) \) does the same as \( \text{concat}(l_1, l_2) \) but it swaps the results before concatenating. The combinator \( \text{or}(l_1, l_2) \) operates by applying either \( l_1 \) or \( l_2 \) to the string. The combinator \( \text{iterate}(l) \) operates by repeatedly applying \( l \) to subparts of a string.

### Lens Typing

The typing judgement for lenses has the form \( l : S \leftrightarrow T \), meaning \( l \) bijectively maps between \( L(S) \) and \( L(T) \). Figure 2 gives the typing relation. Many of the typing derivations require side conditions about unambiguity. These side conditions guarantee that the semantics of the language create a bijective function. For example, if \( l_1 : S_1 \leftrightarrow T_1 \) and \( l_2 : S_2 \leftrightarrow T_2 \), and \( S_1 \) is not unambiguously concatenable with \( S_2 \), then there would exist \( s_1, s'_1 \in L(S_1) \), and \( s_2, s'_2 \in L(S_2) \) where \( s_1 \cdot s_2 = s'_1 \cdot s'_2 \), but \( s_1 \neq s'_1 \), and \( s_2 

\[
\begin{align*}
\text{const}(s_1, s_2) & \Leftrightarrow (s_1, s_2) \\
\text{iterate}(l) & \Leftrightarrow T^l \\
\text{concat}(l_1, l_2) & \Leftrightarrow T_1T_2 \\
\text{swap}(l_1, l_2) & \Leftrightarrow T_2T_1 \\
l_1 : S_1 \leftrightarrow T_1 \\
l_2 : S_2 \leftrightarrow T_2 \\
S_1^{\cdot l} S_2 & \Leftrightarrow T_1^{\cdot l} T_2 \\
l_1 : S_1 \leftrightarrow S_2 \\
l_2 : S_2 \leftrightarrow S_3 \\
l_1 ; l_2 : S_1 \leftrightarrow S_3 \\
l : S_1 \leftrightarrow S_2 \\
l : S_1 \equiv^S S'_1 \\
l : S_2 \equiv^S S'_2
\end{align*}
\]

![Fig. 2. Lens Typing Rules](image-url)

The simplest lens in the combinator language is the constant lens between strings \( s \) and \( t \), \( \text{const}(s, t) \). The lens \( \text{const}(s, t) \), when operated left to right, replaces the string \( s \) with \( t \), and when operated right to left, replaces string \( t \) with \( s \). The identity lens on a regular expression, \( \text{id}_S \), operates in both directions by applying the identity function to strings in \( L(S) \). The composition combinator, \( l_1 \circ l_2 \), operates by applying \( l_1 \) then \( l_2 \) when operating left to right, and applying \( l_2 \) then \( l_1 \) when operating right to left.

Each of the other lenses manipulates structured data. For instance, \( \text{concat}(l_1, l_2) \) operates by applying \( l_1 \) to the left portion of a string, and \( l_2 \) to the right, and concatenating the results. The combinator \( \text{swap}(l_1, l_2) \) does the same as \( \text{concat}(l_1, l_2) \) but it swaps the results before concatenating. The combinator \( \text{or}(l_1, l_2) \) operates by applying either \( l_1 \) or \( l_2 \) to the string. The combinator \( \text{iterate}(l) \) operates by repeatedly applying \( l \) to subparts of a string.
any ambiguous RE can be replaced by an equivalent unambiguous one, these ambiguity constraints do not have an impact on the computational power of the language.

The typing rule for $id_S$ requires a strongly unambiguous regular expression. This unambiguity allows the identity lens to be derivable from other lenses. This requirement does not, however, reduce expressiveness, as any regular expression is equivalent to a strongly unambiguous regular expression [6].

The last rule in Figure 2 is a type equivalence rule that lets the typing rules consider a lens $l : S_1 \leftrightarrow S_2$ to have type $S'_1 \leftrightarrow S'_2$ so long as $S_1 \equiv^\ast S_1$ and $S_2 \equiv^\ast S_2$. Notice that this rule uses star semiring equivalence as opposed to Conway equivalence. In theory, this reduces the expressiveness of the type system; in practice, we have not found it restrictive. We explain and justify the decision to use star semiring equivalence in the next section.

Finally, it is worthwhile at this point to notice that the problem of finding a well-typed lens $l$ given a pair of regular expressions—the lens synthesis problem—would not be difficult if it were not for lens composition and the type equivalence rules. When read bottom up, these two rules apparently require wild guesses at additional regular expressions to continue driving synthesis recursively in a type-directed fashion. In contrast, in the other rules, the shape of the lens is largely determined by the given types. The following sections elaborate on this problem and describe our solution.

3 OVERVIEW

To highlight the difficulties in synthesizing lenses, we use an extended example inspired by the evolution of Microsoft’s Visual Studio Team Services (VSTS). VSTS is a collection of web services for team management – providing a unified location for source control (e.g. a Git server), task management (i.e. providing a means to keep track of TODOs and bugs), and more. In 2014, to increase third party developer interaction, VSTS released new web service endpoints [17]. However,

In practice, we allow regular expressions that aren’t strongly unambiguous to appear in $id_S$, provided that they are expressed as a user defined regular expression. We elide such user-defined regular expression information from the theory for the sake of simplicity.

2

Fig. 3. VSTS Architecture Using Lenses. In (a), we show the proposed architecture of VSTS using lenses. When a legacy client requests a work item, the server retrieves the data in a modern format through the new APIs, then the lens converts it into a legacy format to return to the client. When a legacy client updates a work item, it provides the data in the legacy format to the server. The lens then converts this data into the modern format for the new endpoints to process. Idealized Task representations from legacy and modern web service endpoints are given in (b).
despite VSTS introducing new, modernized web APIs, they must still maintain the old, legacy web
APIs for continued support of legacy clients [27]. Instead of maintaining server code for each
endpoint, we envision an architecture that uses a lens to convert resources of the old form into
resources of the new form and vice versa, as shown in Figure 3a. Writing each of these converters
by hand is slow and error prone. We speed up this process by only requiring users to input regular
expressions and input-output examples. Furthermore, the generated lenses are guaranteed to map
between the provided regular expressions and to act correctly on the provided examples.

Consider a “Work Item,” a resource that represents a task given to a team. Idealized versions
of the representations of work items from the new and old APIs are given in Figure 3b. In our
proposed architecture, if an old client performs an HTTP GET request to receive a work item, the
server first retrieves that work item using the modern API, and then uses the lens’s \texttt{get} function to
convert this task into the legacy format. Similarly, if an old client performs an HTTP PUT request
to update a work item, the server first uses the lens’s \texttt{put} function to convert that data into the
new format, and then inputs the work item in the new representation to the modern APIs. The two
representations contain the same information, but they are presented differently.

For simplicity, let’s consider only finding the mapping between the “Title” field of the task in the
legacy and modern formats. The legacy client accepts inputs of the form

\[
\text{legacy_title} = "<\text{Field Id=2}>\text{text_char}^* </\text{Field}>"
\]

while the modern client accepts inputs of the form

\[
\text{modern_title} = ("\text{Title:}\text{text_char}^* \text{text_char }", ")
| ""
\]

where \texttt{text_char} is a user-defined data type representing what characters can be present in a text
field (like the title field). We would like to be able to synthesize \(l\), a lens that satisfies the typing
judgement \(l : \text{legacy_title} \iff \text{modern_title}\) (i.e. \(l\) maps between the legacy representation
\text{legacy_title}, and the modern representation \text{modern_title}). Because the modern API omits
the title field if it is blank, the lens must perform different actions depending on the number of
text characters present, functionality provided by \texttt{or} lenses. An \texttt{or} lens applies one of two lenses,
depending on which of the lenses’ source types matches the input string.

However, the typing rule for \texttt{or} does not suffice to type check lenses that map between \text{legacy_title}
and \text{modern_title}. While \text{modern_title} is a regular expression with an outermost
disjunction, \text{legacy_title} is a regular expression with an outermost concatenation, so the rule
cannot be directly applied. We address this problem by allowing conversions between equivalent
regular expression types with the type equivalence rule. Using this rule, a type-directed synthesis
algorithm can convert \text{legacy_title} into

\[
\text{legacy_title'} = "<\text{Field Id=2}></\text{Field}>"
| ("<\text{Field Id=2}>\text{text_char text_char}^* </\text{Field}>")
\]

There exist \texttt{or} lenses between \text{legacy_title'} and \text{modern_title}, and the two cases of an empty
and a nonempty number of text characters can be handled separately. However, the need to find
this equivalent type highlights a significant challenge in synthesizing bijective lenses.

\textbf{Challenge 1: Multi-dimensional Search Space.} Since regular expression equivalence is decidable, it
is easy to check whether a given lens \(l\) with type \(S_1 \iff S_2\) also has type \(S_1' \iff S_2'\). During synthesis,
however, deciding when and how to use type conversion is difficult because there are infinitely
many regular expressions that are equivalent to the source and target regular expressions. Does
the algorithm need to consider all of them? In what order? To convert from \text{legacy_title} to
The algorithm must first unroll \texttt{text_char*} into "" | \texttt{text_char text_char*}, and then it must distribute this disjunction on the left and the right.

A related challenge arises from the composition operator, \( l_1 \cdot l_2 \). The typing rule for composition requires that the target type of \( l_1 \) be the source type of \( l_2 \). To synthesize a composition lens between \( S_1 \) and \( S_3 \), a sound synthesizer must find an intermediate type \( S_2 \) and lenses with types \( S_1 \rightleftharpoons S_2 \) and \( S_2 \rightleftharpoons S_3 \). Searching for the correct regular expression \( S_2 \) is again problematic because the search space is infinite.

Thus, naively applying type-directed synthesis techniques involves searching in three infinite dimensions. A complete naive synthesizer must search for (1) a type consisting of two regular expressions equivalent to the given ones but with “similar shapes” and (2) a lens expression that has the given type and is consistent with the user’s examples. Furthermore, whenever composition is part of the expression, naive type-directed synthesis requires a further search for (3) an intermediate regular expression.

Our approach to this challenge is to define a new “DNF syntax” for types and lenses that reduces the synthesis search space in all dimensions. In this new language, regular expressions are written in a disjunctive normal form, where disjunctions are fully distributed over concatenation and where binary operators are replaced by \( n \)-ary ones, eliminating associativity rules. Using DNF regular expressions, when presented with a synthesis problem with type \( (A|B)C \rightleftharpoons A'C|B'C' \), Optician will first convert this type into \( ([A \cdot C] | [B \cdot C]) \rightleftharpoons ([A' \cdot C'] | [B' \cdot C']) \), where \( (\ldots) \) represents \( n \)-ary disjunction and \([\ldots] \) represents \( n \)-ary concatenation. Like DNF regular expressions, DNF lenses are stratified, with disjunctions outside of concatenations, and they use \( n \)-ary operators instead of binary ones. Furthermore, DNF lenses do not need a composition operator, eliminating an entire dimension of search. This stratification and the lack of composition creates a very tight relationship between the structure of a well-typed DNF lens and its DNF regular expression types.

Translating regular expressions into DNF form collapses many equivalent REs into the same syntactic form. However, this translation does not fully normalize regular expressions. Nor do we want it to: If a synthesizer normalized \( \epsilon \mid BB* \) to \( B* \), it would have trouble synthesizing lenses with types like \( \epsilon \mid BB* \rightleftharpoons \epsilon \mid CD* \) where the first occurrence of \( B \) on the left needs to be transformed into \( C \) while the rest of the \( B \)s need to be transformed into \( D \). Normalization to DNF eliminates many, but not all, of the regular expression equivalences that may be needed before a simple, type-directed structural search can be applied—i.e., DNF regular expressions are only pseudo-canonical.

Consequently, a type-directed synthesis algorithm must still search through some equivalent regular expressions. To handle this search, SYNTHDNFLENS is structured as two communicating synthesizers, shown in Figure 4. The first synthesizer, Type\textsc{Prop}, proposes DNF regular expressions equivalent to the input DNF regular expressions. Type\textsc{Prop} uses the axioms of a star semiring to unfold Kleene star operators in one or both types, to obtain equivalent (but larger) DNF regular expression types. The second synthesizer, Rigid\textsc{Synth}, performs a syntax-directed search based on the structure of the provided DNF regular expressions, as well as the input examples. If the second synthesizer finds a satisfying DNF lens, it returns that lens. If the second synthesizer fails to find such a lens, Type\textsc{Prop} learns of that failure, and proposes new candidate DNF regular expression pairs.

\textbf{Star Semiring Equivalence and Rewriting.} One could try to search the space of DNF regular expressions equivalent to the input regular expressions by turning the Conway axioms into (undirected) rewrite rules operating on DNF regular expressions and then trying all possible combinations of rewrites. Doing so would be problematic because the Conway axiomatization itself is both highly nondeterministic and infinitely branching (due to the choice of \( n \) in the dyclicity axiom).

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We also want DNF lenses to be closed under composition – if it is not then we need to be able to synthesize lenses containing composition operators. To be closed under composition, it is sufficient for the equivalence relation used in the type equivalence rule to be the equivalence closure of a rewrite system (→) satisfying four conditions. First, if $S \rightarrow S'$, then $L(S) = L(S')$. Second, if $S \rightarrow S'$ and $S$ is strongly unambiguous, then $S'$ is also strongly unambiguous. The remaining two properties relate the rewrite rules to the typing derivations of DNF lenses, when those typing derivations do not use type equivalence. To express these properties, we use the notation $dl \vdash DS \Leftrightarrow DT$ to mean that if $dl$ is a DNF lens that goes between DNF regular expressions $DS$ and $DT$, then the typing derivation contains no instances of the type equivalence rule. Using this notation, we can express the confluence property, as follows:

**Definition 1 (Confluence).** Whenever $dl \vdash (\downarrow S_1) \Leftrightarrow (\downarrow T_1)$, if $S_1 \rightarrow S_2$ and $T_1 \rightarrow T_2$, there exist regular expressions $S_3$ and $T_3$ and a DNF lens $dl_3$, such that:

1. $S_2 \rightarrow S_3$
2. $T_2 \rightarrow T_3$
3. $dl_3 \vdash (\downarrow S_3) \Leftrightarrow (\downarrow T_3)$
4. $\llbracket dl_3 \rrbracket = \llbracket dl_1 \rrbracket$

We call the final property bisimilarity. Bisimilarity requires two symmetric conditions.

**Definition 2 (Bisimilarity).** Whenever $dl_1 \vdash (\downarrow S_1) \Leftrightarrow (\downarrow T_1)$ and $S_1 \rightarrow S_2$, there exist a regular expression $T_2$ and a DNF lens $dl_2$ such that

1. $T_1 \rightarrow T_2$
2. $dl_2 \vdash (\downarrow S_2) \Leftrightarrow (\downarrow T_2)$
3. $\llbracket dl_2 \rrbracket = \llbracket dl_1 \rrbracket$

To be bisimilar, the symmetric property must also hold for $T_1 \rightarrow T_2$.

Our solution for handling type equivalence is to use $\equiv^s$, the equivalence relation generated by the axioms of a star semiring. This equivalence relation is compatible with our lens synthesis strategy, as orienting these unrolling rules from left to right presents us with a rewrite relation that is both confluent and bisimilar, and whose equivalence closure is $\equiv^s$. The star semiring axioms are the coarsest subset of regular expression equivalences we could find that is generated by a rewrite
relation and is still confluent and bisimilar. We have not been able to prove that this relation is the coarsest such relation possible, but it is sufficient to cover all the test cases in our benchmark suite (see §7). However, it is easy to show that Conway’s axioms (Prodstar in particular) are not bisimilar, which is why we avoid this in our system.

**Challenge 2: Large Types.** DNF lenses are equivalent in expressivity to lenses and the algorithm SynthDNFLens is quite fast. Unfortunately, the conversion to DNF incurs an exponential blowup. In practical examples, the regular expressions describing complex ad hoc data formats may be very large, causing the exponential blowup to have a significant impact on synthesis time. The key to addressing this issue is to observe that users naturally construct large types incrementally, introducing named abbreviations for major subcomponents. For example, in the specification of legacy_title and modern_title, the variable text_char describes which characters can be present in a title. To include a large disjunction representing all valid title characters instead of the concise variable text_char in the definitions of legacy_title and modern_title would be unmaintainable and difficult to read.

Unfortunately, leaving these variables opaque introduces a new dimension of search. In addition to searching through the rewrites on regular expressions, the algorithm must also search through possible substitutions, replacements of variables with their definitions. We designate these two types of equivalences expansions, using “rewrites” to denote expansions that arise from traversing rewrite rules on the regular expressions, and using “substitutions” to denote expansions that arise from replacing a variable with its definition.

Interestingly, Optician can exploit the structure inherent in these named abbreviations to speed up the search dramatically. For example, if text_char appears just once in both the source and the target types, the system hypothesizes that the identity lens can be used to convert between occurrences of text_char. On the other hand, if text_char appears in the source but not in the target, the system recognizes that, to find a lens, text_char must be replaced by its definition. In this way, the positions of names can serve as a guide for applying substitutions and rewrites in the synthesis algorithm. By using these named abbreviations, TypeProp guides the transformation of regular expression types during search by deducing when certain expansions must be taken, or when one of a class of expansions must be taken.

### 4 DNF REGULAR EXPRESSIONS

The first important step in Optician is to convert regular expression types into disjunctive normal form (DNF). A DNF regular expression, abbreviated DNF RE, is an n-ary disjunction of sequences, where a sequence alternates between concrete strings and atoms, and an atom is an iteration of DNF regular expressions. The grammar below describes the syntax of DNF regular expressions (DS, DT), sequences (SQ, TQ), and atoms (A, B) formally.

\[
\begin{align*}
A, B & ::= DS^* \\
SQ, TQ & ::= [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \\
DS, DT & ::= \langle SQ_1 | \ldots | SQ_n \rangle
\end{align*}
\]

Notice that it is straightforward to convert an arbitrary series of atoms and strings into a sequence: if there are multiple concrete strings between atoms, the strings may be concatenated into a single string. If there are multiple atoms between concrete strings, the atoms may be separated by empty strings, which will sometimes be omitted for readability. Notice also that a simple string with no atoms may be represented as a sequence containing just one concrete string. In our implementation, names of user-defined regular expressions are also atoms. However, we elide such definitions from our theoretical analysis.
\(\circ_{SQ} : \text{Sequence} \rightarrow \text{Sequence} \rightarrow \text{Sequence}\)

\[
[s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n] \circ_{SQ} [t_0 \cdot B_1 \ldots \cdot B_m \cdot t_m] = [s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n \cdot t_0 \cdot B_1 \ldots \cdot B_m \cdot t_m]
\]

\(\odot : \text{DNF} \rightarrow \text{DNF} \rightarrow \text{DNF}\)

\[
\langle SQ_1 \mid \ldots \mid SQ_n \rangle \odot \langle TQ_1 \mid \ldots \mid TQ_m \rangle = \langle SQ_1 \odot_{SQ} TQ_1 \mid \ldots \mid SQ_n \odot_{SQ} TQ_m \rangle
\]

\(\ominus : \text{DNF} \rightarrow \text{DNF} \rightarrow \text{DNF}\)

\[
\langle SQ_1 \mid \ldots \mid SQ_n \rangle \ominus \langle TQ_1 \mid \ldots \mid TQ_m \rangle = \langle SQ_1 \mid \ldots \mid SQ_n \mid TQ_1 \mid \ldots \mid TQ_m \rangle
\]

\(D : \text{Atom} \rightarrow \text{DNF}\)

\[
D(A) = \langle [e \cdot A \cdot e] \rangle
\]

\(D(A) = \langle [e \cdot A \cdot e] \rangle\)

\[
\mathcal{L}(\langle [e \cdot A \cdot e] \rangle) = \{s_1 \cdot \ldots \cdot s_n \mid \forall i, s_i \in \mathcal{L}(DS)\}
\]

\[
\mathcal{L}(\langle s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n \rangle) = \{s_0 \cdot t_1 \ldots \cdot t_n \cdot s_n \mid t_i \in \mathcal{L}(A_i)\}
\]

\[
\mathcal{L}(\langle SQ_1 \mid \ldots \mid SQ_n \rangle) = \{s \mid s \in \mathcal{L}(SQ_i) \text{ and } i \in [1, n]\}
\]

A sequence of strings and atoms is sequence unambiguously concatenatable, written \(\langle s_0; A_1; \ldots; A_n; s_n \rangle\), if, when \(s'_i, t'_i \in \mathcal{L}(A_i)\) for all \(i\), then \(s_0 s'_1 \ldots s'_n s_n = s_0 t'_1 \ldots t'_n s_n\) implies \(s'_i = t'_i\) for all \(i\). A DNF regular expression \(S\) is unambiguously iterable, written \(S^*\) if, for all \(n, m \in \mathbb{N}\) and for all strings \(s_1, \ldots, s_n, t_1, \ldots, t_m \in \mathcal{L}(S)\), if \(s_1 \ldots s_n = t_1 \ldots t_m\) then \(n = m\) and \(s_i = t_i\) for all \(i\).

Expressivity of DNF Regular Expressions. Any regular expression may be converted into an equivalent DNF regular expression. To define the conversion function, we rely on several auxiliary functions defined in Figure 5. Intuitively, \(DS \odot DS\) concatenates two DNF regular expressions, producing a well-formed DNF regular expression as a result. Similarly, \(DS \ominus DS\) generates a new DNF regular expression representing the union of two DNF regular expressions. Finally, \(D(A)\) converts a naked atom into a well-formed DNF regular expression. The conversion algorithm itself, written \(\downarrow S\), is defined below.

\[
\downarrow s = \langle [s] \rangle
\]

\[
\downarrow \emptyset = \langle \rangle
\]

\[
\downarrow (S^*) = D((\downarrow S)^*)
\]

\[
\downarrow (S_1 \cdot S_2) = \downarrow S_1 \odot \downarrow S_2
\]

\[
\downarrow (S_1 \mid S_2) = \downarrow S_1 \ominus \downarrow S_2
\]

Using \(\downarrow\), the definition of \textit{legacy_title} gets converted into the DNF regular expression:

\[
\text{dnf}_\text{legacy_title} =
( ["<Field Id=2></Field>"]
| ["<Field Id=2>"\cdot\text_char\cdot""\cdot([text_char])\cdot""<Field Id=2>"] )
\]

and the definition of \textit{modern_title} gets converted into the DNF regular expression:

\[
\text{dnf}_\text{modern_title} =
( ["\text{Title:}\cdot\text_char\cdot""\cdot([text_char])\cdot"","]
| [""" ] )
\]

Theorem 1 (\(\downarrow\) Soundness). For all regular expressions \(S\), \(\mathcal{L}(\downarrow S) = \mathcal{L}(S)\).
which by the internal sequence lenses. As an example, consider a DNF lens that maps between data with type lenses also contain a permutation that provides information about which sequences are mapped to expression is a list of disjuncted sequences, a DNF lens contains a list of or lenses and sequence lenses both contain permutations (The syntax of DNF lenses (Unrollstar) is defined below. DNF ∗ is not a DNF regular expression. Hence, the rule uses DNF concatenation (⊙) and union (⊕) in place of regular expression concatenation and union to ensure a DNF regular expression is constructed. The rule Atom UnrollstarR mirrors the rule UnrollstarR in a similar way.

The rules Atom Structural Rewrite and DNF Structural Rewrite make it possible to rewrite terms involving Kleene star that are nested deep within a DNF regular expression, while ensuring that the resulting term remains in DNF form.

5 DNF LENSES

The syntax of DNF lenses (dl), sequence lenses (sql) and atom lenses (al) is defined below. DNF lenses and sequence lenses both contain permutations (σ) that help describe how these lenses act on data.

\[ al ::= iterate(dl) \]
\[ sql ::= ([[s_0 \cdot t_0] \cdot a_l \cdot \ldots \cdot a_l \cdot (s_n, t_n)], \sigma) \]
\[ dl ::= ([sql_1 | \ldots | sql_n], \sigma) \]

A DNF lens consists of a list of sequence lenses and a permutation. Much like a DNF regular expression is a list of disjuncted sequences, a DNF lens contains a list of ordered sequence lenses. DNF lenses also contain a permutation that provides information about which sequences are mapped to which by the internal sequence lenses. As an example, consider a DNF lens that maps between data with type dnf_legacy_title and data with type dnf_modern_title. In such a lens, the

\[ DS^* \rightarrow_A ([\varepsilon]) \oplus (DS \odot \mathcal{D}(DS^*)) \]
\[ DS^* \rightarrow_A ([\varepsilon]) \oplus (\mathcal{D}(DS^*) \odot DS) \]
permutation $\sigma$ indicates whether data matching "<Field Id=2></Field>" will be translated to data matching ""Title:"·text_char·"·{(text_char)"·","} or "", and likewise for the other sequence in dnf_legacy_title. In this case, we would use the permutation that swaps the order, as the first sequence in dnf_legacy_title gets mapped to the second in dnf_modern_title, and vice-versa. As we will see in a moment, these permutations make it possible to construct a well-typed lens between two DNF regular expressions regardless of the order in which clauses in a DNF regular expression appear, thereby eliminating the need to consider equivalence modulo commutativity of these clauses.

A sequence lens consists of a list of atom lenses separated by pairs of strings, and a permutation. Intuitively, much like a sequence is a list of concatenated atoms and strings, a sequence lens is a list of concatenated atom lenses and string pairs. Sequence lenses also contain a permutation that makes get and put reorder data, allowing sequence lenses to take the job of both concat and swap. If there are $n$ elements in the series then the DNF sequence lens divides an input string up into $n$ substrings. The $i^{th}$ such substring is transformed by the $i^{th}$ element of the series. More precisely, if that $i^{th}$ element is an atom lens, then the $i^{th}$ substring is transformed according to that atom lens. If the $i^{th}$ element is a pair of strings $(s_1, s_2)$ then that pair of strings acts like a constant lens: when used from left-to-right, such a lens translates string $s_1$ into $s_2$; when used from right-to-left, such a lens translates string $s_2$ into $s_1$. After all of the substrings have been transformed, the permutation describes how to rearrange the substrings transformed by the atom lenses to obtain the final output. As an example, consider a sequence lens that maps between data with type "Title:"·text_char·"·{(text_char)"·","} and data with type ""Title:"·text_char·"·{(text_char)"·","}"<Field Id=2>". We desire no reorderings between the atoms text_char and {(text_char)"}, so the permutation associated with this lens would be the identity permutation.

An atom lens is an iteration of a DNF lens; its semantics is similar to the semantics of ordinary iteration lenses. In our implementation, identity transformations between user-defined regular expressions are also atom lenses. However, we elide such definitions from our theoretical analysis.

The semantics of DNF Lenses, sequence lenses and atom lenses is defined formally below.

$$
\text{iterate}(dl) ::= \{s_1 \cdot \ldots \cdot s_n, t_1 \cdot \ldots \cdot t_n | n \in \mathbb{N} \land (s_i, t_i) \in dl\}
$$

$$
\text{(\{(s_0, t_0) \cdot al_1 \cdot \ldots \cdot al_n \cdot (s_n, t_n), \sigma\}) ::= \{s_0 s_1' \cdot \ldots \cdot s_n s_n' \cdot t_0 t_{i(1)}' \cdot \ldots \cdot t_n t_{\sigma(n)}' | (s'_i, t'_i) \in al_i\}}
$$

$$
\text{(\{(sql_1 | \ldots | sql_n), \sigma\}) ::= \{s, t | (s, t) \in sql_i \text{ for some } i\}}
$$

Type Checking. Figure 7 presents the type checking rules for DNF lenses. In order to control where regular expression rewriting may be used (and thereby reduce search complexity), the figure defines two separate typing judgements. The first judgement has the form $dl : DS \Rightarrow DT$. It implies that the lens $dl$ is a well-formed bijective map between the languages of $DS$ and $DT$. This judgement does not include the rule for rewriting the types of the source or target data. The second judgement has the form $dl : DS \Leftrightarrow DT$. It rewrites the source and target types, and then searches for a DNF lens with the rewritten types.

One of the key differences between these typing judgements and the judgements for ordinary lenses are the permutations. For example, in the rule for typing DNF lenses, the permutation $\sigma$ indicates how to match sequence types in the domain ($SQ_1 \ldots SQ_n$) and the range ($TQ_{\sigma(1)} \ldots TQ_{\sigma(n)}$). Permutations are used in a similar way in the typing rule for sequence lenses.

Properties. While DNF lenses have a restrictive syntax, they remain as powerful as ordinary bijective lenses. The following theorems characterize the relationship between the two languages.
\[
\begin{align*}
 dl \vdash DS & \iff DT & DS^! & \iff DT^!
\end{align*} \]
iterate\( (dl) \vdash DS\) \(\iff DT\)

\[
\begin{align*}
 al_1 \vdash A_1 & \iff B_1 & \ldots & al_n \vdash A_n & \iff B_n \\
 \sigma \in S_n & \vdash (s_0; A_1; \ldots; A_n; s_n) & \vdash (t_0; B_{\sigma(1)}; \ldots; B_{\sigma(n)}; t_n) \\
 ((s_0, t_0) \cdot al_1 \cdot \ldots \cdot al_n \cdot (s_n, t_n)), \sigma) & \vdash [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \iff [t_0 \cdot B_{\sigma(1)} \cdot \ldots \cdot B_{\sigma(n)} \cdot t_n] \\
 sq_1 \vdash SQ_1 & \iff TQ_1 & \ldots & sq_n \vdash SQ_n & \iff TQ_n \\
 \sigma \in S_n & \vdash L(SQ_1) \cap L(SQ_j) = \emptyset & \forall i \neq j \\
 ((sq_1 | \ldots | sq_n), \sigma) & \vdash (SQ_1 | \ldots | SQ_n) \iff (TQ_{\sigma(1)} | \ldots | TQ_{\sigma(n)}) \\
 DS' & \rightarrow^* DS & DT' & \rightarrow^* DT & dl \vdash DS \iff DT
\end{align*} \]

Fig. 7. DNF Lens Typing

**Theorem 2** (DNF Lens Soundness). If there exists a derivation of \( dl : DS \iff DT \), then there exist a lens, \( \uparrow dl \), and regular expressions, \( S \) and \( T \), such that \( \uparrow dl : S \iff T \) and \( \uparrow S = DS \) and \( \uparrow T = DT \) and \( \| dl \| = \| dl \| \).

**Theorem 3** (DNF Lens Completeness). If there exists a derivation for \( l : S \iff T \), then there exists a DNF lens \( dl \) such that \( dl : (\uparrow S) \iff (\uparrow T) \) and \( \| l \| = \| dl \| \).

Discussion. DNF lenses are significantly better suited to synthesis than regular bijective lenses. First, they contain no composition operator. Second, the use of equivalence (rewriting) is highly constrained: Rewriting may only be used once at the top-most level as opposed to interleaved between uses of the other rules. Consequently, a type-directed synthesis algorithm may be factored into two discreet steps: one step that searches for an effective pair of regular expressions and a second step that is directed by the syntax of the regular expression types that were discovered in the first step.

6 **ALGORITHM**

Synthesis Overview. Algorithm 1 presents the synthesis procedure. SYNTHLENs takes the source and target regular expressions \( S \) and \( T \), and a list of examples \( exs \), as input. First, SYNTHLENs validates the unambiguity of the input regular expressions, \( S \) and \( T \), and confirm that they parse the input/output examples, \( exs \). Next, the algorithm converts \( S \) and \( T \) into DNF regular expressions \( DS \) and \( DT \) using the \( \| \) operator. It then calls SYNTHDNFLENs on \( DS, DT \), and the examples to create a DNF lens \( dl \). Finally, it uses \( \uparrow \) to convert \( dl \) to a Boomerang lens.

SYNTHDNFLENs starts by creating a priority queue \( Q \) to manage the search for a DNF lens. Each element \( qe \) in the queue is a tuple of the source DNF regular expression \( DS' \), the target DNF regular expression \( DT' \), and a count \( e \) of the number of expansions needed to produce this pair of DNF regular expressions from the originals \( DS \) and \( DT \). (Recall that an expansion is a use of a rewrite rule or the substitution of a user-defined definition for its name.) The priority of each queue element is \( e \); DNF regular expressions that have undergone fewer expansions will get priority. The algorithm initializes the queue with \( DS \) and \( DT \), which have an expansion count of zero. The algorithm then proceeds by iteratively examining the highest priority element from the queue.
Algorithm 1 SynthLens

1: function SynthDNFLens(DS, DT, exs)
2:     Q ← CREATEPQUEUE((DS, DT), 0)
3:     while true do
4:         (qe, Q) ← Pop(Q)
5:         (DS′, DT′, e) ← qe
6:         if (DS′, DT′, e) ∈ exs then
7:             dlo ← RigidSynth(DS′, DT′, exs)
8:             match dlo with
9:                 None →
10:                    qes ← EXPAND(DS, DT, e)
11:                    Q ← ENQUEUEMANY(qes, Q)
12:     end while
13:     return ∪dl

Algorithm 2 Expand

1: function ExpandRequired(DS, DT, e)
2:     CS_{DS} ← GETCURRENTSET(DS)
3:     CS_{DT} ← GETCURRENTSET(DT)
4:     TS_{DS} ← GETTRANSITIVESET(DS)
5:     TS_{DT} ← GETTRANSITIVESET(DT)
6:     foreach (U, i) ∈ CS_{DS} \ TS_{DT} do
7:         r ← true
8:         (DS, e) ← FORCEEXPAND(DS, U, i, e)
9:     endforeach
10:     foreach (U, i) ∈ CS_{DT} \ TS_{DS} do
11:         r ← true
12:         (DT, e) ← FORCEEXPAND(DT, U, i, e)
13:     endforeach
14:     if r then
15:         return ExpandRequired(DS, DT, e)
16:     return (DS, DT, e)

16: function FixProblemElts(DS, DT, e)
17:     CS_{DS} ← GETCURRENTSET(DS)
18:     CS_{DT} ← GETCURRENTSET(DT)
19:     qes ← []
20:     foreach (U, i) ∈ CS_{DT} \ CS_{DS} do
21:         qes ← qes + REVEAL(DS, U, i, e, DT)
22:     endforeach
23:     foreach (U, i) ∈ CS_{DS} \ CS_{DT} do
24:         qes ← qes + REVEAL(DT, U, i, e, DS)
25:     return qes

16: function Expand(DS, DT, e)
26:     (DS, DT, e) ← ExpandRequired(DS, DT, e)
27:     qes ← FixProblemElts(DS, DT, e)
28:     match qes with
29:         [] → return ExpandOnce(DS, DT, e)
30:         _ → return qes

Expand. Intelligent expansion inference is key to the efficiency of Optician. Expand, shown in Algorithm 2, codifies this inference. It makes critical use of the locations of user-defined data types,
measured by their star depth, which is the number of nested ‘*’s the data type occurs beneath. Star depth locations are useful because we can quickly compute the current star depths of user-defined data types (with GetCURRENTSet) and the star depths of user-defined data types reachable via expansions (with GETTRANSITIVESet). Furthermore, the star depths of user-defined data types have the following useful property:

Property 1. If $U$ is present at star depth $i$ in $DS$ and there exists a rewriteless DNF lens $dl$ such that $dl : DS \leftrightarrow DT$, then $U$ is also present at star depth $i$ in $DT$. The symmetric property is true if $U$ is present at star depth $i$ in $DT$.

Property 1 means that if there is a rewriteless DNF lens between two DNF regular expressions, then the same user-defined data types must be present at the same locations in both of the DNF regular expressions. We use this property to determine when certain rules must be applied and to direct the search to rules that make progress towards this required alignment.

Expand has three major components: ExpandRequired, FixProblemElts, and ExpandOnce, which we discuss in turn. ExpandRequired performs expansions that must be taken. In particular, if a user-defined data type $U$ at star depth $i$ is impossible to reach through any number of expansions on the opposite side, then that user-defined data type must be replaced by its definition at that depth. For example, consider trying to find a lens between $\{\text{ legacy.title }\}$ and $\{\text{ modern.title }\}$. No matter how many expansions are performed on modern.title, the user-defined type legacy.title will not be exposed because the set of possible reachable pairs of data types and star depths in modern.title is $\{(\text{modern.title}, 0), (\text{text.char}, 0), (\text{text.char}, 1)\}$. Because no number of expansions will reveal legacy.title on the right, the algorithm must replace legacy.title with its definition on the left in order to find a lens. ExpandRequired continues until it finds all forced expansions.

ExpandRequired finds all the expansions that must be performed, but it does not perform any other expansions. However, there are many situations where it is possible to infer that one of a set of expansions must be performed without forcing any individual expansion. In particular, for any pair of types that have a rewriteless lens between them, for each (user-defined type, star depth) pair $(U, i)$ on one side, that same pair must be present on the other side. FixProblemElts identifies when there is a $(U, i)$ pair present on only one side. After identifying these problem elements, it calls REVEAL to find the expansions that will reveal this element. For example, after $\{\text{ legacy.title }\}$ has been expanded to

$$\{"\langle\text{ Field=2 }>\cdot\{\text{text.char}\}^*\cdot\"\langle/\text{ Field}>\"\}\}$$

and $\{\text{ modern.title }\}$ has been expanded to

$$\{"\langle\text{ Title=\"}>\cdot\{\text{text.char}\}^*\cdot\"\langle/\text{ Title}>\"\}\}$$

we can see that the modern expansion has an instance of text.char at depth 0, where the legacy one does not. For a lens to exist between the two types, text.char must be revealed at star depth 0 in the legacy expansion. Revealing text.char at depth zero will give back two candidate DNF regular expressions, one from an application of Atom UNROLLSTAR, and one from an application of Atom UNROLLSTARL.

Together ExpandRequired and FixProblemElts apply many expansions, but by themselves they are not sufficient. Typically, when FixProblemElts and ExpandRequired do not find all the necessary expansions, the input data formats expect large amounts of similar information. For example, in trying to synthesize the identity transformation between "" | $U$ | $UU(U^*)$ and "" | $U(U^*)$, ExpandRequired and FixProblemElts find no forced expansions. An expansion is necessary, but the set of pairs $\{(U, 0), (U, 1)\}$ is the same for both sides. When this situation arises,
the algorithm uses the EXPANONCE function to conduct a purely enumerative search, implemented by performing all single-step expansions.

**RIGIDSYNTH.** The function RIGIDSYNTH, shown in Algorithm 3, implements the portion of SYNTHLENS that generates a lens from the types and examples without using any equivalences. Intuitively, it aligns the structures of the source and target regular expressions by finding appropriate permutations of nested sequences and nested atoms, taking into account the information contained in the examples. Once it finds an alignment, it generates the corresponding lens.

Searching for aligning permutations requires care, as naively considering all permutations between two DNF regular expressions \(\langle SQ_1 \mid \ldots \mid SQ_n \rangle\) and \(\langle TQ_1 \mid \ldots \mid TQ_n \rangle\) would require time proportional to \(n!\). A better approach is to identify elements of the source and target DNF regular expressions that match and to leverage that information to create candidate permutations.

RIGIDSYNTH performs this identification via orderings on sequences \((\leq_{Seq})\), and atoms \((\leq_{Atom})\). To determine if one expression is less than the other, the algorithm converts each expression into a list of its subterms and returns whether the lexicographic ordering determines the first list less than the second. These orderings are carefully constructed so that equivalent terms have lenses between them. For example, between two sequences, \(SQ\) and \(TQ\), there is a lens \(sql \triangleq SQ \leq TQ\) if, and only if, \(SQ \leq_{Seq} TQ\) and \(TQ \leq_{Seq} SQ\). Through these orderings, aligning the components reduces to merely sorting and zipping lists. Furthermore, through composing the permutations required to sort the sequences, the algorithm discovers the permutation used in the lens.

By incorporating information about how examples are parsed in the orderings, SYNTHLENS guarantees not only that there will be a lens between the regular expressions, but also that the lens will satisfy the examples. For example if \(SQ \leq_{ex} TQ\) and \(TQ \leq_{ex} SQ\) (where \(\leq_{ex}\) is the ordering incorporating example information) then there is not only a sequence lens between \(SQ\) and \(TQ\), but there is one that also satisfies the examples.

As an example, consider trying to find a DNF lens between

\[
\langle \"<Field Id=2></Field>\rangle \\
| \"<Field Id=2>\" \text_char \"." \langle [text_char] \rangle \"." \langle Field Id=2>\rangle \\
\]
and

\[
\langle \"<Field Id=2></Field>\rangle \\
| \["\" \] \rangle
\]

As RIGIDSYNTH considers the legacy DNF regular expression, it orders its two sequences by maintaining the existing order: first \(\langle \"<Field Id=2></Field>\rangle\), then \(\langle \"<Field Id=2>\" \text_char \"." \langle [text_char] \rangle \"." \langle Field Id=2>\rangle\rangle\). In contrast, RIGIDSYNTH reorders the two sequences of the modern DNF regular expression, making \["\""] first, and \["Title:\" \text_char \"." \langle [text_char] \rangle \"." \langle Field Id=2>\rangle\]\ second; the overall permutation is a swap. As a result, the two string sequences become aligned, as do the two complex sequences.

Then, the algorithm calls RIGIDSYNTHSEQ on the two aligned sequence pairs. There are no atoms in both \(\langle \"<Field Id=2></Field>\rangle\) and \["\"\], trivially creating the sequence lens:

\([\langle \"<Field Id=2></Field>\rangle, id]\)

Next, the sequences \(\langle \"<Field Id=2>\" \text_char \["\" \] \rangle \langle [text_char] \rangle \"." \langle Field Id=2>\rangle\rangle\) and \["Title:\" \text_char \["\" \] \langle [text_char] \rangle \"." \langle Field Id=2>\rangle\]\ would be sent to RIGIDSYNTHSEQ. In RIGIDSYNTHSEQ, the atoms would not be reordered, aligning \text_char\ with \text_char, and \langle [text_char] \rangle with \langle [text_char] \rangle. Immediately, RIGIDSYNTHATOM finds the identity transformation on \text_char, and will recurse to find \text{iterate}(((((\"\" \") \cdot id_{text_char} \("\" \") \cdot id), id)))
Algorithm 3 RigidSynth

1: function RigidSynthAtom(A, B, exs)
2:     match (A, B) with
3:         | (U, V) →
4:             if \( U \leq_{\text{Atom}} \) V \& V \leq_{\text{Atom}} U then
5:                 return Some id_U
6:         else
7:                 return None
8:     | (DS', DT') →
9:         match RigidSynth(DS, DT, exs) with
10:             | Some dl → return iterate(dl)
11:             | None → return None
12:     | _ → return None

13: function RigidSynthSeq(SQ, TQ, exs)
14:     \[ s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n \] ← SQ
15:     \[ t_0 \cdot B_1 \cdot \ldots \cdot B_m \cdot t_m \] ← TQ
16:     if \( n \neq m \) then
17:         return None
18:         \( \sigma_1 \leftarrow \text{sorting}(\leq_{\text{exs}}^{\text{Atom}}, [A_1 \cdot \ldots \cdot A_n]) \)
19:         \( \sigma_2 \leftarrow \text{sorting}(\leq_{\text{exs}}^{\text{Atom}}, [B_1 \cdot \ldots \cdot B_m]) \)
20:         \( \sigma \leftarrow \sigma_1^{-1} \circ \sigma_2 \)
21:         ABs ← \text{ZIP}([A_1 \cdot \ldots \cdot A_n], [B_{\sigma(1)} \cdot \ldots \cdot B_{\sigma(n)}])
22:         alos ← \text{MAP}(\text{RigidSynthAtom}(exs), ABs)
23:         match AllSome(alos) with
24:             | Some \([a_{l_1} \cdot \ldots \cdot a_{l_n}]\) → return Some \(([s_0 \cdot t_0 \cdot a_{l_1} \cdot \ldots \cdot a_{l_n} \cdot (s_n \cdot t_n)], \sigma^{-1})\)
25:             | None → return None

26: function RigidSynth(DS, DT, exs)
27:     \langle SQ_1 | \ldots | SQ_n \rangle ← DS
28:     \langle TQ_1 | \ldots | TQ_m \rangle ← DT
29:     if \( n \neq m \) then
30:         return None
31:         \( \sigma_1 \leftarrow \text{sorting}(\leq_{\text{exs}}^{\text{Seq}}, [SQ_1 | \ldots | SQ_n]) \)
32:         \( \sigma_2 \leftarrow \text{sorting}(\leq_{\text{exs}}^{\text{Seq}}, [TQ_1 | \ldots | TQ_n]) \)
33:         \( \sigma \leftarrow \sigma_1^{-1} \circ \sigma_2 \)
34:         STQS ← \text{ZIP}([SQ_1 | \ldots | SQ_n], [TQ_{\sigma(1)} | \ldots | TQ_{\sigma(n)}])
35:         sqlos ← \text{MAP}(\text{RigidSynthSeq}(exs), STQS)
36:         match AllSome(sqlos) with
37:             | Some \([sql_1 | \ldots | sql_n]\) → return Some \(([sql_1 | \ldots | sql_n], \sigma^{-1})\)
38:             | None → return None
for \((\text{text_char})^\ast\)’. Then, these generated atom lenses are combined into a sequence lens. Lastly, the two sequence lenses are used with the swapping permutation to create the final DNF lens.

**Correctness.** We have proven two theorems demonstrating the correctness of our algorithm.

**Theorem 4 (Algorithm Soundness).** For all lenses \(l\), regular expressions \(S\) and \(T\), and examples \(exs\), if \(l = \text{SYNTHLENS}(S, T, exs)\), then \(l : S \Leftrightarrow T\) and for all \((s, t)\) in \(exs\), \((s, t) \in \llbracket l \rrbracket\).

**Theorem 5 (Algorithm Completeness).** Given regular expressions \(S\) and \(T\), and a set of examples \(exs\), if there exists a lens \(l\) such that \(l : S \Leftrightarrow T\) and for all \((s, t)\) in \(exs\), \((s, t) \in \llbracket l \rrbracket\), then \(\text{SYNTHLENS}(S, T, exs)\) will return a lens.

Theorem 4 states that when we return a lens, that lens will match the specifications. Theorem 5 states that if there is a DNF lens that satisfies the specification, then we will return a lens, but not necessarily the same one. However, from Theorem 4, we know that this lens will match the specifications.

**Simplification of Generated Lenses.** While our system takes in only partial specifications, there can be multiple lenses that satisfy the specifications. To help users determine if the synthesized lens is correct, Optician transforms the generated code to make it easily readable. Optician (1) maximally factors the `concat` and `or` s, (2) turns lenses that perform identity transformations into identity lenses, and (3) simplifies the regular expressions the identity lenses take as an argument. Performing these transformations and pretty printing the generated lenses make the synthesized lenses easy to understand.

**Compositional Synthesis.** Most synthesis problems can be divided into subproblems. For example, if the format \(S_1 \cdot S_2\) must be converted into \(T_1 \cdot T_2\), one might first work on the \(S_1 \Leftrightarrow T_1\) and \(S_2 \Leftrightarrow T_2\) subproblems. After those subproblems have been solved, the lenses they generate can be combined into a solution for \(S_1 \cdot S_2 \Leftrightarrow T_1 \cdot T_2\).

Our tool allows users to specify multiple synthesis problems in a single file, and allows the later, more complex problems to use the results generated by earlier problems. This tactic allows Optician to scale to problems of just about any size and complexity with just a bit more user input. This compositional interface also provides users greater control over the synthesized lenses and allows reuse of intermediate synthesized abstractions. The compositional synthesis engine allows lenses previously defined manually by the user, and lenses in the Boomerang standard library to be included in synthesis.

### 7 EVALUATION

We have implemented Optician in 3713 lines of OCaml code. We have integrated our synthesis algorithm into Boomerang, so users can input synthesis tasks in place of lenses. We have published this code on GitHub, with a link given in the non-anonymized supplementary material.

We evaluate our synthesis algorithm by applying it to 39 benchmark programs. All evaluations were performed on a 2.5 GHz Intel Core i7 processor with 16 GB of 1600 MHz DDR3 running macOS Sierra.

**Benchmark Suite Construction.** We constructed our benchmarks by adapting examples from Augeas [25] and Flash Fill [15] and by handcrafting specific examples to test various features of the algorithm.

Augeas is a configuration editing system for Linux that uses lens combinators similar to those in Boomerang. However, it transforms strings on the left to structured trees on the right rather than transforming strings to strings. We adapted these Augeas lenses to our setting by converting
the right-hand sides to strings that correspond to serialized versions of the tree formats. Augeas also supports asymmetric lenses [13], which are more general than the bijective lenses Optician can synthesize. We adapted these examples by adding extra fields to the target to make the transformations bijective and thus suitable for our study. We derived 29 of the benchmark tests by adapting the first 27 lenses in alphabetical order, as well as the lenses `aug/xml-firstlevel` and `aug/xml` that were referenced by the 'A' lenses. Furthermore, the 12 last synthesis problems derived from Augeas were tested after Optician was finalized, demonstrating that the optimizations were not overtuned to perform well on the testing data.

Flash Fill is a system that allows users to specify common string transformations by example [15]. Many of the examples from Flash Fill are non-bijective because the user’s goal is often to extract information. We were able to adapt some examples by adding information to the target so the resulting transformation was bijective. We derived three benchmarks from examples in the Flash Fill paper [15] that were close to bijections.

Finally, we added custom examples to highlight weaknesses of our algorithm (cap-prob and 2-cap-prob) and to test situations for which we thought the tool would be particularly useful (workitem-probs, date-probs, bib-prob, and addr-probs). These examples convert between work item formats, date formats, bibliography formats, and address formats, respectively.

We have both complex and simple synthesis tasks in our benchmark suite. We generate lenses of sizes between 5 AST nodes, for simple problems like changing how dates are represented, and 305 AST nodes, for complex tasks like transforming arbitrary XML of depth 3 or less to a dictionary representation.

Impact of Optimizations. We developed a series of optimizations that improve the performance of the synthesis algorithm dramatically. To determine the relative importance of these optimizations, we developed the 5 different modes that run the synthesis algorithm with various optimizations enabled. These modes are:

**Full**: All optimizations are enabled, and compositional synthesis is used.

**NoCS**: Like **Full**, but compositional synthesis is not used.

**NoFPE**: Like **NoCS**, but FixPROBLEMELTS is never called, expansions are only forced through EXPANDREQUIRED or processed enumeratively through EXPANDONCE.

**NoER**: Like **NoFPE**, but all the expansions taken are generated through enumerative search from EXPANDONCE.

**NoUD**: User-defined data types are no longer kept abstract until needed. All user-defined regular expressions get replaced by their definition at the start of synthesis.

We ran Optician in each mode over our benchmark suite. We summarize the results of these tests in Figure 8. **Full** synthesized all 49 benchmarks, **NoCS** synthesized 48 benchmarks, **NoFPE** synthesized 36 benchmarks, **NoER** synthesized 6 benchmarks, **NoUD** synthesized 8 benchmarks, and **Naïve** synthesized 0 benchmarks. Optician’s optimizations make synthesis effective against a wide range of complex data formats.

Interestingly, **NoER** performs worse than **NoUD**. Adding in user defined data types introduces the additional search through substitutions. The cost of this additional search outweighs the savings that these data type abstractions provide. In particular, because of the large fan-out of possible expansions, **NoER** can only synthesize lenses which require 5 or fewer expansions. However, some lenses require over 50 expansions. Without a way to intelligently traverse expansions, the need to search through substitutions makes synthesis unbearably slow.

In **NoFPE**, we can determine that many expansions are forced, so an enumerative search is often unnecessary. Figure 9 shows that in a majority of examples, all the expansions can be identified
Fig. 8. Number of benchmarks that can be solved by a given algorithm in a given amount of time. **Full** is the full synthesis algorithm. **NoCS** is the synthesis algorithm using all optimizations but without using a library of existing lenses. **NoFPE** is the core DNF synthesis algorithm augmented with user-defined data types with forced expansions performed. **NoER** is the core synthesis augmented with user-defined data types. **NoUD** is the core synthesis algorithm. **FlashExtract** is the existing FlashExtract system. **Flash Fill** is the existing Flash Fill system. **Naïve** is naïve type-directed synthesis on the bijective lens combinators. Our synthesis algorithm performs better than the naïve approach and other string transformation systems, and our optimizations speed up the algorithm enough that all tasks become solvable.

Fig. 9. Number of expansions found using enumerative search for tasks requiring a given number of expansions. **NoCS** is using the full inference algorithm. **NoFPE** only counts forced inferences as found by the EXPANDREQUIRED function. Both systems are able to infer the vast majority of expansions. Full inference only rarely requires enumerative search.
Subtasks Specified During Compositional Synthesis

Fig. 10. Number of subtasks specified during compositional synthesis. Splitting the task into just a few subtasks provides huge performance benefits at the cost of a small amount of additional user work.

as required, minimizing the impact of the large fan-out. While unable to infer every expansion for all the benchmarks, the full algorithm is able to infer quite a bit. In our benchmark suite, \texttt{ExpandRequired} infers a median of 13 and a maximum of 75 expansions.

Merely inferring the forced expansions makes almost all the synthesis tasks solvable. In many cases, \texttt{NoFPE} infers all the expansions. In 22 of the 38 examples solvable by \texttt{NoCS}, all expansions were forced. However, the remaining 16 still require some enumerative search. This enumerative search causes the \texttt{NoFPE} version of the algorithm to struggle with some of the more complex benchmarks. Incorporating \texttt{FixProblems} speeds up these slow benchmarks. When using full inference (\texttt{FixProblems} and \texttt{ExpandRequired}), the synthesis algorithm can recognize that one of a few expansions must be performed. Adding in these types of inferred expansions directs the remaining search even more, both speeding up existing problems and solving previously unmanageable benchmarks.

When combined, these optimizations implement an efficient synthesis algorithm, which can synthesize lenses between a wide range of data formats. However, some of the tasks are still slow, and one remains unsolved. Using compositional synthesis lets the system scale to the most complex synthesis tasks, synthesizing all lenses in under 5 seconds. Additional user interaction is required for compositional synthesis, but the amount of interaction is minimal, as shown in Figure 10. The number of subtasks used was in no way the minimal number of subtasks needed for synthesis under 5 seconds, but rather subtasks were introduced where they naturally arose.

The benchmark that only completes with compositional synthesis is also the slowest benchmark in \texttt{Full}, \texttt{aug/xml}\textsuperscript{3}. Optician can only synthesize a lens for this example when compositional synthesis is used because it is a complex data format, it requires a large number of expansions, and relatively few expansions are forced. When not using compositional synthesis, the algorithm must perform a total of 398 expansions, of which only 105 are forced. The synthesis algorithm is able to force so few expansions because of the highly repetitive nature of the \texttt{aug/xml} specification. XML tags occur at many different levels, and they all use the same user-defined data types. This repetitive nature causes our expansion inference to find only a few of the large number of required expansions. The large fan-out of expansions, combined with the large number of expansions that

\textsuperscript{3}Since xml syntax is context-free, the source and target regular expressions describe only xml expressions up to depth 3.
must be performed, creates a search space too large for our algorithm to effectively search. However, the synthesis algorithm is able to succeed on the easier task of finding the desired transformation when provided with two additional subtasks: synthesis on XML of depth one, and synthesis of XML of depth up to two.

*Importance of Examples.* To evaluate how many user-supplied examples the algorithm requires in practice, we randomly generated appropriate source/target pairs, mimicking what a naive user might do. We did not write the examples by hand out of concern that our knowledge of the synthesis algorithm might bias the selection. Figure 11 shows the number of randomly generated examples it takes to synthesize the correct lens averaged over ten runs. The synthesis algorithm almost never needs any examples: only 5 benchmarks need a nonzero number of examples to synthesize the correct lens and only one, cust/workitem-probs required over 10 randomly generated examples. A clever user may be able to reduce the number of examples further by selecting examples carefully; we synthesized cust/workitem-probs with only 8 examples.

These numbers are low because there are relatively few well-typed bijective lenses between any two source and target regular expressions. As one would expect, the benchmarks where there are multiple ways to map source data to the target (and vice versa) require the most examples. For example, the benchmark cust/workitem-probs requires a large number of examples because it must differentiate between data in different text fields in both the source and target and map between them appropriately. As these text fields are heavily permuted (the legacy format ordered fields by a numeric ID, where the modern format ordered fields alphabetically) and fields can be omitted, a number of examples are needed to correctly identify the mapping between fields.

The average number of examples to infer the correct lens does not tell the whole story. The system will stop as soon as it finds a well-typed lens that satisfies the supplied examples. This inferred lens may or may not correctly handle unseen examples that correspond to unexercised portions of the source and target regular expressions. Figure 11 lists the number of examples that are required to determinize the generation of permutations in RigidSynth. Intuitively, this number represents the maximum number of examples that a user must supply to guide the synthesis engine

![Examples Required for Benchmarks](image-url)
if it always guesses the wrong permutation when multiple permutations can be used to satisfy the specification.

The average number of examples is so much lower than the maximum number of required examples because of correspondences in how we wrote the regular expressions for the source and target data formats. Specifically, when we had corresponding disjunctions in both the source and the target, we ordered them the same way. The algorithm uses the supplied ordering to guide its search, and so the system requires fewer examples. We did not write the examples in this style to facilitate synthesis, but rather because maintaining similar subparts in similar orderings makes the types much easier to read. We expect that most users would do the same.

**Comparison Against Other Tools.** We are the first tool to synthesize bidirectional transformations between data formats, so there is no tool to which we can make an apple-to-apples comparison. Instead, we compare against tools for generating unidirectional transformations instead. Figure 8 includes a comparison against two other well-known tools that synthesize text transformation and extraction functions from examples – Flash Fill and FlashExtract. For this evaluation, we used the version of these tools distributed through the PROSE project [37].

To generate specifications for Flash Fill, we generated input/output specifications by generating random elements of the source language, and running the lens on those elements to generate elements of the target language. These were then fed to Flash Fill.

To generate specifications for FlashExtract, we extracted portions of strings mapped in the generated lens either through an identity transformation or through a previously synthesized lens, whereas strings that were mapped through use of `const` were considered boilerplate and so not extracted.

As these tools were designed for a broader audience, they put less of a burden on the user. These tools only use input/output examples (for Flash Fill), or marked text regions (for FlashExtract), as opposed to Optician’s use of regular expressions to constrain the format of the input and output. By using regular expressions, Optician is able to synthesize significantly more programs than either existing tool.

Flash Fill and FlashExtract have two tasks: to determine how the data is transformed, they must also infer the structure of the data, a difficult job for complex formats. In particular, neither Flash Fill nor FlashExtract was able to synthesize transformations or extractions present under two iterations, a type of format that is notoriously hard to infer. These types of dual iterations are pervasive in Linux configuration files, making Flash Fill and FlashExtract ill suited for many of the synthesis tasks present in our test suite.

Furthermore, as unidirectional transformations, Flash Fill and FlashExtract have a more expressive calculus. To guarantee bidirectionality, our syntax must be highly restrictive, providing a smaller search space to traverse.

### 8 RELATED WORK

In searching for equivalent regular expressions, we focused on Conway’s equational theory rather than alternative axiomatizations such as Kozen’s [21] and Salomaa’s [32]. Kozen and Salomaa’s axiomatizations are not equational theories: applying certain inference rules requires that side conditions must be satisfied. Consequently, using these axiomatizations does not permit a simple search strategy – our algorithm could no longer merely apply rewrite rules because it would need to confirm that the side conditions are satisfied. To avoid these complications, we focused on Conway’s equational theory.
The literature on bidirectional programming languages and on lens-like structures is extensive. We discussed highlights in the introduction; readers can also consult a (slightly dated) survey [8] and recent theoretical perspectives [1, 12].

While we do not know of any previous efforts to synthesize bidirectional transformations, there is a good deal of other recent research on synthesizing unidirectional string transformations [15, 23, 30, 34, 35]. We compared our system to two of these unidirectional string transformers, Flash Fill [15] and FlashExtract [23]. We found that these tools were unsuccessful in synthesizing the complex transformations we are performing – both these tools synthesized under 5 of our 39 examples. Furthermore, neither of these tools were able to infer transformations which occurred under two iterations. Much of this work assumes, like us, that the synthesis engine is provided with a collection of examples. Our work differs in that we assume the programmer supplies both examples and format descriptions in the form of regular expressions. There is a trade-off here. On the one hand, a user must have some programming expertise to write regular expression specifications and it requires some work. On the other hand, such specifications provide a great deal of information to the synthesis system, which decreases the number of examples needed (often to zero), makes the system scale well, and allows it to handle large, complex formats, as shown in §7. By providing these format specifications, the synthesis engine does not have to both infer the format of the data as well as the transformations on it, obviating the need to infer tricky formats like those involving nested iterations. Furthermore, through focusing on bidirectional transformations we limit the space of synthesized functions to bijective ones, reducing the search space.

There are many other recent results showing how to synthesize functions from type-based specifications [2, 11, 14, 28, 31, 33]. These systems enumerate programs of their target language, orienting their search procedures to process only terms that are well-typed. Our system is distinctive in that it synthesizes terms in a language with many type equivalences. Perhaps the most similar is InSynth [16], a system for synthesizing terms in the simply-typed lambda calculus that addresses equivalences on types. Instead of trying to directly synthesize terms of the simply-typed lambda calculus, InSynth synthesizes a well-typed term in the succinct calculus, a language with types that are equivalent “modulo isomorphisms of products and currying” [16]. Our type structure is significantly more complex. In particular, because our types do not have full canonical forms, we use a pseudo-canonical form, which captures part of the equivalence relation over types. To preserve completeness, we push some of the remaining parts of the type equivalence relation into a set of rewriting rules and other parts into the RIGIDSYNTH algorithm itself.

Morpheus [10] is another synthesis system that uses two communicating synthesizers to generate programs. In both Morpheus and Optician, one synthesizer provides an outline for the program, and the other fills in that outline with program details that satisfy the user’s specifications. This approach works well in large search spaces, which require some enumerative search. Our systems differ in that an outline for Morpheus is a sketch—an expression containing holes—whereas an outline for Optician is a pair of DNF regular expressions, i.e., a type. Moreover, in order to implement an efficient search procedure, we had to create both a new type language and a new term language for lenses. Once we did so, we proved our new, more constrained language designed for synthesis was just as expressive as the original, more flexible and compositional language designed for human programmers.

Many synthesis algorithms work on domain-specific languages custom built for synthesis [15, 23, 36, 38]. We too built a custom domain-specific language for synthesis – DNF lenses. We provide the capabilities to convert specifications in an existing language, Boomerang, to specifications as DNF regular expressions, and provide the capabilities to convert our generated DNF lenses to
Boomerang lenses. But we go further than merely providing a converter to Boomerang, we also provide completeness results stating exactly which terms of Boomerang we are able to synthesize.

9 CONCLUSION

Data processing systems often need to convert data back-and-forth between different formats. Domain-specific languages for generating bidirectional programs help prevent data corruption in such contexts, but are unfamiliar and hard to use. To simplify the development of bidirectional applications, we have created the first synthesizer of a bidirectional language, generating lenses from data format specifications and input/output examples. To reduce the size of the synthesis search space, our system introduces a new language of DNF lenses, which are typed by DNF regular expressions. We have proven our new language sound and complete with respect to a declarative specification. We also describe effective optimizations for efficiently searching through the refined space of lenses.

We evaluated our system on a range of practical examples drawn from other systems in the literature including Flash Fill and Augeas. In general, we found our system to be robust, to require few examples, and to finish in seconds, even on complex data formats. We found that our type-directed synthesis algorithm is able to generate data transformations too complex for both existing example-directed systems and for a naive type-directed algorithm, succeeding on 35 more benchmarks than the tested existing alternatives. We attribute its success to a combination of (1) the information provided by format specifications, (2) the structure induced by user-specified names, and (3) the inferences used to guide search. The approaches we used are generalizable both to other bidirectional languages, as well as to other domain-specific languages with large numbers of equivalences on the types.

REFERENCES


A FORMAL DEFINITIONS

Definition 3 (Unambiguous Concatenation Language). If \( L_1 \) and \( L_2 \) are languages, such that for all strings \( s_1, t_1 \in L_1 \), and for all strings \( s_2, t_2 \in L_2 \), if \( s_1 \cdot t_1 = s_2 \cdot t_2 \), then \( L_1 \) is unambiguously concatenable with \( L_2 \), written \( L_1 \cdot 1 \cdot L_2 \).

Definition 4 (Conway’s Regular Expression Equivalences).

\[
\begin{align*}
S \cdot \emptyset &= S + \text{Ident} \\
S \cdot 0 &= \emptyset 0 \text{ Proj}_{R} \\
0 \cdot S &= \emptyset 0 \text{ Proj}_{L} \\
(S \cdot S') \cdot S'' &= S \cdot (S' \cdot S'') \cdot \text{Assoc} \\
(S \cdot S') | S'' &= S | (S' | S'') \mid \text{Assoc} \\
S \cdot T &= T \cdot S \mid \text{Comm} \\
S \cdot (S'| S'') &= (S \cdot S') | (S \cdot S'') \mid \text{Dist}_{R} \\
(S'| S'') \cdot S &= (S' \cdot S) | (S'' \cdot S) \mid \text{Dist}_{L} \\
e \cdot S &= S \cdot \text{Ident}_{L} \\
S \cdot e &= S \cdot \text{Ident}_{R} \\
(S | T)^* &= (S^* \cdot T)^* \cdot S^* \mid \text{Sumstar} \\
(S \cdot T)^* &= e | (S \cdot (T \cdot S)^* \cdot T) \mid \text{Prodstar} \\
(S^*)^* &= S^* \mid \text{Starstar} \\
(S | T)^* &= ((S | T) \cdot T | (S \cdot T^* | S^*) \cdot (S \cdot T^* \mid \ldots | (S \cdot T^*)^*)) \\
\end{align*}
\]

Definition 5 (Definitional Regular Expression Equivalences).

\[
\begin{align*}
S \cdot \emptyset &= S \equiv S + \text{Ident} \\
S \cdot 0 &= S \equiv 0 \text{ Proj}_{R} \\
0 \cdot S &= S \equiv 0 \text{ Proj}_{L} \\
(S \cdot S') \cdot S'' &= S \cdot (S' \cdot S'') \equiv \text{Assoc} \\
(S \cdot S') | S'' &= S | (S' | S'') \mid \text{Assoc} \\
S \cdot T &= T \cdot S \equiv \text{Comm} \\
S \cdot (S'| S'') &= (S \cdot S') | (S \cdot S'') \equiv \text{Dist}_{R} \\
(S'| S'') \cdot S &= (S' \cdot S) | (S'' \cdot S) \equiv \text{Dist}_{L} \\
e \cdot S &= S \equiv \text{Ident}_{L} \\
S \cdot e &= S \equiv \text{Ident}_{R} \\
S^* &= e | (S \cdot S^*) \equiv \text{Unrollstar}_{L} \\
S^* &= e | (S^* \cdot S) \equiv \text{Unrollstar}_{R}
\end{align*}
\]

B PROOFS

The proof is split into separate subsections based on what is being done. The overall goals are to prove soundness and completeness of DNF regular expressions with respect to regular expressions, and soundness and completeness of DNF lenses with respect to lenses.

- Subsection B.1 defines confluence with respect to a property, bisimilarity, and makes some general proofs about those properties. These are used later for the proof of confluence of rewriting with respect to semantics, which is used in lens completeness.
- Subsection B.2 proves some general statements about languages, relating to the relationship between nonintersection of pairs of languages, and sets of languages, and the relationship between shared prefixes and suffixes of pairs of languages. These are used for proving
statements about unambiguity of DNF regular expressions from unambiguous regular expressions, and vice-versa.

- Subsection B.3 proves some intuitive statements about lenses and DNF lenses. These statements are properties like inversion, closure under composition for rewriteless DNF lenses, and proves that bijective lenses and bijective DNF lenses actually express bijections between the languages of their types.
- Subsection B.4 proves soundness and completeness of DNF regular expressions to regular expressions.
- Subsection B.5 proves statements relating to the retention of unambiguity across languages. In particular, it proves statements about \(\downarrow\) and \(\uparrow\), and also proves statements about the retention of unambiguity through rewrites.
- Subsection B.6 proves statements about the retention of language through proofs, and the equivalences of expressibility of various rewrite systems.
- Subsection B.7 proves the soundness of DNF lenses to lenses, using the machinery above.
- Subsection B.8 defines operators on DNF lenses, which provide combinators similar to the combinators of normal lenses. This section also proves statements about these combinators, like how the combinators act similarly to normal lenses.
- Subsection B.9 proves more complex properties about lens operators. These more complex statements are needed because DNF regular expressions don’t have rewrites that order clauses.
- Subsection B.10 proves statements about the ability to build up rewrites on DNF regular expressions composed of less complex ones, from the rewrites of those less complex DNF regular expressions. It also proves the proof of confluence of rewrites.
- Subsection B.11 proves the completeness of dnf lenses with respect to lenses.
- Subsection B.12 proves the algorithm correct.
- Subsection B.13 proves some random statements we make, but don’t formally express, in the paper.

## B.1 Confluence Proofs

This section begins by defining confluence and bisimilarity. Next we prove that if a rewrite system is bisimilar with respect to a property, then the transitive and reflexive closure of that rewrite system is too. Next, a similar statement about transitive and reflexive closure of rewrite systems for confluence is proven, under the conditions that the property confluence is defined with respect to is transitive. Next, propagators are defined, and used in if a rewrite system is confluent with respect to a property with left and right propagators, then the transitive and reflexive closures of that rewrite system is confluent with respect to the same property.

**Definition 6.** Let \(\rightarrow\) and \(p\) be two binary relations on a set \(S\). We say that \(\rightarrow\) is confluent with respect to \(p\), written \(\text{confluent}_p(\rightarrow)\), if, given \(x_1, x_2 \in S\), where \(p(x_1, x_2)\), if \(x_1 \rightarrow x'_1\) and \(x_2 \rightarrow x'_2\), then there exists \(x''_1\) and \(x''_2\) such that \(x'_1 \rightarrow x''_1\), \(x'_2 \rightarrow x''_2\), and \(p(x''_1, x''_2)\).

**Definition 7.** Let \(\rightarrow\) and \(p\) be two binary relations on a set \(S\). We say that \(\rightarrow\) is bisimilar through \(p\), written \(\text{bisimilar}_p(\rightarrow)\), if, given \(x_1, x_2 \in S\), where \(p(x_1, x_2)\), if \(x_1 \rightarrow x'_1\) then there exists some \(x_2\) such that \(x_2 \rightarrow x'_2\) where \(p(x'_1, x'_2)\), and if \(x_2 \rightarrow x'_2\), then there exists some \(x'_1\) such that \(x_1 \rightarrow x'_1\) where \(p(x'_1, x'_2)\).
**Definition 8.** Let $p$ be a binary relation. $p^*$ is the binary relation defined via the inference rules

<table>
<thead>
<tr>
<th>REFLEXIVITY</th>
<th>BASE</th>
<th>TRANSITIVITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^*(x, x)$</td>
<td>$p(x, y)$</td>
<td>$p^<em>(x, y)$ $p^</em>(y, z)$ $p^*(x, z)$</td>
</tr>
</tbody>
</table>

**Lemma 1** (Bisimilarity Preserved through Star left). Let $\text{bisimilar}_p(\to)$. If $p(x, y)$ and $x \to^* x'$ then there exists some $y'$ such that $y \to^* y'$ where $p(x', y')$

**Proof.** By induction on the derivation of $x \to^* x'$.

**Case 1 (Reflexivity).**

$x \to^* x$

Consider the derivation

$y \to^* y$

and by assumption $p(x, y)$.

**Case 2 (Base).**

$x \to x' \quad x \to^* x'$

As $\text{bisimilar}_p(\to)$, $y \to y'$ where $p(x', y)$.

**Case 3 (Transitivity).**

$x \to^* x'' \quad x'' \to^* x' \quad x \to^* x'$

By IH, $y \to y''$ where $p(x'', y'')$. By IH, $y'' \to^* y'$ where $p(x', y')$.

**Lemma 2** (Bisimilarity Preserved through Star right). Let $\text{bisimilar}_p(\to)$. If $p(x, y)$ and $x \to^* x'$ then there exists some $y'$ such that $y \to^* y'$ where $p(x', y')$

**Proof.** Symmetrically to Lemma 1.

**Lemma 3** (Bisimilarity Preserved through Star). If $\text{bisimilar}_p(\to)$, then $\text{bisimilar}_p(\to^*)$.

**Proof.** By application of Lemma 1 and Lemma 2.

**Lemma 4.** If $\text{confluent}_p(\to)$, $\text{bisimilar}_p(\to)$, $p(x, y) \land p(y, z) \Rightarrow p(x, z)$, and $p(x, y) \Rightarrow p(x, x) \land p(y, y)$ then if $p(x, y)$, $x \to^* x_1$, $y \to x_1$, then there exists some $x_2, x_y$ such that $x_1 \to x_2$, $y_1 \to y_2$, and $p(x_2, y_2)$.

**Proof.** By induction on the derivation of $x \to^* x_1$. 

---

Case 1 (Reflexivity).

\[
\frac{x \to^* x}{y \to^* y}
\]

By bisimilar_\(p(\to)\), there exists some \(x_1\) such that \(x \to x_1\) and \(p(x_1, y_1)\). Furthermore,

\[
\frac{y_1 \to^* y_1}{\text{so we are done.}}
\]

Case 2 (Base).

\[
\frac{x \to x_1}{x \to^* x_1}
\]

As confluent_\(p(\to)\), there exists \(x_2, y_2\) such that \(x_1 \to x_2, y_1 \to y_2\), and \(p(x_2, y_2)\). Furthermore

\[
\frac{y_1 \to y_2}{y_1 \to^* y_2}
\]

Case 3 (Transitivity).

\[
\frac{x \to^* x_1 \quad x_1 \to^* x_2}{x \to^* x_2}
\]

By IH, there exists \(x_3, y_2\) such that \(x_1 \to x_3\), and \(y_1 \to^* y_2\), and \(p(x_3, y_2)\).

As \(p(x, y)\), we have \(p(x, x)\). As \(p(x, x)\), and \(x \to^* x_1\), then there exists \(x'\) such that \(p(x_1, x')\), so \(p(x_1, x_1)\). So, by IH, as \(p(x_1, x_1), x_1 \to^* x_2\), and \(x_1 \to x_3\), there exists \(x_4, x_5\) such that \(x_2 \to x_4, x_3 \to^* x_5\), and \(p(x_4, x_5)\).

As \(p(x_3, y_2)\), and \(x_3 \to^* x_5\), then by bisimilar_\(p(\to)\) and Lemma 3, there exists \(y_3\) such that \(y_2 \to^* y_3\), and \(p(x_5, y_3)\). By Transitivity, \(y_1 \to^* y_3\). From before, \(x_2 \to x_4\). Because we have \(p(x_4, x_5)\) and \(p(x_5, y_3)\), we have \(p(x_4, y_3)\).

\[\Box\]

Lemma 5. If confluent_\(p(\to)\), bisimilar_\(p(\to)\), and \(p(x, y) \land p(y, z) \Rightarrow p(x, z)\), and \(p(x, y) \Rightarrow p(x, x) \land p(y, y)\) then if \(p(x, y)\), \(x \to^* x_1\), \(y \to^* x_1\), then there exists some \(x_2, y_2\) such that \(x_1 \to^* x_2, y_1 \to^* y_2\), and \(p(x_2, y_2)\).

Proof. By induction on the derivation of \(y \to^* y_1\).

Case 1 (Reflexivity).

\[
\frac{y \to^* y}{\text{so we are done.}}
\]

Case 2 (Base).

\[
\frac{y \to y_1}{y \to^* y_1}
\]

As confluent_\(p(\to)\), bisimilar_\(p(\to)\), \(y \to y_1, x \to^* x_1\), and \(p^*(x, y)\) if, and only if \(p(x, y)\), by Lemma 4, there exists \(x_2, y_2\) such that \(x_1 \to x_2, y_1 \to^* y_2\), and \(p(x_2, y_2)\). Furthermore

\[
\frac{x_1 \to x_2}{x_1 \to^* x_2}
\]

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Case 3 (Transitivity).

By IH, there exists $x_2, y_3$ such that $x_1 \rightarrow^* x_2$, and $y_1 \rightarrow^* y_3$, and $p(x_2, y_3)$.

As $p(x, y)$, we have $p(y, y)$. As $p(y, y)$, and $y \rightarrow^* y_1$, then there exists $y'$ such that $p(y_1, y')$, so $p(y_1, y_1)$. So, by IH, as $p(y_1, y_1)$, $y_1 \rightarrow^* y_3$, and $y_1 \rightarrow^* y_2$, there exists $y_4, y_5$ such that $y_3 \rightarrow^* y_4$, $y_2 \rightarrow^* y_5$, and $p(y_4, y_5)$.

As $p(x_2, y_3)$, and $y_3 \rightarrow^* y_4$, then by bisimilar$_p(\rightarrow)$ and Lemma 3, there exists $x_3$ such that $x_2 \rightarrow^* x_3$, and $p(x_3, y_4)$. By Transitivity, $x_1 \rightarrow^* x_3$. From before, $y_2 \rightarrow y_3$. As we have $p(x_3, y_4)$ and $p(y_4, y_5)$, we have $p(x_3, y_5)$.

\[\square\]

Definition 9. A property $q$ is a left propagator for $p$ with respect to $\rightarrow$ if bisimilar$_q(\rightarrow)$, confluent$_q(\rightarrow)$, $q(x, y) \wedge q(y, z) \Rightarrow q(x, z)$, $q(x, y) \Rightarrow q(x, x) \wedge q(y, y)$, $p(x, y) \Rightarrow q(x, x)$, and $q(x, y) \wedge p(y, z) \Rightarrow p(x, z)$.

Definition 10. A property $q$ is a right propagator for $p$ with respect to $\rightarrow$ if bisimilar$_p(\rightarrow)$, confluent$_p(\rightarrow)$, $q(x, y) \Rightarrow q(x, z)$, $q(x, y) \Rightarrow q(x, x) \wedge q(y, y)$, $p(x, y) \Rightarrow q(y, y)$, and $p(x, y) \wedge q(y, z) \Rightarrow p(x, z)$.

Lemma 6. Let confluent$_p(\rightarrow)$. Let bisimilar$_p(\rightarrow)$. Let $q_l$ be a left propagator for $p$ with respect to $\rightarrow$. If $p(x_1, x_2), x_1 \rightarrow^* x_1', x_2 \rightarrow^* x_2'$, then there exists some $x_1'', x_2''$ such that $x_1' \rightarrow^* x_1'', x_2' \rightarrow^* x_2''$, and $p(x_1'', x_2'')$.

Proof. By induction on the derivation of $x_1 \rightarrow^* x_1'$.

Case 1 (Reflexivity).

By bisimilar$_p(\rightarrow)$, there exists some $x_1'$ such that $x_1 \rightarrow x_1'$ and $p(x_1', x_1')$. Furthermore,

so we are done.

Case 2 (Base).

As confluent$_p(\rightarrow)$, there exists $x_2, y_2$ such that $x_1 \rightarrow x_2, y_1 \rightarrow y_2$, and $p(x_2, y_2)$. Furthermore

Case 3 (Transitivity).

By IH, there exists $x_3, y_2$ such that $x_1 \rightarrow x_3$, and $y_1 \rightarrow^* y_2$, and $p(x_3, y_2)$.

As q$_l$(a, b) if, and only if q$_l$(a, b), q$_l$(x$_1$, x$_1$).

As p(x, y), we have q$_l$(x, x). As q$_l$(x, x) and x$\rightarrow^* x_1$, there exists $x'$ such that $x \rightarrow^* x'$, and q$_l$(x, x'), which means that q$_l$(x, x). As q$_l$ is a propagator with respect to $\rightarrow$, it fills the properties
required for Lemma 4. So, as \( q_L(x_1, x_1), x_1 \rightarrow^* x_2 \), and \( x_1 \rightarrow x_3 \), there exists \( x_4, x_5 \) such that \( x_2 \rightarrow x_4 \), \( x_3 \rightarrow^* x_5 \), and \( q_L(x_4, x_5) \).

As \( p(x_3, y_2) \), and \( x_3 \rightarrow^* x_5 \), then by \text{bisimilar}_p(\rightarrow) and Lemma 3, there exists \( y_3 \) such that \( y_2 \rightarrow^* y_3 \), and \( p(x_5, y_3) \). By \text{T/r.sc/a.sc/n.sc/s.sc/i.sc/t.sc/i.sc/v.sc/i.sc/t.sc/y.sc} 

\[ x \rightarrow^* y \]

By \text{bisimilar}_p(\rightarrow), and Lemma 3, there exists some \( y_1 \) such that \( y \rightarrow^* y_1 \) and \( p(x_1, y_1) \). Furthermore,

\[ y_1 \rightarrow^* y_1 \]

so we are done.

\text{Case 2 (Base).}

\[ y \rightarrow y_1 \]

\[ y_1 \rightarrow^* y_1 \]

As \text{confluent}_p(\rightarrow), \text{bisimilar}_p(\rightarrow), y \rightarrow y_1, x \rightarrow^* x_1 \), and \( p^*(x, y) \) if, and only if \( p(x, y) \), by Lemma 6, there exists \( x_2, y_2 \) such that \( x_1 \rightarrow x_2, y_1 \rightarrow^* y_2 \), and \( p(x_2, y_2) \). Furthermore

\[ x_1 \rightarrow x_2 \]

\[ x_1 \rightarrow^* x_2 \]

\text{Case 3 (Transitivity).}

\[ y \rightarrow^* y_1 \]

\[ y_1 \rightarrow^* y_2 \]

By IH, there exists \( x_2, y_3 \) such that \( x_1 \rightarrow^* x_2 \), and \( y_1 \rightarrow y_3 \), and \( p(x_2, y_3) \).

As \( p(x, y) \), we have \( q_R(y, y) \). As \( q_R(x, x) \) and \( y \rightarrow^* y_1 \), there exists \( y' \) such that \( y \rightarrow^* y' \), and \( q_R(y_1, y') \), which means that \( q_R(y_j, y_i) \). As \( q_L \) is a propagator with respect to \( 

\[ y \rightarrow^* y_2 \]

\text{Definition 11.} Let \( p \) be a binary relation. \( \equiv_p \) is the binary relation defined via the inference rules

\begin{align*}
\text{Base} & \quad \frac{p(x, y)}{\equiv_p (x, y)} & \text{Reflexivity} & \quad \frac{\equiv_p (x, x)}{p(x, x)} & \text{Transitivity} & \quad \frac{\equiv_p (x, y) \quad \equiv_p (y, z)}{p(x, z)} & \text{Symmetry} & \quad \frac{p(x, y)}{\equiv_p (y, x)}
\end{align*}
B.2 Language Proofs

These proofs prove similar things to unambiguity, but on general languages.

Lemma 7. Let \( L_1, \ldots, L_n, L'_1, \ldots, L'_m \) be nonempty languages. If \( \langle L_1; \ldots; L_n \rangle, \langle L'_1; \ldots; L'_m \rangle \), and \( \{s_1 \cdot \ldots \cdot s_n \mid s_i \in L_i \} \cdot \{s'_1 \cdot \ldots \cdot s'_m \mid s'_i \in L'_i \} \), then \( \langle L_1; \ldots; L_n; L'_1; \ldots; L'_m \rangle \)

Proof. Let \( \langle L_1; \ldots; L_n \rangle, \langle L'_1; \ldots; L'_m \rangle \), and \( \{s_1 \cdot \ldots \cdot s_n \mid s_i \in L_i \} \cdot \{s'_1 \cdot \ldots \cdot s'_m \mid s'_i \in L'_i \} \) Let \( s_i, t_i \in L_i \), \( s'_i, t'_i \in L'_i \). Let \( s_1 \cdot \ldots \cdot s_n \cdot s'_1 \cdot \ldots \cdot s'_m = t_1 \cdot \ldots \cdot t_n \cdot t'_1 \cdot \ldots \cdot t'_m \). Because \( \{s_1 \cdot \ldots \cdot s_n \mid s_i \in L_i \} \cdot \{s'_1 \cdot \ldots \cdot s'_m \mid s'_i \in L'_i \} \), we know \( s_1 \cdot \ldots \cdot s_n = t_1 \cdot \ldots \cdot t_n \) and \( s'_1 \cdot \ldots \cdot s'_m = t'_1 \cdot \ldots \cdot t'_m \). Because \( \langle L_1; \ldots; L_n \rangle, s'_i = t'_i \). So \( \langle L'_1; \ldots; L'_m \rangle \)

Lemma 8. Let \( L_1, \ldots, L_n, L'_1, \ldots, L'_m \) be nonempty languages. \( \langle L_1; \ldots; L_n \rangle, \langle L'_1; \ldots; L'_m \rangle \), and \( \{s_1 \cdot \ldots \cdot s_n \mid s_i \in L_i \} \cdot \{s'_1 \cdot \ldots \cdot s'_m \mid s'_i \in L'_i \} \) if, and only if \( \langle L_1; \ldots; L_n; L'_1; \ldots; L'_m \rangle \)

Proof.

Case 1 (\( \Rightarrow \)). By Lemma 7.

Case 2 (\( \Leftarrow \)). Let \( s, t \in \{s_1 \cdot \ldots \cdot s_n \mid s_i \in L_i \} \). Let \( s', t' \in \{s_1 \cdot \ldots \cdot s_n \mid s_i \in L'_i \} \). Let \( s \cdot s' = t \cdot t' \). \( s = s_1 \cdot \ldots \cdot s_n \) where \( s_i \in L_i, t = t_1 \cdot \ldots \cdot t_n \) where \( t_i \in L_i, s'_i = s'_1 \cdot \ldots \cdot s'_m \) where \( s'_i \in L'_i \), and \( t' = t'_1 \cdot \ldots \cdot t'_m \) where \( t'_i \in L'_i \). \( s \cdot s' = t_1 \cdot \ldots \cdot s_n \cdot s'_1 \cdot \ldots \cdot s'_m \) and \( t \cdot t' = s_1 \cdot \ldots \cdot s_n \cdot s'_1 \cdot \ldots \cdot s'_m \).

By assumption \( s_i = t_i \) and \( s'_i = t'_i \). This means \( s = t \) and \( s' = t' \).

Let \( s_i, t_i \in L_i \), and let \( s_1 \cdot \ldots \cdot s_n = t_1 \cdot \ldots \cdot t_n \). Consider some strings \( s'_i \in L'_i, s_1 \cdot \ldots \cdot s_n \cdot s'_1 \cdot \ldots \cdot s'_m = t_1 \cdot \ldots \cdot t_n \cdot s'_1 \cdot \ldots \cdot s'_m \).

By assumption, \( s_i = t_i \), as desired.

Lemma 9. Let \( L_1, \ldots, L_n, L'_1, \ldots, L'_m \) be languages. Let \( L_{i,j} = \{s \cdot t \mid s \in L_i \land t \in L'_j \} \). Let \( A = \bigcup_{i \in [1,n]} L_i \neq \emptyset \). Let \( B = \bigcup_{i \in [1,m]} L'_i \neq \emptyset \). \( i \neq j \Rightarrow L_i \cap L'_j = \emptyset \) and \( A \cdot B \)

if, and only if \( (i_1,j_1) \neq (i_2,j_2) \Rightarrow L''_{i_1,j_1} \cap L''_{i_2,j_2} = \emptyset \) and for all \( i \in [1,n], j \in [1,m] \), we have \( L_i \cdot L'_j \)

Proof.

Case 1 (\( \Rightarrow \)). Let \( i \neq j \Rightarrow L_i \neq L'_j \). \( i \neq j \Rightarrow L_i \neq L'_j \) and \( A \cdot B \)

We shall prove \( (i_1,j_1) \neq (i_2,j_2) \Rightarrow L''_{i_1,j_1} \cap L''_{i_2,j_2} = \emptyset \) by contrapositive. Let \( s \in L''_{i_1,j_1} \land L''_{i_2,j_2} \). This means that \( s = s_{i_1} \cdot s_{j_1} \) for some \( s_{i_1} \in L_{i_1} \) and some \( s_{j_1} \in L_{j_1} \), and that \( s = s_{i_2} \cdot s_{j_2} \) for some \( s_{i_2} \in L_{i_2} \) and some \( s_{j_2} \in L_{j_2} \).

Because \( A \cdot B \) \( s_{i_1} = s_{i_2} \) and \( s_{j_1} = s_{j_2} \). Because each of \( A \) and \( B \) are pairwise disjoint, this means \( i_1 = i_2 \) and \( j_1 = j_2 \).

Let \( s_i, t_i \in L_i \). Let \( s_j, t_j \in L'_j \). Let \( s_i \cdot s_j = t_i \cdot t_j \). By definition, \( s_i, t_i \in A \) and \( s_j, t_j \in B \). By assumption, \( A \cdot B \), so \( s_i = t_i \) and \( s_j = t_j \).

Case 2 (\( \Leftarrow \)). Let \( (i_1,j_1) \neq (i_2,j_2) \Rightarrow L''_{i_1,j_1} \cap L''_{i_2,j_2} = \emptyset \) and for all \( i \in [1,n], j \in [1,m] \), we have \( L_i \cdot L'_j \).

We prove \( i \neq j \Rightarrow L_i \cap L'_j = \emptyset \) by contrapositive. Let \( L_i \cap L'_j \neq \emptyset \). Let \( s \in L_i \cap L'_j \). Let \( t \in B \).

\( t \in L'_j \) for some \( k \in [1,m], s \cdot t \in L''_{i,k} \) and \( s \cdot t \in L''_{j,k} \). By assumption \( (i,k) = (j,k) \), so \( i = j \).

We prove \( i \neq j \Rightarrow L_i \cap L'_j = \emptyset \) in the same way.

Let \( s_1, s_2 \in A, t_1, t_2 \in B \), and \( s_1 \cdot t_1 = s_2 \cdot t_2 \). Let \( s_1 \in L_i \) for some \( i \), and \( s_2 \in L_j \) for some \( j \). \( \Rightarrow t_1 \in L'_k \) for some \( k \), and \( t_2 \in L'_k \) for some \( l \). This means \( s_1 \cdot t_1 = L''_{i,k} \), \( s_2 \cdot t_2 \in L''_{j,k} \). Because \( s_1 \cdot t_1 = s_2 \cdot t_2 \),\( (i,k) = (j,l) \). So as \( s_1 \in L_i, s_2 \in L_j, t_1 \in L'_k, t_2 \in L'_l, \) and \( L_i \cdot L'_k \), \( s_1 = s_2 \) and \( t_2 = t_2 \).
Lemma 10. Let $A = \{L_1, \ldots, L_n\}$, $B = \{L'_1, \ldots, L'_m\}$, $C = \{L''_1, \ldots, L''_m\}$, be sets of languages. Such that $A \cup B = C$, $(\bigcup_{i \in [1,n]} L_i) \cap (\bigcup_{j \in [1,m]} L'_j) = \emptyset$, for all $i, j \in [1, n]$, $i \neq j$  $\Rightarrow \text{Language}_i \cap L_j = \emptyset$, and for all $i, j \in [1, m]$, $i \neq j$  $\Rightarrow L'_i \cap L'_j = \emptyset$ if, and only if for all $i, j \in [1, n + m]$, $i \neq j$  $\Rightarrow L''_i \cap L''_j = \emptyset$.

Proof.

Case 1 ($\Rightarrow$). Let $L''_i \subseteq L''_j \subseteq C$, where $i \neq j$. If $L''_i \subseteq A$ and $L''_j \subseteq A$, then, by pigeonhole principle, there exists an $i', j'$ where $i' \neq j'$ such that $L''_i = L''_{i'}$ and $L''_j = L''_{j'}$. By assumption, $L''_{i'} \cap L''_{j'} = \emptyset$, so $|\text{Language}_{i''} \cap L''_{j''} = \emptyset|$. Similarly for if $L''_i \subseteq B$ and $L''_j \subseteq B$. If $L''_i \subseteq A$ and $L''_j \subseteq B$. $(\bigcup_{i \in [1,n]} L_i) \cap (\bigcup_{j \in [1,m]} L'_j) = \emptyset$. By application of distributivity $\bigcup_{(k,l) \in [1,n] \times [1,m]} L''_k \cap L''_l = \emptyset$. This means that for all $(k, l) \in [1, n] \times [1, m]$, $L_k \cap L'_l = \emptyset$. In particular, $L''_i \cap L''_j = \emptyset$. Case 2 ($\Leftarrow$). Let $i, j \in [1, n]$ and $i \neq j$. By pigeonhole principle, there exists some $i', j'$ where $i' \neq j'$ such that $L_i = L'_{i'}$ and $L_j = L'_{j'}$. By assumption, $L''_i \cap L''_j = \emptyset$, so $L_i \cap L_j = \emptyset$. Similarly for $i, j \in [1, m]$. Assume there exists some $s \in (\bigcup_{i \in [1,n]} L_i) \cap (\bigcup_{j \in [1,m]} L'_j)$. Then $s \in L_i$ for some $i \in [1, n]$, and $s \in L'_j$ for some $j \in [1, m]$. There exists some $i', j'$ where $i' \neq j'$ in $[1, n + m]$ such that $L_i = L'_{i'}$ and $L_j = L'_{j'}$. But, by assumption, $L''_i \cap L''_j$, so we have a contraction. So there is no $s \in (\bigcup_{i \in [1,n]} L_i) \cap (\bigcup_{j \in [1,m]} L'_j)$, so $(\bigcup_{i \in [1,n]} L_i) \cap (\bigcup_{j \in [1,m]} L'_j) = \emptyset$.

B.3 Lens and DNF Basic Property Proofs

There are many intuitive facts about DNF lenses. For example, without rewrites, they are closed under composition. Furthermore, we can express the identity transformation on DNF lenses. Well-typed DNF lenses and normal lenses express bijections between their types. These properties are proven in this section, and used throughout the paper.

Lemma 11 (DNF Lens Inversion).

1. If $dl : DS \Rightarrow DT$, then there exists some $DS', DT'$ such that $dl : DS \Rightarrow DT$, $DS \Rightarrow^* DS'$, and $DT \Rightarrow^* DT'$.
2. If $dl : DS \Rightarrow DT$, then there exists some $n \in \mathbb{N}$, $SQ_1, \ldots, SQ_n$, $TQ_1, \ldots, TQ_n$, $\sigma \in S_n$, and $sql_1, \ldots, sql_n$ such that for all $i \in [1, n]$, $sql_i : SQ_i \Rightarrow TQ_i$, $dl = (\langle sql_0 \mid \ldots \mid sql_n \rangle, \sigma)$, $DS = \langle SQ_1 \mid \ldots \mid SQ_n \rangle$, and $DT = \langle TQ_{\sigma(1)} \mid \ldots \mid TQ_{\sigma(n)} \rangle$.
3. If $sql : SQ \Rightarrow TQ$, there exists some $n \in \mathbb{N}$, $A_1, \ldots, A_n, B_1, \ldots, B_n, String_0, \ldots, String_n, \tau_0, \ldots, \tau_n$, $\sigma \in S_n$, and $al_1, \ldots, al_n$ such that for all $i \in [1, n]$, $al_i : A_i \Rightarrow B_i$, $sql = \langle (\langle String_0, \ldots, String_n \rangle, \tau_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot (\langle \tau_0, \ldots, \tau_n \rangle, \sigma), SQ = [String_0, \ldots, String_n], and TQ = [\tau_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot (\langle \tau_0, \ldots, \tau_n \rangle, \tau_0]$. 
4. If $al : A \Rightarrow B$, then there exists some $dl, DS, DT$, such that $dl : DS \Rightarrow DT$, $al = \text{iterate}(dl)$, $A = DS^*$, and $B = DT^*$.

Proof.

1. Let $dl : DS \Rightarrow DT$. The only rule that introduces a typing of this form is REWRITE DNF REGEX LENS. Because of this, there exists some $DS', DT'$ such that $dl : DS \Rightarrow DT'$, $DS \Rightarrow^* DS'$, and $DT \Rightarrow^* DT'$, to build up the typing

$$
\begin{array}{ccc}
\text{dl} : DS \Rightarrow DT & \text{DS} \Rightarrow^* DS' & \text{DT} \Rightarrow^* DT' \\
\end{array}
$$


(2) Let \( dl : DS \iff DT \). The only rule that introduces a typing of this form is DNF LENS. Because of this there exists some \( n \in \mathbb{N}, SQ_1, \ldots, SQ_n, TQ_1, \ldots, TQ_n, \sigma \in S_n \), and \( s_l, \ldots, s_l_n \) such that for all \( i \in [1, n] \), \( s_l_i : SQ_i \iff TQ_i, dl = ((SQ_0 \mid \ldots \mid SQ_n, \sigma), DS = (SQ_1 \mid \ldots \mid SQ_n), \)
\[ DT = \langle TQ_{\sigma(1)} \mid \ldots \mid TQ_{\sigma(n)} \rangle, i \neq j \Rightarrow SQ_i \cap SQ_j = \emptyset, \text{ and } i \neq j \Rightarrow TQ_i \cap TQ_j = \emptyset, \]
to build up the typing
\[ s_l_i : SQ_i \iff TQ_i \quad i \neq j \Rightarrow SQ_i \cap SQ_j = \emptyset \quad i \neq j \Rightarrow TQ_i \cap TQ_j = \emptyset \]
\[ dl : DS \iff DT \]

(3) Let \( s_l : SQ \iff TQ \). The only rule that introduces a typing of this form is SEQUENCE LENS. Because of this there exists some \( n \in \mathbb{N}, A_1, \ldots, A_n, B_1, \ldots, B_n, S_{\text{string}}, \ldots, s_n, t_0, \ldots, t_n, \)
\( \sigma \in S_n \), and \( a_1, \ldots, a_n \) such that for all \( i \in [1, n] \), \( a_l_i : A_i \iff B_i, s_l = \langle (s_0, t_0) \cdot a_1 \cdot \ldots \cdot a_1 (s_n, t_n), \sigma \rangle, SQ = [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n], \)
\( \text{ and } TQ = [t_0 \cdot A_{\sigma(1)} \cdot \ldots \cdot A_{\sigma(n)} \cdot t_n] \) to build up the typing
\[ a_l_i : A_i \iff B_i \quad 1 \langle(s_0; A_1; \ldots; A_n; s_n), (t_0; B_{\sigma(1)}; \ldots; B_{\sigma(n)}; t_n) \rangle \\
\langle [(s_0, t_0) \cdot a_1 \cdot \ldots \cdot a_1 (s_n, t_n)], \sigma \rangle : [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \iff [t_0 \cdot A_{\sigma(1)} \cdot \ldots \cdot A_{\sigma(n)} \cdot t_n] \]

(4) Let \( a_l : A \iff B \). The only rule that introduces a typing of this form is ATOM LENS. Because of this, there exists some \( dl, DS, DT, \) such that \( dl : DS \iff DT, a_l = \text{iterate}(dl), A = DS^*, \)
\( \text{ and } B = DT^* \) to build up the typing
\[ dl : DS \iff DT \quad \text{DNFRegex}^{*l} \quad \text{DNFRegexAlt}^{*l} \quad \text{iterate}(dl) : DS^* \iff DT^* \]

Lemma 12. If \( l : S \iff T \), then \( \llbracket l \rrbracket \) is a bijection between \( \mathcal{L}(S) \) and \( \mathcal{L}(T) \).

Proof. By induction on the typing derivation of the lens

Case 1 (Const).
\[ s_1 \in S^* \quad s_2 \in S^* \]
\[ \mathcal{L}(s_1) = \{s_1\} \]
\[ \mathcal{L}(s_2) = \{s_2\} \]
\[ \text{Semantics} \{\text{const}(s_1, s_2) = \{(s_1, s_2)\} \}

Case 2 (Identity).
\[ S \text{ is strongly unambiguous} \quad id_S : S \iff S \]
\[ \mathcal{L}(S) = \mathcal{L}(S) \]
The identity relation on \( \mathcal{L}(S) \) is a bijection.

Case 3 (Iterate).
\[ l : S \iff T \quad S^* \quad T^* \]
\[ \text{iterate}(l) : S^* \iff T^* \]
Let \( s_1, s_2 \in \mathcal{L}(S^*), \) and \( (s_1, t) \in \llbracket \text{iterate}(l) \rrbracket, \) and \( (s_2, t) \in \llbracket \text{iterate}(l) \rrbracket. \)
So \( s_1 = s_{1,1} \cdots s_{1,n}, t = t_1 \cdots t_n, \) and \( (s_{1,i}, t_i) \in \llbracket l \rrbracket. \) So \( s_2 = s_{2,1} \cdots s_{2,m}, t = t'_1 \cdots t'_m, \) and \( (s_{2,i}, t'_i) \in \llbracket l \rrbracket. \) By \( T^*, \) this means that \( m = n, \) and \( t_i = t'_i. \) So \( (s_{1,i}, t_i), \) and \( (s_{2,i}, t_i) \) are both in \( \llbracket l \rrbracket. \) As \( l \) is a bijection, by IH, \( s_{1,i} = s_{2,i}, \) so \( s_1 = s_2. \)
Similarly for $t_1, t_2 \in \mathcal{L}(T^*)$.

Let $s \in \mathcal{L}(S^*), s = s_1 \ldots s_n$, where $s_1 \in \mathcal{L}(S)$. By IH, as $l$ is a bijection, there exists $t_i \in \mathcal{L}(S)$ such that $(s_i, t_i) \in \mathbb{I}$. So $(s_1, \ldots, t_n) \in \mathbb{iterate}(l)$, and $t_1, \ldots, t_n \in \mathcal{L}(T^*)$.

Similarly for $t \in \mathcal{L}(T^*)$.

Case 4 (ConcatLensType).

\[
\begin{align*}
    l_1 : S_1 &\iff T_1 \\
    l_2 : S_2 &\iff T_2 \\
    S_1 \cdot S_2 &\iff T_1 \cdot T_2 \\
    \text{concat}(l_1, l_2) : S_1 S_2 &\iff T_1 T_2
\end{align*}
\]

Let $s_1, s_2 \in \mathcal{L}(S_1 \cdot S_2)$, and $(s_1, t) \in \mathbb{concat}(l_1, l_2)$, and $(s_2, t) \in \mathbb{concat}(l_1, l_2)$.

So $s_1 = s_{1,1} \cdot s_{1,2}, t = t_1 t_2$, and $(s_{1,i}, t_i) \in \mathbb{I}$. So $s_2 = s_{2,1} \cdot s_{2,2}, t = t'_1 \cdot t'_2$, and $(s_{2,i}, t'_i) \in \mathbb{I}$. By $T_1 \cdot T_2, t_i = t'_i$. So $(s_{1,i}, t_i)$, and $(s_{2,i}, t'_i)$ are both in $\mathbb{I}$. As $l_i$ is a bijection, by IH, $s_{1,i} = s_{2,i}$, so $s_i = s_i$.

Similarly for $t_1, t_2 \in \mathcal{L}(T_1 \cdot T_2)$.

Let $s \in \mathcal{L}(S_1 \cdot S_2), s = s_1 \cdot s_2$, where $s_1 \in \mathcal{L}(S_1)$. By IH, as $l_i$ is a bijection, there exists $t_i \in \mathcal{L}(DT)$ such that $(s_i, t_i) \in \mathbb{I}$. So $(s_1, t_1 \cdot t_2) \in \mathbb{concat}(l_1, l_2)$.

Similarly for $t \in \mathcal{L}(T_1 \cdot T_2)$.

Case 5 (Swap).

\[
\begin{align*}
    l_1 : S_1 &\iff T_1 \\
    l_2 : S_2 &\iff T_2 \\
    S_1 \cdot S_2 &\iff T_1 \cdot T_2 \\
    \text{swap}(l_1, l_2) : S_1 S_2 &\iff T_2 T_1
\end{align*}
\]

Let $s_1, s_2 \in \mathcal{L}(S_1 \cdot S_2)$, and $(s_1, t) \in \mathbb{swap}(l_1, l_2)$, and $(s_2, t) \in \mathbb{swap}(l_1, l_2)$.

So $s_1 = s_{1,1} \cdot s_{1,2}, t = t_2 t_1$, and $(s_{1,i}, t_i) \in \mathbb{I}$. So $s_2 = s_{2,1} \cdot s_{2,2}, t = t'_2 \cdot t'_1$, and $(s_{2,i}, t'_i) \in \mathbb{I}$. By $T_2 \cdot T_1, t_i = t'_i$. So $(s_{1,i}, t_i)$, and $(s_{2,i}, t'_i)$ are both in $\mathbb{I}$. As $l_i$ is a bijection, by IH, $s_{1,i} = s_{2,i}$, so $s_i = s_i$.

Similarly for $t_1, t_2 \in \mathcal{L}(T_1 \cdot T_2)$.

Let $s \in \mathcal{L}(S_1 \cdot S_2), s = s_1 \cdot s_2$, where $s_1 \in \mathcal{L}(S_1)$. By IH, as $l_i$ is a bijection, there exists $t_i \in \mathcal{L}(T_1)$ such that $(s_i, t_i) \in \mathbb{I}$. So $(s_1, t_2 \cdot t_1) \in \mathbb{swap}(l_1, l_2)$, and $t_2 \cdot t_1 \in \mathcal{L}(S_2 \cdot S_1)$.

Similarly for $t \in \mathcal{L}(T_2 \cdot T_1)$.

Case 6 (Or).

\[
\begin{align*}
    l_1 : S_1 &\iff T_1 \\
    l_2 : S_2 &\iff T_2 \\
    \mathcal{L}(S_1) \cap \mathcal{L}(S_2) &\neq \emptyset \\
    \mathcal{L}(T_1) \cap \mathcal{L}(T_2) &\neq \emptyset \\
    \text{or}(l_1, l_2) : S_1 \mid S_2 &\iff T_1 \mid T_2
\end{align*}
\]

Let $s_1, s_2 \in \mathcal{L}(S_1 \mid S_2)$, and $(s_1, t) \in \mathbb{or}(l_1, l_2)$, and $(s_2, t) \in \mathbb{or}(l_1, l_2)$.

So $(s_1, t) \in \mathbb{I}_1$ or $(s_1, t) \in \mathbb{I}_2$. So $(s_2, t) \in \mathbb{I}_1$ or $(s_2, t) \in \mathbb{I}_2$.

As $\mathcal{L}(T_1) \cap \mathcal{L}(T_2) = \emptyset$, $t_i$ is in only one of $\mathcal{L}(T_1)$ or $\mathcal{L}(T_2)$.

Let $t_i \in \mathcal{L}(T_i)$. This means that $(s_i, t) \in \mathbb{I}_1$ and $(s_i, t) \in \mathbb{I}_2$. As $l_i$ is a bijection, by IH, $s_1 = s_2$.

Similarly if $t \in \mathcal{L}(T_2)$.

Similarly for $t_1, t_2 \in \mathcal{L}(T_2 \cdot T_1)$.

Let $s \in \mathcal{L}(S_1 \mid S_2)$. So $s$ is either in $\mathcal{L}(S_1)$ or $\mathcal{L}(S_2)$. If $s \in \mathcal{L}(S_1)$, then as $l_1$ is a bijection between $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$, there exists $t \in \mathcal{L}(\text{RegexpAlt}_1)$ such that $(s, t) \in \mathbb{I}_1$, so $(s, t) \in \mathbb{or}(l_1, l_2)$, and $t \in \mathcal{L}(T_1 \mid T_2)$. Similarly if $s \in \mathcal{L}(S_2)$.

Similarly for $t \in \mathcal{L}(T_2 \cdot T_1)$. 

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
**Case 7 (Compose).**

\[
l_1 : S_1 ⇔ S_2 \quad l_2 : S_2 ⇔ S_3
\]

\[
l_1 \circ l_2 : S_1 ⇔ S_3
\]

By IH, \([l_1] \) is a bijection between \(L(S_1)\) and \(L(S_2)\). By IH, \([l_2] \) is a bijection between \(L(S_2)\) and \(L(S_3)\). From math, this means that their composition is also a bijection.

□

**Lemma 13.**

- If \(dl : DS ⇔ DT\), then \([dl] \) is a bijection between \(L(DS)\) and \(L(DT)\).
- If \(sql : SQ ⇔ TQ\), then \([sql] \) is a bijection between \(L(SQ)\) and \(L(TQ)\).
- If \(al : A ⇔ B\), then \([al] \) is a bijection between \(L(A)\) and \(L(B)\).

**Proof.** By mutual induction on the typing derivations of DNF lenses, sequence lenses, and atom lenses.

**Case 1 (Iterate).**

\[
dl : DS ⇔ DT \quad DS^J \quad DT^J
\]

iterate\(dl) : DS^J ⇔ DT^J

Let \(s_1, s_2 ∈ L(DS^J), and (s_1, t) ∈ \([iterate(dl)]\), and (s_2, t) ∈ \([iterate(dl)]\).

So \(s_1 = s_1 \cdot \ldots \cdot s_{m,1} \cdot t = t_1 \cdot \ldots \cdot t_m\), and \((s_1, t_1) ∈ \([dl]\). So \(s_2 = s_2 \cdot \ldots \cdot s_{m,2} \cdot t = t_1' \cdot \ldots \cdot t_m'\), and \((s_2, t_1') ∈ \([dl]\). By \(DT^J\), this means that \(m = n\), and \(t_1 = t_1'\). So \((s_1, t_1)\) and \((s_2, t_1)\) are both in \([dl]\). As \(dl\) is a bijection, by IH, \(s_1, t_1 = s_2, t_1\), so \(s_1 = s_2\).

Similarly for \(t_1, t_2\) in \(L(DT^J)\).

Let \(s ∈ L(DS^J), s = s_1 \cdot \ldots \cdot s_n, where s_i ∈ L(DS). By IH, as \(dl\) is a bijection, there exists \(t_1 ∈ L(DT)\) such that \((s_1, t_1) ∈ \([dl]\). So \((s, t_1 \cdot \ldots \cdot t_n) ∈ \([iterate(dl)]\), and \(t_1 \cdot \ldots \cdot t_n ∈ L(DT)\).

Similarly for \(t \in L(DT^J)\).

**Case 2 (SequenceLens).**

\[
\ldots \quad al_n : A_n ⇔ B_n \quad σ ∈ S_n \quad \ldots \quad \langle s_0, A_0 \cdot \ldots \cdot A_n, s_n \rangle \quad \langle t_0, B_{σ(1)} \cdot \ldots \cdot B_{σ(n)}, t_n \rangle
\]

\[
(\langle s_0, t_0, \ldots, s_n, t_n \rangle, σ) \quad \langle s_0, A_0 \cdot \ldots \cdot A_n, s_n \rangle \quad \langle t_0, B_{σ(1)} \cdot \ldots \cdot B_{σ(n)}, t_n \rangle
\]

Let \(s_1, s_2 ∈ L(\langle s_0, A_1 \cdot \ldots \cdot A_n, s_n \rangle)\), and \((s_1, t) ∈ \([\langle s_0, t_0, \ldots, s_n, t_n \rangle, σ]\), and \((s_2, t) ∈ \([\langle s_0, t_0, \ldots, s_n, t_n \rangle, σ]\).

So \(s_1 = s_0 \cdot s_1 \cdot \ldots \cdot s_{n,1} \cdot s_{n,1} \cdot t = t_0 \cdot t_{1,σ(i)} \cdot \ldots \cdot t_{1,σ(n)} \cdot t_n\), and \((s_1, t_1) ∈ \([al]\). So \(s_2 = s_0 \cdot s_2 \cdot \ldots \cdot s_{n,2} \cdot s_{n,2} \cdot t = t_0 \cdot t_{2,σ(1)} \cdot \ldots \cdot t_{2,σ(n)} \cdot t_n\), and \((s_2, t_1) ∈ \([dl]\). By \(\langle t_0, B_{σ(1)} \cdot \ldots \cdot B_{σ(n)}, t_n \rangle\), this means that \(t_1 = t_2\). So \((s_1, t_1)\) and \((s_2, t_1)\) are both in \([dl]\). As \(al\) is a bijection, by IH, \(s_1, t_1 = s_2, t_1\), so \(s_1 = s_2\).

Similarly for \(t_1, t_2\) in \(L(DT^J)\).

**Case 3 (DNFLens).**

\[
sql_1 : SQ_1 ⇔ TQ_1 \quad \ldots \quad sql_n : SQ_n ⇔ TQ_n
\]

\[
σ ∈ S_n \quad i \neq j ⇒ L(SQ_i) \cap L(SQ_j) = \emptyset \quad i \neq j ⇒ L(TQ_i) \cap L(TQ_j) = \emptyset
\]

\[
\langle sql_1 | \ldots | sql_n, σ \rangle \quad \langle SQ_1 | \ldots | SQ_n \rangle \quad \langle TQ_{σ(1)} | \ldots | TQ_{σ(n)} \rangle
\]
Let \( s_1, s_2 \in \mathcal{L}(\langle SQ_1 \mid \ldots \mid SQ_n \rangle) \), and \( (s_1, t) \in \mathcal{L}(\langle \langle sq_1 \mid \ldots \mid sq_n \rangle, \sigma \rangle) \), and \( (s_2, t) \in \mathcal{L}(\langle \langle sq_1 \mid \ldots \mid sq_n \rangle, \sigma \rangle) \).

So \( \exists i, s_1 \in \mathcal{L}(\langle SQ_i \rangle) \), \( t \in \mathcal{L}(\langle TQ_{\sigma(j)} \rangle) \), and \( (s_1, t) \in \mathcal{L}(\langle sq_1 \rangle) \). So \( \exists i, s_1 \in \mathcal{L}(\langle SQ_i \rangle) \), \( t \in \mathcal{L}(\langle TQ_{\sigma(j)} \rangle) \), and \( (s_1, t) \in \mathcal{L}(\langle sq_1 \rangle) \). As \( i \neq j \Rightarrow \mathcal{L}(\langle TQ_i \rangle) \cap \mathcal{L}(\langle TQ_j \rangle) = \emptyset \), \( i = j \). So \( s_1 \in \mathcal{L}(\langle SQ_i \rangle) \), \( s_2 \in \mathcal{L}(\langle SQ_i \rangle) \). As by IH, \( sq_i \) is a bijection, \( s_1 = s_2 \).

Similarly for \( t_1, t_2 \in \mathcal{L}(\langle DT^* \rangle) \).

Let \( s \in \mathcal{L}(\langle SQ_1 \mid \ldots \mid SQ_n \rangle) \). So there exists an \( i \) such that \( s \in \mathcal{L}(\langle SQ_i \rangle) \). By IH, as \( SQ_i \) is a bijection, there exists \( t \in \mathcal{L}(\langle TQ_i \rangle) \) such that \( (s, t) \in \mathcal{L}(\langle sq_1 \rangle) \). So \( (s, t) \in \mathcal{L}(\langle \langle sq_1 \mid \ldots \mid sq_n \rangle, \sigma \rangle) \), and \( t \in \mathcal{L}(\langle \langle TQ_{\sigma(i)} \rangle \mid \ldots \mid TQ_{\sigma(n)} \rangle) \).

Similarly for \( t \in \mathcal{L}(\langle DT^* \rangle) \).

\[\Box\]

**Lemma 14** (Closure of Rewriteless Regular Expressions under Composition).

1. If there are two atom lenses \( al_1 : A_1 \Rightarrow A_2 \) and \( al_2 : A_2 \Rightarrow A_3 \), then there exists an atom lens \( al : A_1 \Rightarrow A_3 \), such that \( \mathcal{L}(al) = \{ (s_1, s_3) \mid \exists s_2 (s_1, s_2) \in \mathcal{L}(al_1) \land (s_2, s_3) \in \mathcal{L}(al_2) \} \).
2. If there are two sequence lenses \( sq_1 : SQ_1 \Rightarrow SQ_2 \) and \( sq_2 : SQ_2 \Rightarrow SQ_3 \), then there exists an sequence lens \( sq : SQ_1 \Rightarrow SQ_3 \), such that \( \mathcal{L}(sq) = \{ (s_2, s_3) \mid \exists s_1 (s_1, s_2) \in \mathcal{L}(sq_1) \land (s_2, s_3) \in \mathcal{L}(sq_2) \} \).
3. If there are two DNF lenses \( dl_1 : DS_1 \Rightarrow DS_2 \) and \( dl_2 : DS_2 \Rightarrow DS_3 \), then there exists a DNF lens \( dl : DS_1 \Rightarrow DS_3 \), such that \( \mathcal{L}(dl) = \{ (s_1, s_3) \mid \exists s_2 (s_1, s_2) \in \mathcal{L}(dl_1) \land (s_2, s_3) \in \mathcal{L}(dl_2) \} \).

**Proof.** By mutual induction

**Case 1 (Atom Lenses).** Let \( DS_1^*, DS_2^*, DS_3^* \) be three atoms, and \( iterate(dl_1) : DS_1^* \Rightarrow DS_2^* \) with \( iterate(dl_2) : DS_2^* \Rightarrow DS_3^* \) lenses between them. By induction assumption, there exists the typing of a lens

\[
dl : DS_1 \Rightarrow DS_3
\]

such that \( \mathcal{L}(dl_1) = \{ (s_1, s_3) \mid \exists s_2 (s_1, s_2) \in \mathcal{L}(dl_1) \land (s_2, s_3) \in \mathcal{L}(dl_2) \} \).

\( iterate(dl_1) \) and \( iterate(dl_2) \) came from \textsc{atom lens}, so \( DS_1^*, DS_2^*, \) and \( DS_3^* \).

Consider the lens

\[
dl : DS_1 \Rightarrow DS_3 \quad DS_1^{\uparrow} \quad DS_2^{\uparrow} \quad DS_3^{\uparrow}
\]

\( iterate(dl) : DS_1^* \Rightarrow DS_3^* \)

This lens has the semantics

\[
iterate(dl) = \{ (s_{1,1} \cdot \ldots \cdot s_{1,n}, s_{3,1} \cdot \ldots \cdot s_{3,n}) \mid (s_{1,1}, s_{3,1}) \in \mathcal{L}(dl_1) \land (s_{2,1}, s_{3,1}) \in \mathcal{L}(dl_2) \}
\]

\[
= \{ (s_{1,1} \cdot \ldots \cdot s_{1,n}, s_{3,1} \cdot \ldots \cdot s_{3,n}) \mid \exists s_2(s_{1,1}, s_{2,1}) \in \mathcal{L}(dl_1) \land (s_{2,1}, s_{3,1}) \in \mathcal{L}(dl_2) \}
\]

\[
= \{ (s_{1,1} \cdot \ldots \cdot s_{1,n}, s_{3,1} \cdot \ldots \cdot s_{3,n}) \mid \exists s_2(s_1, s_2) \in \mathcal{L}(iterate(dl_1)) \land (s_2, s_3) \in \mathcal{L}(iterate(dl_2)) \}
\]

**Case 2 (Sequence Lenses).** Let \( [s_{1,0} : A_{1,1} \cdot \ldots \cdot A_{1,n} \cdot s_{1,n}] \) and \( [s_{2,0} : A_{2,\sigma_1(1)} \cdot \ldots \cdot A_{2,\sigma_1(n)} \cdot s_{2,n}] \) and \( [s_{3,0} : A_{3,\sigma_2(1)} \cdot \ldots \cdot A_{3,\sigma_2(n)} \cdot s_{3,n}] \) be sequences, with \( (\langle s_{1,0}, s_{2,0} \rangle \cdot al_{1,1} \cdot \ldots \cdot al_{1,n} \cdot (s_{1,n}, s_{2,n})) \), \( \sigma_1 \) and \( (\langle s_{2,0}, s_{3,0} \rangle \cdot al_{2,1} \cdot \ldots \cdot al_{2,n} \cdot (s_{2,n}, s_{3,n})) \), \( \sigma_2 \) be lenses between them. By induction assumption, there is a typing of lenses
Lemma 15 (Expressibility of Identity on Strongly Unambiguous DNF Regex, Clauses, and Atoms).
(1) If $DS$ is a strongly unambiguous DNF Regular expression, then there exists a DNF lens $dl : DS \iff DS$, such that $\[dl\] = \{(s, s) \mid s \in \mathcal{L}(DS)\}$, where $dl$ typing includes no rewrite rules.

(2) If $SQ$ is a strongly unambiguous sequence, then there exists a sequence lens $sql : SQ \iff SQ$, such that $\[sql\] = \{(s, s) \mid s \in \mathcal{L}(DS)\}$, where $sql$ typing includes no rewrite rules.

(3) If $A$ is a strongly unambiguous atom, then there exists an atom lens $al : A \iff A$, such that $\[al\] = \{(s, s) \mid s \in \mathcal{L}(DS)\}$, where $al$ typing includes no rewrite rules.

**Proof.** By mutual induction on the structure of the DNF regular expression, atom, and clause.

**Case 1 (Star).** Let $A = DS^\ast$. As $A$ is strongly unambiguous, $DS$ is strongly unambiguous, and $DS^\ast$.

By IH, there exists $dl : DS \iff DS$ such that $\[dl\] = \{(s, s) \mid s \in \mathcal{L}(DS)\}$. Consider the atom lens

$$iterate(dl) : DS \iff DS^\ast$$

with typing as desired.

$$\[iterate(dl)\] = \{(s_1 \ldots s_n, t_1 \ldots t_n) \mid (s_i, t_i) \in \[dl\]\}.$$ So through semantics of $dl$, $\[iterate(dl)\] = \{(s_1 \ldots s_n, s_1 \ldots s_n) \mid s \in \mathcal{L}(DS)\}$, so through the definition of $DS^\ast$, $\[iterate(dl)\] = \{(s, s) \mid s \in \mathcal{L}(DS^\ast)\}$

**Case 2 (MultiConcat).** Let $SQ = [s_0 \cdot A_1 \ldots A_n \cdot s_n]$. As $SQ$ is strongly unambiguous, for all $i$, $A_i$ is strongly unambiguous, and $\{s_0; A_1; \ldots; A_n; s_n\}$

By IH, for all $i$, there exists $al_i : A_i \iff A_i$ such that $\[al_i\] = \{(s, s) \mid s \in \mathcal{L}(A_i)\}$.

Consider the typing

$$al_1 : A_1 \iff A_1$$

\[
\cdots \quad al_n : A_n \iff A_n \quad id \in S_n \quad \vdash (s_0; A_1; \ldots; A_n; s_n) \quad \vdash (s_0; A_1; \ldots; A_n; s_n)
\]

as desired.

\[
\[\{((s_0, s_0) : al_1 \ldots al_n (s_n, s_n), id)\} = \{(s_0, t_1 \ldots t_n, s_0, t_1' \ldots t_n', s_0, t_1' \ldots t_n' \mid (t_i, t_i') \in \[al_i\]\} \}
\]

So, through the definition of $al_i$, $\[\{((s_0, s_0) : al_1 \ldots al_n (s_n, s_n), id)\} = \{(s_0, t_1 \ldots t_n, s_0, t_1 \ldots t_n', s_0, t_1' \ldots t_n' \mid t_i \in \mathcal{L}(A_i)\}$. So, through the definition of $\{s_0 \cdot A_1 \ldots A_n \cdot s_n\}$, $\\[\[\{((s_0, s_0) : al_1 \ldots al_n (s_n, s_n), id)\} = \{(s, s) \mid s \in \mathcal{L}((s_0 \cdot A_1 \ldots A_n \cdot s_n)\}$,

as desired.

**Case 3 (MultiOr).** Let $DS = \langle SQ_1 \mid \ldots \mid SQ_n \rangle$. As $DS$ is strongly unambiguous, for all $i$, $SQ_i$ is strongly unambiguous, and $i \neq j \Rightarrow \mathcal{L}(SQ_i) \cap \mathcal{L}(SQ_j) = \emptyset$.

By IH, for all $i$, there exists $al_i : A_i \iff A_i$ such that $\[al_i\] = \{(s, s) \mid s \in \mathcal{L}(A_i)\}$.

Consider the typing

$$sql_i : SQ_i \iff SQ_i$$

\[
\cdots \quad sql_1 : SQ_1 \iff SQ_1 \quad id \in S_n \quad \vdash (s_1, \ldots, s_n) \iff (s_1, \ldots, s_n)
\]

as desired.

\[
\[\{\langle sql_1 \mid \ldots \mid sql_n, id\} = \{(s, t) \mid \exists i. (s, t) \in \[sql_i\]\} \}
\]

So, through the definition of $sql_i$, $\[\{\langle sql_1 \mid \ldots \mid sql_n, id\} = \{(s, s) \mid \exists i. s \in \mathcal{L}(SQ_i)\}$. So, through the definition of $\langle SQ_1 \mid \ldots \mid SQ_n \rangle$, $\[\{\langle sql_1 \mid \ldots \mid sql_n, id\} = \{(s, s) \mid s \in \mathcal{L}(\langle SQ_1 \mid \ldots \mid SQ_n \rangle)\}$, as desired.

\[\square\]

**Definition 12 (Strong Unambiguity on DNF Regular Expressions).**
Synthesizing Bijective Lenses

1:43

⟨SQ₁ | . . . | SQₙ⟩ is strongly unambiguous if SQᵢ is strongly unambiguous for all i, and i ≠ j ⇒ L(SQᵢ) ∩ L(SQⱼ) = ∅.

[s₀ · A₁ · . . . · Aₙ · SQₙ] is strongly unambiguous if Aᵢ is strongly unambiguous, and i=(s₀; A₁; . . . ; Aₙ; sₙ).

DS* is strongly unambiguous if DS is strongly unambiguous, and DS⁺.

Lemma 16 (Strong Unambiguity in Lens Types). If l : S ⇔ T, then S is strongly unambiguous, and T is strongly unambiguous.

Proof. By induction on the typing derivation of l

Case 1 (Const).

<table>
<thead>
<tr>
<th></th>
<th>s₁ ∈ Σ⁺</th>
<th>s₂ ∈ Σ⁺</th>
</tr>
</thead>
<tbody>
<tr>
<td>const(s₁, s₂) : s₁ ⇔ s₂</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Base strings are strongly unambiguous.

Case 2 (Concat).

<table>
<thead>
<tr>
<th></th>
<th>l₁ : S₁ ⇔ T₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>l₂ : S₂ ⇔ T₂</td>
<td></td>
</tr>
<tr>
<td>S₁ ·¹ S₂</td>
<td>T₁ ·¹ T₂</td>
</tr>
</tbody>
</table>

concat(l₁, l₂) : S₁S₂ ⇔ T₁T₂

So by IH, S₁, S₂, T₁, and T₂ are all strongly unambiguous.

As S₁ ·¹ S₂, S₁ · S₂ is strongly unambiguous.

As T₁ ·¹ T₂, T₁ · T₂ is strongly unambiguous.

Case 3 (Iterate).

<table>
<thead>
<tr>
<th></th>
<th>l₁ : S ⇔ T</th>
</tr>
</thead>
<tbody>
<tr>
<td>l₂ : S ⇔ T</td>
<td></td>
</tr>
<tr>
<td>S₁*</td>
<td>T₁*</td>
</tr>
</tbody>
</table>

iterate(l₁) : S* ⇔ T*

So, by IH, S and T are both strongly unambiguous.

As S*, S⁺ is strongly unambiguous.

As T⁺, T* is strongly unambiguous.

Case 4 (Swap).

<table>
<thead>
<tr>
<th></th>
<th>l₁ : S₁ ⇔ T₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>l₂ : S₂ ⇔ T₂</td>
<td></td>
</tr>
<tr>
<td>S₁ ·¹ S₂</td>
<td>T₂ ·¹ T₁</td>
</tr>
</tbody>
</table>

swap(l₁, l₂) : S₁S₂ ⇔ T₂T₁

So by IH, S₁, S₂, T₁, and T₂ are all strongly unambiguous.

As S₁ ·¹ S₂, S₁ · S₂ is strongly unambiguous.

As T₂ ·¹ T₁, T₂ · T₁ is strongly unambiguous.

Case 5 (Or).

<table>
<thead>
<tr>
<th></th>
<th>l₁ : S₁ ⇔ T₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>l₂ : S₂ ⇔ T₂</td>
<td></td>
</tr>
<tr>
<td>L(S₁) ∩ L(S₂) = ∅</td>
<td>L(T₁) ∩ L(T₂) = ∅</td>
</tr>
</tbody>
</table>

or(l₁, l₂) : S₁ | S₂ ⇔ T₁ | T₂

So by IH, S₁, S₂, T₁, and T₂ are all strongly unambiguous.

As L(S₁) ∩ L(S₂) = ∅, S₁ | S₂ is strongly unambiguous.

As L(T₂) ∩ L(T₁) = ∅, T₂ | T₁ is strongly unambiguous.
1:44 Anders Miltner, Kathleen Fisher, Benjamin Pierce, David Walker, and Steve Zdancewic

Case 6 (Compose).

\[ l_1 : S_1 \leftrightarrow S_2 \quad l_2 : S_2 \leftrightarrow S_3 \]

\[ l_1 ; l_2 : S_1 \leftrightarrow S_3 \]

By IH, \( S_1 \) is strongly unambiguous, and \( S_3 \) is strongly unambiguous.

Case 7 (Identity).

\[ S \text{ is strongly unambiguous} \]

\[ id_S : S \leftrightarrow S \]

By assumption, \( S \) is strongly unambiguous.

\[ \square \]

**Lemma 17** (Strong Unambiguity in Rewriteless DNF Lens Types).

- If \( dl : DS \leftrightarrow DT \), then \( DS \) is strongly unambiguous, and \( DT \) is strongly unambiguous.
- If \( sql : SQ \leftrightarrow TQ \), then \( SQ \) is strongly unambiguous, and \( TQ \) is strongly unambiguous.
- If \( al : A \leftrightarrow B \), then \( A \) is strongly unambiguous, and \( B \) is strongly unambiguous.

**Proof.** By mutual induction on the typing derivation of \( dl \), \( sql \), and \( al \).

Case 1 (AtomLens).

\[ dl : DS \leftrightarrow DT \quad DS^{\ast t} \quad DT^{\ast t} \]

\[ \text{iterate}(dl) : DS^{\ast} \leftrightarrow DT^{\ast} \]

By IH, \( DS \) and \( DT \) are strongly unambiguous.

As \( DS^{\ast t} \), \( DS^{\ast} \) is strongly unambiguous.

As \( DT^{\ast t} \), \( DT^{\ast} \) is strongly unambiguous.

Case 2 (SequenceLens).

\[ \ldots \quad al_n : A_n \leftrightarrow B_n \quad \sigma \in S_n \quad al_1 : A_1 \leftrightarrow B_1 \quad [s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n] \quad [l_0 \cdot B_{\sigma(1)} \ldots \cdot B_{\sigma(n)} \cdot t_n] \]

\[ \{[(s_0, t_0) \cdot al_1 \ldots \cdot al_n \cdot (s_n, t_n)], \sigma \} : [s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n] \leftrightarrow [l_0 \cdot B_{\sigma(1)} \ldots \cdot B_{\sigma(n)} \cdot t_n] \]

By IH, \( A_1 \) and \( B_1 \) are strongly unambiguous for all \( i \).

As \( [s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n] \), we have \( [s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n] \) is strongly unambiguous.

As \( [l_0 \cdot B_{\sigma(1)} \ldots \cdot B_{\sigma(n)} \cdot t_n] \), we have \( [l_0 \cdot B_{\sigma(1)} \ldots \cdot B_{\sigma(n)} \cdot t_n] \) is strongly unambiguous.

Case 3 (DNFLens).

\[ \sigma \in S_n \quad \quad \quad sql_1 : SQ_1 \leftrightarrow TQ_1 \quad \ldots \quad sql_n : SQ_n \leftrightarrow TQ_n \]

\[ (sql_1 \mid \ldots \mid sql_n, \sigma) : \quad \langle SQ_1 \mid \ldots \mid SQ_n \rangle \leftrightarrow \langle TQ_{\sigma(1)} \mid \ldots \mid TQ_{\sigma(n)} \rangle \]

By IH, \( SQ_j \) and \( TQ_j \) are strongly unambiguous for all \( i \).

As \( i \neq j \Rightarrow \mathcal{L}(SQ_j) \cap \mathcal{L}(SQ_j) = \emptyset \), we have \( \langle SQ_1 \mid \ldots \mid SQ_n \rangle \) is strongly unambiguous.

As \( i \neq j \Rightarrow \mathcal{L}(TQ_j) \cap \mathcal{L}(TQ_j) = \emptyset \), we have \( \langle TQ_{\sigma(1)} \mid \ldots \mid TQ_{\sigma(n)} \rangle \) is strongly unambiguous.

\[ \square \]

**Lemma 18** (Closure of Rewriteless DNF Lenses Under Inversion).

1. If \( dl : DS \leftrightarrow DT \), then there exists a dnf lens \( dl^{-1} : DT \leftrightarrow DS \) such that \( ||DS^{-1}|| = \{(t, s) \mid (s, t) \in ||dl||\} \)

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
2. If \( \text{sql} \vdash SQ \iff TQ \iff SQ \) such that \( \|\text{sql}^{-1}\| = \{(t, s) \mid (s, t) \in \|\text{sql}\|\} \).

3. If \( al : A \iff B \), then there exists an atom lens \( al^{-1} : B \iff A \) such that \( \|al^{-1}\| = \{(t, s) \mid (s, t) \in \|al\|\} \).

**Proof.** By mutual induction on the typing derivation of \( dl, \text{sql}, \) and \( al. \)

**Case 1 (DNF Lens).**

\[
\begin{align*}
\text{sql}_1 \vdash SQ_1 & \iff TQ_1, \quad \cdots \quad \text{sql}_n \vdash SQ_n \iff TQ_n \\
\sigma \in S_n \quad i \neq j & \Rightarrow \mathcal{L}(SQ_i) \cap \mathcal{L}(SQ_j) = \emptyset \\
& \sigma \in S_n \quad i \neq j \Rightarrow \mathcal{L}(TQ_i) \cap \mathcal{L}(TQ_j) = \emptyset \\
\end{align*}
\]

\[
\frac{}{(\langle \text{sql}_1 \mid \cdots \mid \text{sql}_n \rangle, \sigma) \vdash \langle SQ_1 \mid \cdots \mid SQ_n \rangle \iff \langle TQ_{\sigma(1)} \mid \cdots \mid TQ_{\sigma(n)} \rangle}
\]

By IH, there exists \( \text{sql}_i^{-1} \vdash TQ_i \iff SQ_i \) where \( \|\text{sql}_i^{-1}\| = \{(t, s) \mid (s, t) \in \|\text{sql}_i\|\} \).

Consider the typing

\[
\begin{align*}
\text{sql}_{i(1)}^{-1} \vdash TQ_{\sigma(1)} & \iff SQ_{\sigma(1)}, \quad \cdots \quad \text{sql}_{i(n)}^{-1} \vdash TQ_{\sigma(n)} \iff SQ_{\sigma(n)} \\
\sigma^{-1} \in S_n \quad i \neq j & \Rightarrow \mathcal{L}(TQ_i) \cap \mathcal{L}(TQ_j) = \emptyset \\
& \sigma^{-1} \in S_n \quad i \neq j \Rightarrow \mathcal{L}(SQ_i) \cap \mathcal{L}(SQ_j) = \emptyset \\
\end{align*}
\]

\[
\frac{}{(\langle \text{sql}_{i(1)}^{-1} \mid \cdots \mid \text{sql}_{i(n)}^{-1} \rangle, \sigma^{-1}) \vdash \langle TQ_{\sigma^{-1}(1)} \mid \cdots \mid TQ_{\sigma^{-1}(n)} \rangle \iff \langle SQ_{\sigma^{-1}(1)} \mid \cdots \mid SQ_{\sigma^{-1}(n)} \rangle}
\]

So \( (\langle \text{sql}_{i(1)}^{-1} \mid \cdots \mid \text{sql}_{i(n)}^{-1} \rangle, \sigma^{-1}) \vdash \langle TQ_{\sigma(1)} \mid \cdots \mid TQ_{\sigma(n)} \rangle \iff \langle SQ_1 \mid \cdots \mid SQ_n \rangle \), or in other words \((\langle \text{sql}_{i(1)}^{-1} \mid \cdots \mid \text{sql}_{i(n)}^{-1} \rangle, \sigma^{-1}) \vdash DT \iff DS\), as desired.

\[
\|\langle \text{sql}_{i(1)}^{-1} \mid \cdots \mid \text{sql}_{i(n)}^{-1} \rangle, \sigma^{-1} \| = \{(s, t) \mid \exists i. (s, t) \in \|\text{sql}_{i(1)}^{-1}\|\} = \{(t, s) \mid \exists i. (s, t) \in \|\text{sql}_i\|\} = \{(t, s) \mid (s, t) \in \|dl\|\}
\]

**Case 2 (Sequence Lens).**

\[
\begin{align*}
\text{al}_1 \vdash A_1 & \iff B_1 \\
\cdots \quad \text{al}_n \vdash A_n & \iff B_n \\
\sigma \in S_n \quad [t_0 \cdot A_1 \cdots \cdot A_n \cdot s_n]^{-1} & [t_0 \cdot B_1 \cdots \cdot B_n \cdot t_n] \\
\end{align*}
\]

\[
\frac{}{([\langle (s_0, t_0) \cdot A_1 \cdots \cdot A_n \cdot (s_n, t_n) \rangle, \sigma]) \vdash [s_0 \cdot A_1 \cdots \cdot A_n \cdot s_n] \iff [t_0 \cdot B_1 \cdots \cdot B_n \cdot t_n]}
\]

By IH, there exists \( al_i^{-1} \vdash B_i \iff A_i \) where \( \|al_i^{-1}\| = \{(t, s) \mid (s, t) \in \|al_i\|\} \).

Consider the typing

\[
\begin{align*}
\text{al}_{i(1)}^{-1} \vdash B_{\sigma(1)} & \iff A_{\sigma(1)}, \quad \cdots \quad \text{al}_{i(n)}^{-1} \vdash B_{\sigma(n)} \iff A_{\sigma(n)} \\
\sigma^{-1} \in S_n \quad [t_0 \cdot B_1 \cdots \cdot B_n \cdot t_n]^{-1} & [s_0 \cdot A_1 \cdots \cdot A_n \cdot s_n] \\
\end{align*}
\]

\[
\frac{}{([\langle (t_0, s_0) \cdot A_1 \cdots \cdot A_n \cdot (t_n, s_n) \rangle, \sigma]) \vdash [s_0 \cdot A_1 \cdots \cdot A_n \cdot s_n] \iff [t_0 \cdot B_1 \cdots \cdot B_n \cdot t_n]}
\]

So \((\langle (t_0, s_0) \cdot \text{al}_{i(1)}^{-1} \cdots \cdot \text{al}_{i(n)}^{-1} \rangle, \sigma^{-1}) \vdash [t_0 \cdot TQ_{\sigma(1)} \cdots \cdot TQ_{\sigma(n)} \cdot t_n] \iff [s_0 \cdot SQ_1 \cdots \cdot SQ_n]\), or in other words \((\langle (t_0, s_0) \cdot \text{al}_{i(1)}^{-1} \cdots \cdot \text{al}_{i(n)}^{-1} \rangle, \sigma^{-1}) \vdash DT \iff SQ\), as desired.

\[
\|\langle \text{al}_{i(1)}^{-1} \mid \cdots \mid \text{al}_{i(n)}^{-1} \rangle, \sigma^{-1} \| = \{(t_0^i, s_0^i) \cdots \cdot t_{\sigma(n)}^i \cdot s_n^i \mid \forall i. (s_i^i, t_i^i) \in \|\text{al}_i\|\} = \{(t, s) \mid (s, t) \in \|\text{sql}\|\}, \text{as desired.}
\]

**Case 3 (Atom Lens).**

\[
\frac{dl \vdash DS \iff DTDS^\ast}{\text{iterate}(dl) \vdash DS^\ast \iff DT^\ast}
\]

By IH, there exists \( dl^{-1} \vdash DT \iff DS \) where \( \|dl^{-1}\| = \{(t, s) \mid (s, t) \in \|dl\|\} \).

Consider the typing

\[
\frac{dl^{-1} \vdash DT \iff DSDT^\ast \iff DS^\ast}{\text{iterate}(dl^{-1}) \vdash DT^\ast \iff DS^\ast}
\]
So iterate($dl^{-1}$) $\implies$ DT$^*$ $\iff$ DS$^*$, or in other words (iterate($dl^{-1}$)) $\implies$ AtomAlt $\iff$ Atom as desired.

$\|	ext{iterate}(dl^{-1})\| = \{(s_0 \ldots s_n, t_0 \ldots t_m) \mid \forall i. (s_i, t_i) \in \|dl^{-1}\|\} = \{(t_0 \ldots t_n, s_0 \ldots s_n) \mid \forall i. (s_i, t_i) \in \|dl\|\} = \{(t, s) \mid (t, s) \in \|sql\|\},$ as desired.

\[ \square \]

### B.4 DNF Regular Expression and Regular Expression Proofs

This subsection is in the aims of proving that $\downarrow$ preserves language, and $\uparrow$ is its left inverse. We use this throughout. These functions are built on the DNF regular expression operators of $\odot$, $\oplus$, and $\downarrow$. In this section, we prove that these operators do as we expect them to, and use these lemmas throughout the paper.

**Lemma 19** (Equivalence of $\odot_{SQ}$ and $\cdot$). If $L(S) = L(SQ)$, and $L(T) = L(TQ)$, then $L(S \cdot T) = L(SQ \odot_{SO} TQ)$.

**Proof.** Let $SQ = [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n]$, and let

\[
L(SQ \odot_{SO} TQ) = L(s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n \cdot t_0 \cdot B_1 \cdot \ldots \cdot B_m \cdot t_m)
\]

\[= \{s_0 \cdot s'_1 \cdot \ldots \cdot s'_n \cdot t_0 \cdot t'_1 \cdot \ldots \cdot t'_m \cdot t_m \mid s'_i \in L(A_i) \land t'_i \in L(B_i)\} \]

\[= \{s \cdot t \mid s \in L(SQ) \land t \in L(TQ)\} = \{s \cdot t \mid s \in L(S) \land t \in L(T)\} = L(S \cdot T) \]

**Lemma 20** (Equivalence of $\odot$ and $\cdot$). If $L(S) = L(DS)$, and $L(T) = L(DT)$, then $L(S \cdot T) = L(DS \odot DT)$.

\[L(DS \odot DT) = L((SQ_i \odot_{SO} TQ_j) \text{ for } i \in [1, n]) \]

\[= \{s \mid s \in SQ_i \odot_{SO} TQ_j\} \]

\[
= \{s \cdot t \mid s \in L(SQ_i) \land t \in L(TQ_j)\} \text{ for } i \in [1, n]
\]

\[= \{s \cdot t \mid s \in L(DS) \land t \in L(TQ_m)\} = \{s \cdot t \mid s \in L(S) \land t \in L(T)\} = L(S \cdot T) \]

\[\square\]

**Lemma 21** (Equivalence of $A$ and $D(A)$). $L(A) = L(D(A))$

**Proof.** $L(D(A)) = L([\epsilon \cdot A \cdot \epsilon])$

$L([\epsilon \cdot A \cdot \epsilon]) = \{s \mid s \in L([\epsilon \cdot A \cdot \epsilon])\}$

$L([\epsilon \cdot A \cdot \epsilon]) = \{s \cdot \epsilon \mid s \in L(A)\} = \{s \mid s \in L(A)\} = L(A)$.

This means $L([\epsilon \cdot A \cdot \epsilon]) = \{s \mid s \in L(A)\} = L(A)$.

\[\square\]

**Lemma 22** (Equivalence of $\oplus$ and $\mid$). If $L(S) = L(DS)$, and $L(T) = L(DT)$, then $L(S \mid T) = L(DS \oplus DT)$.

**Proof.** Let $DS = (SQ_0 \mid \ldots \mid SQ_n)$, and let $DT = (TQ_0 \mid \ldots \mid TQ_m)$

$\text{Vol. 1, No. 1, Article 1. Publication date: January 2018.}$
\[ L(DS \oplus DT) = L((SQ_0 | \ldots | SQ_n | TQ_1 | \ldots | TQ_m)) = \{ s \mid s \in SQ_i \lor s \in TQ_j \} \]
where \( i \in [1, n], j \in [1, m] \)
\( = \{ s \mid s \in L(DS) \lor s \in L(DT) \} \)
\( = \{ s \mid s \in L(S) \lor s \in L(T) \} \)
\( = L(S \mid T) \)

\[ \square \]

**Theorem 7.** For all regular expressions \( S \), \( L(\| S) = L(S) \).

**Proof.** By structural induction.

Let \( S = s. L(\| (s)) = L(\{ s \}) = \{ s \} = L(s) \).

Let \( S = \emptyset. L(\| (\emptyset)) = L(\emptyset) = \{ \} = L(\emptyset) \).

Let \( S = S'^n \). By induction assumption, \( L(\| (S')) = L(S') \).

\( L(\| (DS'^n)) = L(\{ \| (S') \}) \)
\( = \{ s \mid s \in L(\{ \| (S') \}) \} \)
\( = \{ s \mid s \in L(\| (S')) \} \)
\( = \{ s_1 \ldots s_n \mid n \in \mathbb{N} \land s_i \in L(S') \} \)
\( = L(S'^n) \)

Let \( S = S_1 \cdot S_2 \). By induction assumption, \( L(\| (S_1)) = L(S_1) \), and \( L(\| (S_2)) = L(S_2) \).

\( \| (S_1 \cdot S_2) = \| (S_1) \cup \| (S_2) \).

Let \( S = S_1 | S_2 \). By induction assumption, \( L(\| (S_1)) = L(S_1) \), and \( L(\| (S_2)) = L(S_2) \).

\( \| (S_1 | S_2) = \| (S_1) \oplus \| (S_2) \).

**\[ \square \]**

**Lemma 23.** Let \( [s_0 \cdot A_1 \ldots A_n \cdot s_n] \) be a sequence, and

\( \| (\langle A_i \rangle) = \langle A_i \rangle \).

Then,

\( \| (\langle [s_0 \cdot A_1 \ldots A_n \cdot s_n] \rangle) = \langle [s_0 \cdot A_1 \ldots A_n \cdot s_n] \rangle \).

**Proof.** By induction on \( n \).

Let \( n = 0. SQ = [s_0] \).

\( \| (\langle [s_0] \rangle) = \| (s_0) = \langle [s_0] \rangle \).

Let \( n > 0. SQ = [s_0 \cdot A_1 \ldots A_n \cdot s_n] \).

\( \| (\langle [s_0 \cdot A_1 \ldots A_n \cdot s_n] \rangle) \)
\( \| (\langle [s_0 \cdot A_1 \ldots A_{n-1} \cdot s_{n-1}] \rangle \cdot \langle A_n \rangle \cdot s_n) = \)
\( \| (\langle [s_0 \cdot A_1 \ldots A_{n-1} \cdot s_{n-1}] \rangle) \cup (\langle A_n \rangle \cdot s_n) \)
\( \| (\langle [s_0 \cdot A_1 \ldots A_{n-1} \cdot s_{n-1}] \rangle) \cup (\langle A_n \rangle \cdot s_n) \)
\( \langle [A_n] \rangle \cup \langle [s_n] \rangle = \langle [s_0 \cdot A_1 \ldots A_n \cdot s_n] \rangle \).

**\[ \square \]**

**Lemma 24.** Let \( \langle SQ_1 \mid \ldots | SQ_n \rangle \) be a sequence, and

\( \| (\langle SQ_i \rangle) = \langle SQ_i \rangle \).

Then,

\( \| (\langle [SQ_1 \mid \ldots | SQ_n] \rangle) = \langle SQ_1 \mid \ldots | SQ_n \rangle \).

**Proof.** By induction on \( n \).

Let \( n = 0 \| (\langle \rangle \rangle) = \| (\langle \emptyset \rangle \rangle = \langle \emptyset \rangle \).

Let \( n > 0 \| (\langle [SQ_1 \ldots SQ_n] \rangle) = \| (\langle [SQ_1 \ldots SQ_{n-1}] \rangle \cdot \langle SQ_n \rangle) = \| (\langle [SQ_1 \ldots SQ_{n-1}] \rangle) \cup (\langle SQ_n \rangle) \)
\( \| (\langle SQ_n \rangle) = \langle SQ_1 \ldots SQ_n \rangle \).

**\[ \square \]**

**Lemma 25** (Elimination of \( \| \circ \)).
(1) \(\downarrow (\uparrow (A)) = [A]\)
(2) \(\downarrow (\uparrow (SQ)) = \langle SQ \rangle\)
(3) \(\downarrow (\uparrow (DS)) = DS\)

Proof. By mutual induction

Let \(DS^*\) be an atom. \(\downarrow (\uparrow (DS^*)) = \downarrow (\uparrow (DS)^*) = (\downarrow (\uparrow (DS))) = \langle DS^* \rangle\)

Let \([s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n]\) be a sequence. \(\downarrow (\uparrow ([s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n]))\). By induction assumption, for each \(A_i, \downarrow (\uparrow (A_i)) = \langle A_i \rangle\). By Lemma 23, \(\downarrow (\uparrow ([s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n])) = \langle [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \rangle\).

Let \(\langle SQ_1 \cdot \ldots \cdot SQ_n \rangle\) be a DNF regular expression. By induction assumption, for each \(SQ_i, \downarrow (\uparrow (SQ_i)) = \langle SQ_i \rangle\). By Lemma 24, \(\downarrow (\uparrow ([SQ_1 \cdot \ldots \cdot SQ_n])) = \langle SQ_1 \cdot \ldots \cdot SQ_n \rangle\).

\(\square\)

Lemma 26. \((DS_1 \oplus DS_2) \oplus DS_3 = DS_1 \oplus (DS_2 \oplus DS_3)\)

Proof. Let \(DS_1 = \langle SQ_{1,1} | \ldots | SQ_{1,n_1} \rangle\) and \(DS_2 = \langle SQ_{2,1} | \ldots | SQ_{2,n_2} \rangle\) and \(DS_3 = \langle SQ_{3,1} | \ldots | SQ_{3,n_3} \rangle\).

\((DS_1 \oplus DS_2) \oplus DS_3 = \langle SQ_{1,1} | \ldots | SQ_{1,n_1} | SQ_{2,1} | \ldots | SQ_{2,n_2} \rangle \oplus DS_3 = \langle \langle SQ_{1,1} | \ldots | SQ_{1,n_1} | SQ_{2,1} | \ldots | SQ_{2,n_2} \rangle \oplus DS_3 \rangle = \langle DS_1 \oplus \langle SQ_{2,1} | \ldots | SQ_{2,n_2} \rangle | SQ_{3,1} \rangle | \ldots | \langle SQ_{3,n_3} \rangle \rangle = \langle DS_1 \oplus (DS_2 \oplus DS_3) \rangle\)

\(\square\)

Lemma 27. \((SQ_1 \circ_{SQ} SQ_2) \circ_{SQ} SQ_3 = SQ_1 \circ_{SQ} (SQ_2 \circ_{SQ} SQ_3)\)

Proof. Let \(SQ_1 = [s_{1,0} \cdot A_{1,1} \cdot \ldots \cdot A_{1,n_1} \cdot s_{1,n_1}]\), \(SQ_2 = [s_{2,1} \cdot A_{2,1} \cdot \ldots A_{2,n_2} \cdot s_{2,n_2}]\), and \(SQ_3 = [s_{3,1} \cdot A_{3,1} \cdot \ldots A_{3,n_3} \cdot s_{3,n_3}]\).

\((SQ_1 \circ_{SQ} SQ_2) \circ_{SQ} SQ_3 = [s_{1,1} \cdot A_{1,1} \cdot \ldots A_{1,n_1} \cdot s_{1,n_1} \cdot s_{2,1} \cdot A_{2,1} \cdot \ldots A_{2,n_2} \cdot s_{2,n_2}] \circ_{SQ} SQ_3 = [s_{1,1} \cdot A_{1,1} \cdot \ldots A_{1,n_1} \cdot s_{1,n_1} \cdot s_{2,0} \cdot A_{2,1} \cdot \ldots A_{2,n_2} \cdot s_{2,n_2} \cdot s_{3,0} \cdot A_{3,1} \cdot \ldots A_{3,n_3} \cdot s_{3,n_3}] = SQ_1 \circ_{SQ} [s_{2,0} \cdot A_{2,1} \cdot \ldots A_{2,n_2} \cdot s_{2,n_2} \cdot s_{3,0} \cdot A_{3,1} \cdot \ldots A_{3,n_3} \cdot s_{3,n_3}] = SQ_1 \circ_{SQ} (SQ_2 \circ_{SQ} SQ_3)\)

\(\square\)

Lemma 28. \((DS_1 \circ_{DS} DS_2) \circ_{DS} DS_3 = DS_1 \circ_{DS} (DS_2 \circ_{DS} DS_3)\)

Proof. Let \(DS_1 = \langle SQ_{1,1} | \ldots | SQ_{1,n_1} \rangle\), \(DS_2 = \langle SQ_{2,1} | \ldots | SQ_{2,n_2} \rangle\), and \(DS_3 = \langle SQ_{3,1} | \ldots | SQ_{3,n_3} \rangle\).

\((DS_1 \circ_{DS} DS_2) \circ_{DS} DS_3 = \langle SQ_{1,1} \circ_{DS} SQ_{2,1} | \ldots | SQ_{1,n_1} \circ_{DS} SQ_{2,n_1} \rangle \circ_{DS} DS_3 = \langle \langle SQ_{1,1} \circ_{DS} SQ_{2,1} \rangle \circ_{DS} SQ_{3,1} | \ldots | \langle SQ_{1,n_1} \circ_{DS} SQ_{2,n_1} \rangle \circ_{DS} SQ_{3,n_1} \rangle \rangle = \langle DS_1 \circ_{DS} \langle SQ_{2,1} \circ_{DS} SQ_{3,1} \rangle | \ldots | \langle SQ_{2,n_2} \circ_{DS} SQ_{3,n_2} \rangle \rangle = DS_1 \circ_{DS} (DS_2 \circ_{DS} DS_3)\)

\(\square\)

Lemma 29. \(\langle \rangle \circ_{DS} DS_1 = DS_1\)

Proof. By inspection.

\(\square\)

Lemma 30. \(DS_1 \circ_{\langle \rangle} DS_1 = DS_1\)

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Lemma 31. \( \langle [e] \rangle \odot DS = DS \)

**Proof.** By inspection. \( \square \)

Lemma 32. \( DS \odot \langle [e] \rangle = DS \)

**Proof.** Done similarly to Lemma 31. \( \square \)

Lemma 33. \( \langle \rangle \odot DS = \langle \rangle \)

**Proof.** By inspection. \( \square \)

Lemma 34. \( DS \odot \langle \rangle = \langle \rangle \)

**Proof.** By inspection. \( \square \)

Lemma 35. \( (DS_1 \oplus DS_2) \odot DS_3 = (DS_1 \odot DS_3) \oplus (DS_2 \odot DS_3) \)

**Proof.** Let \( DS_1 = \langle SQ_{1,1} \mid \ldots \mid SQ_{1,n_1} \rangle \).
Let \( DS_2 = \langle SQ_{2,1} \mid \ldots \mid SQ_{2,n_2} \rangle \).
Let \( DS_3 = \langle SQ_{3,1} \mid \ldots \mid SQ_{3,n_3} \rangle \).
\((DS_1 \oplus DS_2) \odot DS_3 = (SQ_{1,1} \mid \ldots \mid SQ_{1,n_1} \mid SQ_{2,1} \mid \ldots \mid SQ_{2,n_2}) \odot (SQ_{3,1} \mid \ldots \mid SQ_{3,n_3}) \). So, through application of \( \odot \), \( (SQ_{1,1} \odot SQ_{3,1}) \mid \ldots \mid SQ_{2,1} \odot SQ_{3,1}) \mid \ldots \mid SQ_{2,n_2} \odot SQ_{3,n_3}) \). This equals \( SQ_{1,1} \odot SQ_{3,1} \mid \ldots \mid SQ_{2,1} \odot SQ_{3,1} \mid \ldots \mid SQ_{2,n_2} \odot SQ_{3,1} \mid \ldots \mid SQ_{2,n_2} \odot SQ_{3,n_3} \), which is \( (DS_1 \odot DS_3) \oplus (DS_2 \odot DS_3) \). \( \square \)

Lemma 36. \( (SQ) \odot (DS_1 \oplus DS_2) = (SQ) \odot (DS_1) \oplus (SQ) \odot (DS_2) \)

**Proof.** Let \( DS_1 = \langle SQ_{1,1} \mid \ldots \mid SQ_{1,n_1} \rangle \). Let \( DS_2 = \langle SQ_{2,1} \mid \ldots \mid SQ_{2,n_2} \rangle \).
\( (SQ) \odot (DS_1 \oplus DS_2) = (SQ) \odot ((SQ_{1,1} \mid \ldots \mid SQ_{1,n_1}) \oplus (SQ_{2,1} \mid \ldots \mid SQ_{2,n_2})) \). So, through application of \( \odot \), \( (SQ \odot SQ_{1,1} \mid \ldots \mid SQ \odot SQ_{1,n_1}) \odot (SQ \odot SQ_{2,1} \mid \ldots \mid SQ \odot SQ_{2,n_2}) \). This equals \( SQ \odot SQ_{1,1} \mid \ldots \mid SQ \odot SQ_{1,n_1} \odot SQ \odot SQ_{2,1} \mid \ldots \mid SQ \odot SQ_{2,n_2} \), through the definitions, equals \( ((SQ) \odot DS_1) \oplus ((SQ) \odot DS_2) \). \( \square \)

Lemma 37 (\( \equiv^s \) is finer than \( \equiv \)). If \( S \equiv^s T \), then \( S \equiv T \)

**Proof.** By induction on the derivation of \( \equiv^s \)

Case 1 (+ \( Ident \)). Through the use of \( \equiv^s \) \( + \) \( Ident \).

Case 2 (0 \( Proj_R \)). Through the use of \( \equiv^s \) \( 0 \) \( Proj_R \).

Case 3 (0 \( Proj_L \)). Through the use of \( \equiv^s \) \( 0 \) \( Proj_L \).

Case 4 (\( \cdot Assoc \)). Through the use of \( \equiv^s \) \( \cdot \) \( Assoc \).

Case 5 (\( \mid Assoc \)). Through the use of \( \equiv^s \) \( \mid \) \( Assoc \).

Case 6 (\( \mid Comm \)). Through the use of \( \equiv^s \) \( \mid \) \( Comm \).
Case 7 (Dist$_R$). Through the use of $\equiv$’s Dist$_R$.

Case 8 (Dist$_L$). Through the use of $\equiv$’s Dist$_L$.

Case 9 ( Ident$_L$). Through the use of $\equiv$’s · Ident$_L$.

Case 10 ( Ident$_R$). Through the use of $\equiv$’s · Ident$_R$.

Case 11 (Unrollstar$_L$). Let $S \equiv^s T$ through an application of Unrollstar$_L$.

So, without loss of generality, from symmetry, we can say $S = S^*$ and $T = \epsilon \mid (S' \cdot S^*)$.

Consider the derivations

\[
\begin{align*}
S' &\equiv S' \cdot \epsilon \\
S'^* &\equiv (S' \cdot \epsilon)^*
\end{align*}
\]

\[
S' \cdot \epsilon^* \equiv \epsilon \mid (S' \cdot (\epsilon \cdot S')^* \cdot \epsilon)
\]

\[
S' \cdot (\epsilon \cdot S')^* \cdot \epsilon \equiv S' \cdot (\epsilon \cdot S')^*
\]

\[
\epsilon \mid (S' \cdot (\epsilon \cdot S')^* \cdot \epsilon) \equiv \epsilon \mid (S' \cdot (\epsilon \cdot S')^*)
\]

\[
S' \cdot \epsilon \equiv S'
\]

\[
\vdots
\]

\[
\epsilon \mid (S' \cdot (\epsilon \cdot S')^*) \equiv \epsilon \mid (S' \cdot S'^*)
\]

Through repeated application of equational theory transitivity, $S \equiv T$.

Case 12 (Unrollstar$_R$). Let $S \equiv^s T$ through an application of Unrollstar$_L$.

So, without loss of generality, from symmetry, we can say $S = S^*$ and $T = \epsilon \mid (S' \cdot S^*)$.

Consider the derivations

\[
\begin{align*}
S' &\equiv \epsilon \cdot S' \\
S'^* &\equiv (\epsilon \cdot S')^*
\end{align*}
\]

\[
\epsilon \cdot S'^* \equiv \epsilon \mid (\epsilon \cdot (S' \cdot \epsilon)^* \cdot S')
\]

\[
\epsilon \cdot (S' \cdot \epsilon)^* \cdot S' \equiv (S' \cdot \epsilon)^* \cdot S'
\]

\[
\epsilon \mid (\epsilon \cdot (S' \cdot \epsilon)^* \cdot S') \equiv \epsilon \mid (S' \cdot S' \epsilon)^*)
\]

\[
S' \cdot \epsilon \equiv S'
\]

\[
\vdots
\]

\[
\epsilon \mid ((S' \cdot \epsilon)^* \cdot S') \equiv \epsilon \mid (S' \cdot S'^*)
\]

Through repeated application of equational theory transitivity, $S \equiv T$. □
B.5 Unambiguity Property Proofs
Unambiguity is critical in the typing derivations, so unambiguity proofs are similarly critical. In this section, we prove requirements for maintaining unambiguity. The bulk of the work for many of these is proven in the language unambiguity proofs in Subsection B.2. However, this combines these together to prove things like unambiguity is maintained through application of the definitional equivalence rules.

Lemma 38. If $S \mid T$ be strongly unambiguous, then $\mathcal{L}(S) \cap \mathcal{L}(T) = \{\}$, and both $S$ and $T$ are strongly unambiguous.

Proof. If $S \mid T$ is strongly unambiguous, then either $\mathcal{L}(S \mid T) = \{\}$, or $\mathcal{L}(S) \cap \mathcal{L}(T) = \{\}$, and $S$ and $T$ are both strongly unambiguous.

If the former, then we are done.

If the latter, then both $\mathcal{L}(S) = \{\}$ and $\mathcal{L}(T) = \{\}$. This means they are both strongly unambiguous. Furthermore, $\{\} \cap \{\} = \{\}$, so $\mathcal{L}(S) \cap \mathcal{L}(T) = \{\}$. □

Lemma 39. Let $S \equiv^S T$. If $S$ is strongly unambiguous, then $T$ is strongly unambiguous.

Proof. If $\mathcal{L}(S) = \{\}$, then $\mathcal{L}(T) = \{\}$, by Lemma 37.

For the case where $\mathcal{L}(S) \neq \{\}$, we proceed by induction on the derivation of equivalence of $S$ and $T$.

Case 1 ($+ \text{Ident}$ left to right). Let the last step of the derivation be $+ \text{Ident}$ left to right. $S \equiv^S S \mid \emptyset$.

$\emptyset$ is strongly unambiguous, as its language is empty. $S$ is strongly unambiguous by assumption $\mathcal{L}(S) \cap \mathcal{L}(\emptyset) = \mathcal{L}(S) \cap \{\} = \emptyset$, so $T$ is strongly unambiguous.

Case 2 ($+ \text{Ident}$ right to left). Let the last step of the derivation be $+ \text{Ident}$ right to left. $T \mid \emptyset \equiv^S T$.

If $\mathcal{L}(T \mid \emptyset) = \{\}$ then $\mathcal{L}(T) = \{\}$, so $T$ is strongly unambiguous.

Otherwise $T$ is strongly unambiguous, which is what is desired.

Case 3 ($0 \text{Proj}_R$ both directions). Let the last step of the derivation be $0 \text{Proj}_R$. The language of both sides is $\{\}$, by Lemma 37.

Case 4 ($0 \text{Proj}_L$ both directions). Let the last step of the derivation be $0 \text{Proj}_L$. The language of both sides is $\{\}$, by Lemma 37.

Case 5 ($\cdot \text{Assoc}$ left to right). Let the last step of the derivation be $\cdot \text{Assoc}$ left to right. $(S_1 \cdot S_2) \cdot S_3 \equiv^S S_1 \cdot (S_2 \cdot S_3)$.

Because $S_1$ is strongly unambiguous. $S_1 \cdot S_2$ and $(S_1 \cdot S_2) \cdot S_3$

Let $s_2, t_2 \in S_2$, let $s_3, t_3 \in S_3$, and let $s_2 \cdot s_3 = t_2 \cdot t_3$. Consider $s_1$ in $S_1$ which exists as $\mathcal{L}(S) \neq \{\}$.

$(s_1 \cdot s_2) \cdot s_3 = (s_1 \cdot t_2) \cdot t_3$, so $s_3 = t_3$ and $s_1 \cdot s_2 = s_1 \cdot t_2$, so $s_2 = t_2$.

Let $s_2 \cdot s_3 \in S_2 \cdot S_3$, $t_2 \cdot t_3 \in S_2 \cdot S_3$, $t_1 \cdot t_2 \in S_1$, and let $s_1 \cdot (s_2 \cdot s_3) = t_1 \cdot (t_2 \cdot t_3)$. This means $(s_1 \cdot s_2) \cdot s_3 = (t_1 \cdot t_2) \cdot t_3)$.

So by assumption, $s_3 = t_3$, and $s_1 \cdot s_2 = t_1 \cdot t_2$. So by assumption, $s_1 = t_1$ and $s_2 = t_2$. So, $s_2 \cdot s_3 = t_2 \cdot t_3$, and $s_1 = t_1$.

Case 6 ($\cdot \text{Assoc}$ right to left). Very similarly to left to right.

Case 7 ($\mid \text{Assoc}$ left to right). $S_1 \mid (S_2 \mid S_3) \equiv^S (S_1 \mid S_2) \mid S_3$.

$\mathcal{L}(S_1) \cap \mathcal{L}(S_2 \mid S_3) = \{\}$. This means that $\mathcal{L}(S_1) \cap (\mathcal{L}(S_2) \cup \mathcal{L}(S_3)) = \{\}$, so through distributivity, $\mathcal{L}(S_1) \cap \mathcal{L}(S_2) \cup \mathcal{L}(S_1) \cap \mathcal{L}(S_3) = \{\}$. This means $\mathcal{L}(S_1) \cap \mathcal{L}(S_2) = \{\}$ and $\mathcal{L}(S_1) \cap \mathcal{L}(S_3) = \{\}$.

If $\mathcal{L}(S_2 \mid S_3) = \{\}$, then the language of each is empty, so they are each strongly unambiguous. This means $S_1 \mid S_2$ is strongly unambiguous.
Furthermore, \( L(S_1) \cap L(S_2) \cup L(S_2) \cap L(S_3) = \{ \} \) as each of the intersections is empty. So the whole thing is unambiguous.

**Case 8** (| Assoc right to left). Done very similarly to the left to right case.

**Case 9** (| Comm). \( S_1 \mid S_2 \equiv^s S_2 \mid S_1 \) So if the languages are empty, then they are both empty. Otherwise, \( S_1 \) is strongly unambiguous, and \( S_2 \) is strongly unambiguous, and \( L(S_1) \cap L(S_2) = \{ \} \). So \( L(S_2) \cap L(S_1) = \{ \} \), and so \( S_2 \mid S_1 \) is strongly unambiguous.

**Case 10** (Dist \( R \) left to right). \( S_1 \cdot (S_2 \mid S_3) \equiv^s (S_1 \cdot S_2) \mid (S_1 \cdot S_3) \).

If \( L(S_1 \cdot (S_2 \mid S_3)) = \{ \} \), then \( (S_1 \cdot S_2) \mid (S_1 \cdot S_3) = \{ \} \), and we are done.

If the language is nonempty, so too are the languages of each side, so \( S_1 \) is nonempty, and \( S_2 \mid S_3 \) is nonempty, and \( S_1 \) is strongly unambiguous, and \( S_2 \mid S_3 \) is strongly unambiguous.

\( S_2 \mid S_3 \) being strongly unambiguous implies \( S_2 \) is strongly unambiguous, \( S_3 \) is strongly unambiguous, and \( L(S_2) \cap L(S_3) = \{ \} \), by Lemma 38.

Let \( s_1, t_1 \in L(S_1), s_2, t_2 \in L(S_2), s_1 \cdot s_2 = t_1 \cdot t_2, s \). Then \( t_1 \in L(S_1 \mid S_2), \) and \( t_2 \in L(S_1 \mid S_2) \). By assumption of strong unambiguity, where the languages are not empty, \( s_1 = s_2 \) and \( t_1 = t_2 \).

Similarly for \( s_1, t_1 \in L(S_1), s_3, t_3 \in L(S_3) \).

Assume there exists some \( s \in L(S_1 \cdot S_2) \cap L(S_1 \cdot S_3) \). This means \( s = s_1 \cdot s_2, \) for \( s_1 \in L(S_1) \) and \( s_2 \in L(S_2), \) uniquely. It means \( s = t_1 \cdot t_2, \) for \( t_1 \in L(S_1) \) and \( t_2 \in L(S_2) \). From assumption, as \( s \in L(S_1 \cdot (S_2 \mid S_3)), s_1 = t_1 \) and \( s_2 = t_3 \). Contradiction, as \( L(S_2) \cap L(S_3) = \{ \} \). So there is no string in the intersection, or in other words \( L(S_1 \cdot S_2) \cap L(S_1 \cdot S_3) = \{ \} \).

As such, \( (S_1 \cdot S_2) \mid (S_1 \cdot S_3) \) is strongly unambiguous.

**Case 11** (Dist \( R \) right to left). \( (S_1 \cdot S_2) \mid (S_1 \cdot S_3) \equiv^s S_1 \cdot (S_2 \mid S_3) \).

If \( L(S_1) = \{ \} \), then the language of the entire \( S \) is empty, and we are done. Otherwise assume \( L(S_1) \neq \{ \} \).

From assumption \( S_1 \cdot S_2 \) is strongly unambiguous, \( S_1 \cdot S_3 \) is strongly unambiguous, and \( L(S_1 \cdot S_2) \cap L(S_1 \cdot S_3) = \{ \} \).

Assume there exists some \( s \in L(S_1) \cap L(S_1) \). Let \( s_1 \in L(S_1) \). This makes \( s_1 \cdot s \in L(S_1 \cdot S_2) \cap L(S_1 \cdot S_3) \). This is a contradiction, so \( L(S_2) \cap L(S_3) = \{ \} \).

Let \( s_1, t_1 \in L(S_1) \). Let \( s, t \in L(S_2) \cap S_3 \). Let \( s_1 \cdot s = t_1 \cdot t \). Assume \( s \in L(S_2) \). Then \( t \in L(S_2), \) as otherwise \( S \) is not strongly unambiguous. So as \( s_1 \cdot s \in L(S_1 \cdot S_2), \) and \( t_1 \cdot t \in L(S_1 \cdot S_3), \) by assumption, \( s_1 = t_1, \) and \( s = t \). If \( s \in L(S_2) \), then \( s_1 \in L(S_2), \) and the same argument applies.

**Case 12** (Dist \( L \) both directions). Proceeds the same as Dist \( R \).

**Case 13** (Ident \( L \) left to right). \( e \cdot S' \equiv^s S' \)

If they have empty languages, we are done.

If nonempty, then \( S' \) is strongly unambiguous, and we are done.

**Case 14** (Ident \( L \) right to left). \( S' \equiv^s e \cdot S' \)

Both \( S' \) and \( e \) are strongly unambiguous, by assumption and definition, respectively.

Furthermore, let \( s_1, t_1 \in L(e), \) and \( s_2, t_2 \in L(S'), \) and \( s_1 \cdot s_2 = t_1 \cdot t_2, s_1 = t_1 = e, \) so \( s_1 = t_2, \) which makes \( s_2 = t_2. \)

**Case 15** (Ident \( R \) both directions). Very similar to Ident \( L \).

**Case 16** (Unrollstar \( L \) left to right). \( S'' \equiv^s e \mid (S' \cdot S'') \)

Let \( s \in L(e) \cap L(S' \cdot S'') \). So \( s = e. \) So \( e \in L(S') \). Contradiction, as if \( e \) in \( L(S'), \) then if \( s_1 \cdot s_n = t_1 \cdot t_m, \) \( n \) no longer must equal \( m, \) as arbitrarily many \( e s \) can be input.

\( e \) is strongly unambiguous.
If \( \mathcal{L}(S') = \emptyset \), then \( S' \cdot S'' \) also has an empty language, and is strongly unambiguous.

If the language is nonempty, \( S' \) is strongly unambiguous.

Let \( s_1, s_2 \in \mathcal{L}(S') \), \( t_1, t_2 \in \mathcal{L}(S'') \). Let \( s_1 \cdot t_1 = t_2 \cdot t_2 \cdot t_2 \cdot t_3 \cdot t_n \) and \( s_2 = t_2 \cdot t_2 \cdot \ldots \cdot t_m \), where \( t_{1,i} \). Consider \( s_1 \cdot t_1 \cdot \ldots \cdot t_n \) and \( s_2 \cdot t_2 \cdot \ldots \cdot t_m \). As \( S' \) is unambiguously iterable, \( n + 1 = m + 1 \), and \( s_1 = s_2 \) and \( t_{1,i} = t_{2,i} \). This means that \( t_1 = t_2 \). So Regex' is unambiguously concatenable with \( S'' \).

**Case 17 (Unrollstar_R right to left).** \( \epsilon \mid (S' \cdot S'') \equiv S'' \)

If \( \mathcal{L}(S') = \emptyset \), then it is vacuously unambiguously concatenable, and \( S' \) is strongly unambiguous, so \( S'' \) is strongly unambiguous.

Let \( \mathcal{L}(S') \) not be empty.

Let \( s_1 \cdot \ldots \cdot s_n = t_1 \cdot \ldots \cdot t_m \), and \( s_i, t_i \in \mathcal{L}(S') \). We want to show that \( n = m \) and \( s_i = t_i \). This can be done by induction on \( n \).

If \( n = 0 \), then \( m = 0 \), as otherwise \( m > 0 \), which would imply that \( \epsilon \in \mathcal{L}(S') \), making \( S \) not strongly unambiguous.

If \( n \neq 0 \), then by the unambiguous concatenability of \( S' \) and \( S'' \), \( s_1 = t_1 \), and \( s_2 \cdot \ldots \cdot s_n = t_2 \cdot \ldots \cdot t_m \), and the IH applies.

**Case 18 (Unrollstar_R both directions).** Done similarly to Unrollstar_L.

**Case 19 (All structural cases).** As \( \equiv \) is finer than \( \equiv \), the subparts will have the same languages. If the language of \( S \) is empty, then we are done, otherwise, each subpart will be strongly unambiguous, by the induction hypothesis. As the top level unambiguity condition is based on the language, and the languages of the subparts are equal, the top level unambiguity condition will be satisfied.

**Case 20 (Transitivity of Equational Theory).** If \( S \equiv S' \) and \( S' \equiv S'' \), then by IH, \( S' \) is strongly unambiguous, and by IH again, \( T \) is strongly unambiguous.

\[ \square \]

**Lemma 40.** If \( DS_1 \odot (DS_2 \oplus DS_3) \) is strongly unambiguous, then \( (DS_1 \odot DS_2) \oplus (DS_1 \odot DS_3) \) is strongly unambiguous.

**Proof.** Let \( DS_1 = \{ SQ_{1,1} | \ldots | SQ_{1,n_1} \} \).

Let \( DS_2 = \{ SQ_{2,1} | \ldots | SQ_{2,n_2} \} \).

Let \( DS_3 = \{ SQ_{3,1} | \ldots | SQ_{3,n_3} \} \).

\( DS_2 \oplus DS_3 = \{ SQ_{2,1} | \ldots | SQ_{2,n_2} \oplus SQ_{3,1} | \ldots | SQ_{3,n_3} \} \).

\( DS_1 \odot (DS_2 \oplus DS_3) = \{ SQ_{1,1} \odot SQ_{2,1} | \ldots | SQ_{1,1} \odot SQ_{2,n_2} \oplus SQ_{1,1} \odot SQ_{3,1} | \ldots | SQ_{1,1} \odot SQ_{3,n_3} \} \).

From this as strongly unambiguous, \( SQ_{1,i} \odot SQ_{i,k} \) is strongly unambiguous for all \( i, j, k \). Furthermore, by strong unambiguity, if \( (i_1, j_1, k_1) \neq (i_2, j_2, k_2) \), then \( SQ_{i_1,j_1,k_1} \cap SQ_{i_2,j_2,k_2} \) is strongly unambiguous.

The same process can be repeated to show that assuming \( (DS_1 \odot DS_2) \oplus (DS_1 \odot DS_3) \) is strongly unambiguous, we can show \( DS \odot (DS_2 \oplus DS_3) \) is strongly unambiguous. \[ \square \]
B.6 Rewrite Equivalence Proofs

This subsection goes through equivalence associated with the rewrites. In this section, proofs of rewrites not altering the language are proven. Furthermore, it is shown that the Definitional Equivalence Rules are finer than the base axioms. We prove that both $\| \|$ and $\|\|$ maintain unambiguity. Parallel Rewrites with Swap are shown to be equivalent to the definitional equivalence rules. We prove that the reflexive and transitive closure of rewrites is equivalent in expressibility to the reflexive and transitive closure of parallel rewrites. Lastly, we prove that if the DNF versions of two regular expressions are can be written to each other, then those two regular expressions are definitionally equivalent.

**Lemma 41** (Single Rewrites Respecting Language).

- If $A \rightarrow_B DS$, then $L(A) = L(DS)$
- If $DS \rightarrow_B DT$, then $L(DS) = L(DT)$

**Proof.** By mutual induction on the derivation of $\rightarrow$ and $\rightarrow_B$

**Case 1** (Atom UnrollStarL).

$$DS^* \rightarrow_B \langle [\epsilon] \rangle \oplus (DS \odot D(DS^*))$$

Let $\uparrow DS = S$.

$S^* \equiv \epsilon \mid (S \cdot S^*)$, by Lemma 37. By Theorem 1, $L([S] \cdot [S]) = L([\epsilon \mid (S \cdot S^*)])$. So $L(D([S] \cdot [S])) = L([\epsilon \mid (S \cdot S)])$. So by Lemma 21, and application of $\|$, $L(DS^*) = L([\epsilon] \oplus DS \odot D(DS^*))$, as desired.

**Case 2** (Atom UnrollStarR).

$$DS^* \rightarrow_B \langle [\epsilon] \rangle \oplus (D(DS^*) \odot DS)$$

Let $\uparrow DS = S$.

$S^* \equiv \epsilon \mid (S \cdot S^*)$, by Lemma 37. By Theorem 1, $L([S] \cdot [S]) = L([\epsilon \mid (S \cdot S)])$. So $L(D([S] \cdot [S])) = L([\epsilon \mid (S \cdot S)])$. So by Lemma 21, and application of $\|$, $L(DS^*) = L([\epsilon] \oplus D(DS^*) \odot DS)$, as desired.

**Case 3** (Atom Structural Rewrite).

$$DS \rightarrow_B DT$$

$$DS^* \rightarrow_B D(DT^*)$$

$L(DS) = L(DT)$, so $L(DS^*) = L(DT^*)$. Through application of Lemma 21, $L(DS^*) = L(D(DT^*))$.

**Case 4** (DNF Structural Rewrite).

$$A_j \rightarrow_B DS$$

$$\langle SQ_1 \mid \ldots \mid SQ_{i-1} \rangle \oplus \langle \{s_0 \cdot A_1 \cdot \ldots \cdot s_{j-1} \} \odot D(A_j) \odot \{s_j \cdot \ldots \cdot s_m \} \oplus \langle SQ_{i+1} \mid \ldots \mid SQ_n \rangle \rightarrow \langle SQ_1 \mid \ldots \mid SQ_{i-1} \rangle \oplus \langle \{s_0 \cdot A_1 \cdot \ldots \cdot s_{j-1} \} \odot DS \odot \{s_j \cdot \ldots \cdot s_m \} \oplus \langle SQ_{i+1} \mid \ldots \mid SQ_n \rangle$$

As $L(D(A_j)) = L(DS)$, by IH and Lemma 21, and because the left side is the same as the right, except with $D(A_j)$ replacing $DS$, the two languages are the same.

$\Box$

**Lemma 42** (Rewrites Respecting Language). If $DS \rightarrow^* DT$, then $L(DS) = L(DT)$

**Proof.** By induction on the derivation of $\rightarrow^*$
Case 1 (Reflexivity).

\[ DS \rightarrow^* DS \]

\( \mathcal{L}(DS) = \mathcal{L}(DS) \) so we’re done.

Case 2 (Base).

\[ DS \rightarrow DT \]

\[ DS \rightarrow^* DT \]

By Lemma 41, as \( DS \rightarrow DT \), \( \mathcal{L}(DS) = \mathcal{L}(DT) \).

Case 3.

\[ DS \rightarrow^* DS' \]

\[ DS' \rightarrow^* DT \]

\[ DS \rightarrow^* DT \]

By IH, \( \mathcal{L}(DS) = \mathcal{L}(DS') \). By IH, \( \mathcal{L}(DS') = \mathcal{L}(DT) \). So \( \mathcal{L}(DS) = \mathcal{L}(DT) \).

\[ \square \]

**Lemma 43.** If \( \downarrow \downarrow S = \langle \rangle \), and \( S \equiv T \), then \( \downarrow \downarrow T = \langle \rangle \).

**Proof.** By induction on the proof of equivalence

Case 1 (Structural Equality Rule). Then \( T = S \), so \( \downarrow \downarrow T = \downarrow \downarrow S = \langle \rangle \).

Case 2 (+ Ident left to right). \( S \equiv S \mid \emptyset \). \( \downarrow (S \mid \emptyset) = \downarrow S \downarrow \emptyset = \langle \rangle \downarrow \langle \rangle = \langle \rangle \).

Case 3 (+ Ident right to left). \( T \equiv T \mid \emptyset \). \( \downarrow (T \mid \emptyset) = \langle \rangle \). So by definition, \( \downarrow T \downarrow \emptyset = \langle \rangle \). Again by definition, \( \downarrow T \downarrow \langle \rangle = \langle \rangle \). So by Lemma 30, \( \downarrow T = \langle \rangle \)

Case 4 (0 \( \text{Proj}_R \) left to right). \( T = \emptyset \) so \( \downarrow \downarrow T = \langle \rangle \)

Case 5 (0 \( \text{Proj}_L \) right to left). \( T = S \cdot \emptyset \), so \( \downarrow \downarrow T = \downarrow S \downarrow \emptyset = \downarrow S \downarrow \langle \rangle \), so by Lemma 33, \( \downarrow \downarrow T = \langle \rangle \).

Case 6 (0 \( \text{Proj}_L \) both directions). Done similarly to 0 \( \text{Proj}_R \).

Case 7 (\( \cdot \) Assoc left to right). \( (S_1 \cdot S_2) \cdot S_3 \equiv S_1 \cdot (S_2 \cdot S_3) \). Through definitions, and Lemma 27, \( \langle \rangle = \downarrow ((S_1 \cdot S_2) \cdot S_3) = (\downarrow S_1 \downarrow (\downarrow S_2 \downarrow S_3)) = (\downarrow S_1 \downarrow S_2) \downarrow S_3 = \downarrow (S_1 \cdot (S_2 \cdot S_3)) \)

Case 8 (\( \cdot \) Assoc right to left). Analogously to left to right

Case 9 (\( \mid \) Assoc left to right). \( (S_1 \mid S_2) \mid S_3 \equiv S_1 \mid (S_2 \mid S_3) \). Through definitions, and Lemma 26, \( \langle \rangle = \downarrow ((S_1 \mid S_2) \mid S_3) = (\downarrow (S_1 \mid S_2) \downarrow S_3) = (\downarrow S_1 \downarrow S_2) \downarrow S_3 = \downarrow (S_1 \cdot (S_2 \cdot S_3)) \)

Case 10 (\( \mid \) Assoc right to left). Analogously to left to right

Case 11 (\( \mid \) Comm). \( S_1 \mid S_2 \equiv S_1 \mid (S_2 \mid S_3) \). By the definition of \( \cdot \), \( \downarrow S_1 = \langle \rangle \), and \( \downarrow S_2 = \langle \rangle \). \( \downarrow (S_2 \mid S_1) = \downarrow S_2 \uparrow \downarrow S_1 = \langle \rangle \downarrow \langle \rangle = \langle \rangle \).

Case 12 (\( \text{Dist}_R \) left to right). \( S_1 \cdot (S_2 \mid S_3) \equiv (S_1 \cdot S_2) \mid (S_1 \cdot S_3) \)

\( \downarrow S_1 \cdot (S_2 \mid S_3) = \downarrow S_1 \downarrow (\downarrow S_2 \uparrow \downarrow S_3) = \langle \rangle \). By the definition of \( \cdot \), this means \( \downarrow S_1 = \langle \rangle \), or \( \downarrow S_2 \uparrow \downarrow S_3 = \langle \rangle \).

If \( \downarrow S_1 = \langle \rangle \), then by Lemma 33, \( \downarrow S_1 \downarrow S_2 = \langle \rangle \) and \( \downarrow S_1 \downarrow S_3 = \langle \rangle \), so \( \downarrow ((S_1 \cdot S_2) \mid (S_1 \cdot S_3)) = (\downarrow S_1 \downarrow S_2) \downarrow (\downarrow S_1 \downarrow S_3) = \langle \rangle \).

If \( \downarrow S_2 \downarrow \downarrow S_3 = \langle \rangle \), then by definition of \( \cdot \), \( \downarrow S_2 = \langle \rangle \) and \( \downarrow S_3 = \langle \rangle \). By Lemma 34, \( \downarrow S_2 \downarrow S_3 = \langle \rangle \) and \( \downarrow S_1 \downarrow \downarrow S_3 = \langle \rangle \), so \( \downarrow ((S_1 \cdot S_2) \mid (S_1 \cdot S_3)) = (\downarrow S_2 \downarrow S_3) \downarrow (\downarrow S_1 \downarrow S_3) = \langle \rangle \).


Case 13 (Dist\textsubscript{R} right to left). \((S_1 \cdot S_2) \mid (S_1 \cdot S_3) \equiv S_1 \cdot (S_2 \mid S_3)\)
\[
\downarrow ((S_1 \cdot S_2) \mid (S_1 \cdot S_3)) = (\downarrow S_1 \odot \downarrow S_2) \oplus (\downarrow S_1 \odot \downarrow S_3) = \langle \rangle. \]
By the definition of \(\oplus\), this means \(\downarrow S_1 \odot \downarrow S_2 = \langle \rangle\), and \(\downarrow S_1 \odot \downarrow S_3 = \langle \rangle\).
As \(\downarrow S_1 \odot \downarrow S_2 = \langle \rangle\).
If \(\downarrow S_1 = \langle \rangle\), then by Lemma 33, \(\downarrow S_1 \odot (\downarrow S_2 \oplus \downarrow S_3) = \langle \rangle\), so \(S_1 \cdot (S_2 \mid S_3) = \langle \rangle\).
If \(\downarrow S_1 \neq \langle \rangle\), then \(\downarrow S_2 = \langle \rangle\) and \(\downarrow S_3 = \langle \rangle\). This means \(\downarrow S_2 \oplus \downarrow S_3 = \langle \rangle\). So, by Lemma 33, \(\downarrow S_1 \odot (\downarrow S_2 \oplus \downarrow S_3) = \langle \rangle\), so \(\downarrow (S_1 \cdot (S_2 \mid S_3))\).

Case 14 (Dist\textsubscript{L} both directions). Proceeds analogously to Dist\textsubscript{R}.

Case 15 (Ident\textsubscript{L} left to right). \(\varepsilon \cdot T \equiv T\). By assumption, \(\downarrow (\varepsilon \cdot T) = \langle \rangle\) This means \(\downarrow e \odot \downarrow T = \langle \rangle\). By Lemma 31, \(\downarrow e \odot \downarrow T = \downarrow T\), so \(\downarrow T = \langle \rangle\).

Case 16 (Ident\textsubscript{L} right to left). \(S \equiv \varepsilon \cdot S\). By assumption, \(\downarrow S = \langle \rangle\). By Lemma 31, \(\downarrow (\varepsilon \cdot S) = \downarrow S\), so \(\downarrow S = \langle \rangle\).

Case 17 (Ident\textsubscript{R} both directions). Done analogously to \(\cdot \text{Ident\textsubscript{L}}\).

Case 18 (Sumstar, Prodstar, Starstar, Dicyc, Structural Star Equality). In all of these cases, the regular expression on the left is of the form \(S''\), for some \(S', \varepsilon \in S''\) for all \(S'\). However, \(L(\langle \rangle) = \{\}\), and by Theorem 1, \(L(\downarrow S) = L(S)\). This means that \(L(S') \neq \{\}\), for all \(S'\), so these rules do not apply.

Case 19 (Structural Or Equality).
\[
\begin{array}{c}
S_1 \equiv T_1 \quad S_2 \equiv T_2 \\
\downarrow S_1 \mid S_2 \equiv \downarrow T_1 \mid T_2 \\
\hline
\end{array}
\]
\(\downarrow (S_1 \mid S_2) = \downarrow S_1 \odot \downarrow S_2 = \langle \rangle\). By the definition of \(\odot\), \(\downarrow S_1 = \langle \rangle\) and \(\downarrow S_2 = \langle \rangle\). So, by induction, \(\downarrow T_1 = \langle \rangle\) and \(\downarrow T_2 = \langle \rangle\). So \(\downarrow (T_1 \mid T_2) = \downarrow (T_1 \mid T_2) = \langle \rangle\).

Case 20 (Structural Concat Equality).
\[
\begin{array}{c}
S_1 \equiv T_1 \quad S_2 \equiv T_2 \\
\downarrow S_1 \cdot S_2 \equiv \downarrow T_1 \cdot T_2 \\
\hline
\end{array}
\]
\(\downarrow (S_1 \cdot S_2) \equiv \downarrow S_1 \odot \downarrow S_2 = \langle \rangle\). By the definition of \(\odot\), \(\downarrow S_1 = \langle \rangle\) or \(\downarrow S_2 = \langle \rangle\). So, by induction, \(\downarrow T_1 = \langle \rangle\) or \(\downarrow T_2 = \langle \rangle\). So \(\downarrow T_1 \odot \downarrow T_2 = \downarrow (T_1 \cdot T_2) = \langle \rangle\).

Case 21 (Transitivity of Equational Theories).
\[
\begin{array}{c}
S \equiv S' \quad S' \equiv T \\
\hline
S \equiv T \\
\end{array}
\]
By IH, \(\downarrow S' = \langle \rangle\). So, by IH, \(\downarrow T = \langle \rangle\). □

Lemma 44. If \(L(S) = \{\}\), then \(\downarrow S = \langle \rangle\)

Proof. \(L(\emptyset) = \{\}\). We know \(L(S) = \{\}\), so \(S \equiv \emptyset\). \(\downarrow \emptyset = \langle \rangle\). So, by Lemma 43, \(\downarrow S = \langle \rangle\). □

Lemma 45. If \(S\) is strongly unambiguous as a regular expression, then \(\downarrow S\) is strongly unambiguous as a DNF regular expression.

Proof. We proceed by induction.

Case 1 (Base). \(\downarrow s = \langle [s] \rangle\), which is strongly unambiguous.

Case 2 (Empty). \(\downarrow \emptyset = \langle \rangle\), which is strongly unambiguous.
Case 3 (Star). Let $S = \llbracket S'' \rrbracket$ be strongly unambiguous. $\llbracket S'' \rrbracket = D(\llbracket S' \rrbracket)$ By IH, $\llbracket S' \rrbracket$ is strongly unambiguous. Furthermore, $L(S') = L(\llbracket S' \rrbracket)$ is unambiguously iterable, so $\llbracket S' \rrbracket$ is strongly unambiguous. This means that $D(\llbracket S' \rrbracket)$ is strongly unambiguous.

Case 4 (Concat). Let $S = S_1 \cdot S_2$ be strongly unambiguous.

If $L(S) = \{\}$, by Lemma 44, $\llbracket S \rrbracket = \{\}$, which is strongly unambiguous.

Let $\llbracket S_1 \rrbracket = \langle SQ_1 \mid \ldots \mid SQ_n \rangle$. Let $\llbracket S_2 \rrbracket = \langle TQ_1 \mid \ldots \mid TQ_m \rangle$. If $L(S) \neq \{\}$, this means that $L(\llbracket S_1 \rrbracket) \neq \{\}$, and $L(\llbracket S_2 \rrbracket) \neq \{\}$. This means that $SQ_i$ is nonempty, and so is $TQ_i$, for all $i$. Furthermore, as $S$ is strongly unambiguous, and $L(S) \neq \{\}$, $S_1$ and $S_2$ are strongly unambiguous, which means so too are $\langle SQ_1 \mid \ldots \mid SQ_n \rangle$ and $\langle TQ_1 \mid \ldots \mid TQ_m \rangle$, and so too are $SQ_i$ and $TQ_i$.

As $\llbracket S_1 \rrbracket \neq \{\}$, $\llbracket S_2 \rrbracket$, $i \neq j \Rightarrow SQ_i \cap SQ_j = \emptyset$, and $i \neq j \Rightarrow TQ_i \cap TQ_j = \emptyset$ I know from Lemma 9, $(i_1, j_1) \neq (i_2, j_2) \Rightarrow L(SQ_{i_1} \circ SQ_{j_1}) \cap L(SQ_{i_2} \circ SQ_{j_2}) = \{\}$. and $L(SQ_i) \neq \{\}$.

Let $SQ_i = [s_{i,0}, A_{i,1}, \ldots, A_{i,n_i}, s_{i,n_i}]$ and $TQ_i = [t_{i,0}, B_{i,1}, \ldots, B_{i,n_i}, t_{i,n_i}]$. Furthermore, $SQ_i$ and $TQ_i$ have nonempty languages. By Lemma 8, $(s_{i,0}, A_{i,1}, \ldots, A_{i,n_i}, s_{i,n_i}) \circ (t_{i,0}, B_{i,1}, \ldots, B_{i,n_i}, t_{i,n_i})$

As $SQ_i$ and $TQ_i$ are strongly unambiguous, we know $A_{i,j}$ and $B_{i,j}$ are strongly unambiguous. So, as $(s_{i,0}, A_{i,1}, \ldots, A_{i,n_i}, s_{i,n_i}) \circ (t_{i,0}, B_{i,1}, \ldots, B_{i,n_i}, t_{i,n_i}), SQ_i \cap TQ_i$ is strongly unambiguous.

Furthermore, as $SQ_i \circ TQ_j$ is strongly unambiguous and $(i_1, j_1) \neq (i_2, j_2) \Rightarrow L(SQ_{i_1} \circ SQ_{j_1}) \cap L(SQ_{i_2} \circ SQ_{j_2}) = \{\}$, then $(SQ_1 \circ TQ_1 \mid \ldots \mid SQ_n \circ TQ_m)$.

Case 5 (Or). Let $S = S_1 \mid S_2$ be strongly unambiguous.

If $L(S) = \{\}$, by Lemma 44, $\llbracket S \rrbracket = \{\}$, which is strongly unambiguous.

Otherwise, $S_1$ and $S_2$ are strongly unambiguous, and $L(S_1) \cap L(S_2) = \{\}$. This means $\llbracket S_1 \rrbracket$ and $\llbracket S_2 \rrbracket$ are also strongly unambiguous, by IH.

Let $\llbracket S_1 \rrbracket = \langle SQ_1 \mid \ldots \mid SQ_n \rangle$. Let $\llbracket S_2 \rrbracket = \langle TQ_1 \mid \ldots \mid TQ_n \rangle$. Let $SQ'_i = \langle SQ_1 \mid \ldots \mid TQ_{i-1} \rangle$ if $i \leq n$

As $\llbracket S_1 \rrbracket$ and $\llbracket S_2 \rrbracket$ are strongly unambiguous, $i \neq j \Rightarrow L(SQ_i) \cap L(SQ_j) = \{\}$ and $i \neq j \Rightarrow L(TQ_i) \cap L(TQ_j) = \{\}$. Furthermore, as $\bigcup_{i \in [1,n]} L(SQ_i) \cap \bigcup_{j \in [1,n]} L(SQ_j)$, from Lemma 10, $i \neq j \Rightarrow L(SQ'_i) \cap L(SQ'_j) = \{\}$, and as each $SQ_i$ and $TQ_i$ is strongly unambiguous, $\llbracket S_1 \bigoplus \llbracket S_2 \rrbracket$ is strongly unambiguous.

Lemma 46. If $DS = \langle SQ_1 \mid \ldots \mid SQ_n \rangle$ is strongly unambiguous, and for all $i, \llbracket S_i \rrbracket$ is strongly unambiguous, then $\llbracket DS \rrbracket$ is strongly unambiguous.

Proof. By induction on $n$

Case 1 ($n = 0$). $\llbracket \{\} \rrbracket = \emptyset$, which is strongly unambiguous.

Case 2 ($n > 0$). $\llbracket S_1 \mid \ldots \mid S_{n-1} \mid S_n \rrbracket = \llbracket S_1 \mid \ldots \mid S_{n-1} \rrbracket \mid \llbracket S_n \rrbracket$. By IH, $\llbracket S_1 \mid \ldots \mid S_{n-1} \rrbracket$ is strongly unambiguous. Furthermore, as $DS$ is strongly unambiguous, by Lemma 10, $L(\llbracket S_1 \mid \ldots \mid S_{n-1} \rrbracket \cap \llbracket S_n \rrbracket) = \emptyset$, so $\llbracket S_n \rrbracket$ is strongly unambiguous, so the entire thing is strongly unambiguous.

Lemma 47. If $SQ = [s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n]$ is strongly unambiguous, and for all $i, \llbracket A_i \rrbracket$ is strongly unambiguous, then $\llbracket SQ \rrbracket$ is strongly unambiguous.

Proof. By induction on $n$

Case 1 ($n = 0$). $\llbracket [s_0] \rrbracket = s_0$, which is strongly unambiguous.
Case 2 \( (n > 0) \). \[ \| s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n = \| s_0 \cdot A_1 \cdot \ldots \cdot A_{n-1} \cdot s_{n-1} \cdot \| A_n \cdot \| s_n \]. From \| A_n \] and \| s_n \, we know \| A_n \cdot s_n \ because the second part will always be \( s_n \), so the first part must be the same. By IH, \| SQ_1 \| \ldots | SQ_{n-1} \) is strongly unambiguous. Furthermore, as SQ is strongly unambiguous, by Lemma 8, \( L(\| s_0 \cdot A_1 \cdot \ldots \cdot A_{n-1} \cdot s_{n-1} \cdot A_n \cdot s_n) \), so as each side is also is strongly unambiguous, the entire thing is strongly unambiguous.

\[ \square \]

Lemma 48.

- If DS is strongly unambiguous as a DNF regular expression, then \( \| DS \) is strongly unambiguous as a regular expression
- If SQ is strongly unambiguous as a sequence, then \( \| SQ \) is strongly unambiguous as a sequence
- If A is strongly unambiguous as an atom, then \( \| A \) is strongly unambiguous as an atom

Proof.

Case 1 (MultiOr). Let DS = \( \langle SQ_1 | \ldots | SQ_n \rangle \). By IH, \( \| SQ \) is strongly unambiguous. By Lemma 46, \( \| DS \) is strongly unambiguous.

Case 2 (MultiConcat). Let SQ = \[ s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n \]. By IH, \( \| A \) is strongly unambiguous. By Lemma 47, \( \| SQ \) is strongly unambiguous.

Case 3 (StarAtomType). Let \( A = DS^* \). By IH, \( \| DS \) is strongly unambiguous. As DS* is strongly unambiguous, \( DS^* \), so \( L(DS^*) \), so \( (\| DS^*) \). So DS* is strongly unambiguous.

\[ \square \]

Definition 13 (Parallel Rewriting Without Reordering).

\[
\begin{align*}
& \text{ATOM UNROLLSTAR} & & \text{ATOM UNROLLSTAR_R} \\
& DS^* \xrightarrow{\| A ([e])} (DS \odot ([DS^*])) & & DS^* \xrightarrow{\| A ([e])} ([DS^*] \odot DS) \\
& \text{PARALLEL ATOM STRUCTURAL REWRITE} & & \text{PARALLEL ATOM STRUCTURAL REWRITE} \\
& DS \xrightarrow{} DS^* & & DS^* \xrightarrow{} ([DS^*]) \\
& DS = \langle SQ_1 | \ldots | SQ_n \rangle & & DS = \langle s_{i,0} | A_{i,1} \cdot \ldots \cdot A_{i,n_j} \cdot s_{i,n_j} \rangle \\
& \forall i, j, A_i, j \xrightarrow{} DS_{i,j} & & DS_i = \langle s_{i,0} \rangle \otimes DS_{i,1} \otimes \ldots \otimes DS_{i,n_j} \odot \langle s_{i,n_j} \rangle \\
& DS \xrightarrow{} DS_1 \oplus \ldots \oplus DS_n & & DS \xrightarrow{} DS \\
& \text{IDENTITY REWRITE} & & \text{IDENTITY REWRITE} \\
& DS \xrightarrow{} DS & & DS \xrightarrow{} DS
\end{align*}
\]
Definition 14 (Parallel Rewriting With Reordering).

\[
\text{ATOM UNROLLSTARL}
\]

\[
\begin{array}{l}
\text{ATOM UNROLLSTARLN} \\
\text{DNF Reorder} \\
\text{Definition 49 ( \(\leftrightarrow\) Maintained Under Iteration). Let } DS \leftrightarrow DT, \text{ then } \langle DS^* \rangle \leftrightarrow \langle DT^* \rangle. \\
\text{Proof. Consider the derivation}
\end{array}
\]

\[
\begin{array}{l}
\frac{DS \leftrightarrow DT}{DS^* \leftrightarrow_A \langle [DT^*] \rangle} \\
\frac{\langle DS^* \rangle \leftrightarrow \langle [DT^*] \rangle}{\langle DS^* \rangle} \\
\end{array}
\]

Lemma 50. If \(DS \leftrightarrow DS\) through an application of Identity Rewrite, then \(DS \leftrightarrow DS\) through an application of Parallel DNF Structural Rewrite.

Proof. Let \(DS \leftrightarrow DS\) through an application of Identity Rewrite.

Let \(DS = \langle SQ_1 \mid \ldots \mid SQ_n \rangle\). Let \(SQ_i = [s_{i,0} \cdot A_{i,1} \cdot \ldots \cdot A_{i,n_i} \cdot s_{i,n_i}]\). By Lemma 70, \(A_{i,j} \leftrightarrow A D(A_{i,j})\). Define \(DS \leftrightarrow DS\) as \(D(DS_i)\).

Define \(DS_i\) as \(\langle [s_{i,0}] \rangle \circ DS_{i,1} \circ \ldots \circ DS_{i,n_i} \circ \langle [s_{i,n_i}] \rangle\), so as \(DS_{i,j} = D(A_{i,j})\), through the definition of \(\circ\), \(DNFRegEx_i = \langle SQ_i \rangle\).

By the definition of \(\oplus\), \(\langle SQ_1 \rangle \oplus \ldots \oplus \langle SQ_n \rangle = \langle SQ_1 \mid \ldots \mid SQ_n \rangle = DS\).

So \(DS \leftrightarrow DS\), with the final rule being an application of Identity Rewrite.

Lemma 51 ( \(\leftrightarrow\) Maintained Under \(\oplus\)). Let \(DS \leftrightarrow DS'\) and \(DT \leftrightarrow DT'\) then \(DS \oplus DT \leftrightarrow DS' \oplus DT'\).

Proof. By Lemma 50, a derivation with the final rule being an application of Identity Rewrite, can be converted into a derivation with the final rule being an application of Parallel DNF Structural Rewrite. So we can assume that the final rule of each is an application of Parallel DNF Structural Rewrite.
Consider the derivation
\[
D(T) = \langle TQ_i \mid \ldots \mid SQ_m \rangle \quad \forall i. TQ_i = [t_{i,0} \cdot B_{i,1} \cdot \ldots \cdot B_{i,m_i} \cdot t_{i,m_i}]
\]

\[
\forall i, j. B_{i,j} \mapsto_A DT_{i,j} \quad \forall i. DT_i = ([t_{i,0}] \circ DT_{i,1} \circ \ldots \circ DT_{i,n_i} \circ [t_{i,n_i}] )
\]

\[
DT \mapsto DT_1 \circ \ldots \circ DT_n
\]

Define \( A''_{i,j} = \begin{cases} A_{i,j} & \text{if } i \leq n \\ B_{i-n,j} & \text{if } i > n \end{cases} \)

Define \( s''_{i,j} = \begin{cases} s_{i,j} & \text{if } i \leq n \\ t_{i-n,j} & \text{if } i > n \end{cases} \)

Define \( n''_i = \begin{cases} n_i & \text{if } i \leq n \\ m_{i-n} & \text{if } i > n \end{cases} \)

Define \( SQ''_i = [s''_{i,0} \cdot A''_{i,1} \cdot \ldots \cdot A''_{i,n''_i} \cdot s''_{i,n''_i}] \). By inspection, \( SQ''_i = \begin{cases} SQ_i & \text{if } i \leq n \\ TQ_{i-n} & \text{if } i > n \end{cases} \).

Define \( DS'' = \langle SQ''_1 \mid \ldots \mid SQ''_{n+m} \rangle \). By inspection, \( DS'' = DS \oplus DT \).

Define \( DS''_{i,j} = \begin{cases} DS_{i,j} & \text{if } i \leq n \\ DT_{i-n,j} & \text{if } i > n \end{cases} \).

Define \( DS''_{i,j} = \{ DS_i \mid i \leq n \} \). By inspection \( A''_{i,j} \mapsto DS''_{i,j} \).

This means that \( DS'' \oplus \ldots \oplus DS''_{n+m} = (DS_1 \oplus \ldots \oplus DS_n \oplus DT_1 \oplus \ldots \oplus DT_m) = DS' \oplus DT' \).

Consider the derivation
\[
DS'' = \langle SQ''_1 \mid \ldots \mid SQ''_{n+m} \rangle \quad \forall i. SQ''_i = [s''_{i,0} \cdot A''_{i,1} \cdot \ldots \cdot A''_{i,n''_i} \cdot s''_{i,n''_i}]
\]

\[
\forall i, j. A''_{i,j} \mapsto_A DS''_{i,j} \quad \forall i. DS''_i = \langle [s''_{i,0}] \circ DS''_{i,1} \circ \ldots \circ DS''_{i,n_i} \circ [s''_{i,n_i}] \rangle
\]

\[
DS'' \mapsto DS''_1 \circ \ldots \circ DS''_{n+m}
\]

\( \square \)

**Lemma 52** (\( \mapsto^n \) Maintained Under Iteration). Let \( DS \mapsto^n DT \), then \( \mathcal{D}(DS') \mapsto^n \mathcal{D}(DT') \).

**Proof.** By induction on the derivation of \( \mapsto^n \).

**Case 1 (Reflexivity).**

\[
\begin{array}{c}
DS \mapsto^n DS
\end{array}
\]

By reflexivity rule

\[
\mathcal{D}(DS') \mapsto^n \mathcal{D}(DS')
\]

**Case 2 (Base).**

\[
DS \mapsto DT
\]

By Lemma 49, \( \mathcal{D}(DS') \mapsto \mathcal{D}(DT') \)

Consider the derivation

\[
\mathcal{D}(DS') \mapsto \mathcal{D}(DT')
\]

\[
\frac{\mathcal{D}(DS') \mapsto^n \mathcal{D}(DT')} \mathcal{D}(DS') \mapsto^n \mathcal{D}(DT')
\]

**Case 3 (Transitivity).**

\[
\begin{array}{c}
DS \mapsto^n DS' \quad DS' \mapsto^* DT
\end{array}
\]

\[
DS \mapsto^* DT
\]
By IH, $\mathcal{D}(DS^*) \leftrightarrow^* \mathcal{D}(DS^*)$ and $\mathcal{D}(DS^*) \leftrightarrow^* \mathcal{D}(DT^*)$.

\[ \square \]

**Lemma 53** (Equivalence of $\uparrow \circ \downarrow$). $(\uparrow \circ \downarrow)S \equiv^S S$

**Proof.** By induction on the structure of $S$

**Case 1** (Base). $(\uparrow \circ \downarrow)s = \uparrow(\downarrow s) = 0 \mid s$

$0 \mid s \equiv^S s$

**Case 2** (Empty). $(\uparrow \circ \downarrow)\emptyset = \uparrow(\downarrow \emptyset) = \emptyset$

$0 \equiv^S \emptyset$

**Case 3** (Star). $(\uparrow \circ \downarrow)S^* = \uparrow(\{e \cdot (\downarrow S^*)^* \cdot e\}) = 0 \mid (e \cdot ((\uparrow \circ \downarrow)S^*)^* \cdot e)$ Then, through application of equation theory transitivity, $+ \text{Ident}, \cdot \text{Ident}_L$, and $\cdot \text{Ident}_R$. We get $(\uparrow \circ \downarrow)S^* \equiv^S ((\uparrow \circ \downarrow)S)^*$ By application of the IH, and transitivity, we get $(\uparrow \circ \downarrow)S^* \equiv^S S^*$

**Case 4** (Concat). Let $(\uparrow \circ \downarrow)(S_1 \cdot S_2) = (\uparrow(\downarrow S_1 \circ \downarrow S_2)$. Let $\downarrow S_1 = \langle SQ_1 \mid \ldots \mid SQ_n \rangle$ and $\downarrow S_2 = \langle TQ_1 \mid \ldots \mid TQ_m \rangle$.

$S_1 \equiv^S (\uparrow \circ \downarrow)(S_1) = (\emptyset \mid (\uparrow SQ_1 \mid \ldots \mid (\uparrow SQ_n \ldots))) \equiv^S \uparrow SQ_1 \mid \ldots \mid \uparrow SQ_n$ and $S_2 \equiv^S (\uparrow \circ \downarrow)(S_2) = (\emptyset \mid (\uparrow TQ_1 \mid \ldots \mid (\uparrow TQ_m \ldots))) \equiv^S \uparrow TQ_1 \mid \ldots \mid \uparrow TQ_m$

So by structural Concat identity, and transitivity, $S_1 \cdot S_2 \equiv^S (\uparrow SQ_1 \mid \ldots \mid \uparrow SQ_n) \cdot (\uparrow TQ_1 \mid \ldots \mid \uparrow TQ_m)$

Through repeated application of $\text{Dist}_R$ and $\text{Dist}_L$, $S_1 \cdot S_2 \equiv^S (\uparrow SQ_1 \cdot \uparrow TQ_1 \mid \ldots \mid \uparrow SQ_n \cdot \uparrow TQ_m)$.

Now, I want to show $\uparrow SQ_1 \cdot \uparrow TQ_j \equiv^S (\uparrow SQ_1 \circ \uparrow TQ_j)$. Let $SQ_1 = [s_{i,0} \cdot A_{i,1} \cdot \ldots \cdot A_{i,n_i} \cdot s_{i,n_i}]$, and $TQ_j = [t_{j,0} \cdot B_{j,1} \cdot \ldots \cdot A_{j,m_j} \cdot s_{j,n_j}]$. $\uparrow SQ_1 \circ \uparrow TQ_j = \uparrow [s_{i,0} \cdot A_{i,1} \cdot \ldots \cdot A_{i,n_i} \cdot s_{i,n_i} \cdot t_{j,0} \cdot B_{j,1} \cdot \ldots \cdot A_{j,m_j} \cdot s_{j,n_j} \ldots)]$ So through repeated application of $\text{Assoc}$, $\uparrow (SQ_1 \circ SQ_j) \equiv^S (s_{i,0} \cdot (A_{i,1} \cdot (\ldots (A_{i,n_i} \cdot s_{i,n_i}) \ldots))) \cdot (t_{j,0} \cdot (B_{j,1} \cdot (\ldots (B_{j,n_j} \cdot s_{j,n_j}) \ldots)))$.

Because of this $S_1 \cdot S_2 \equiv^S (\uparrow SQ_1 \circ SQ_j \mid \ldots \mid \uparrow SQ_n \circ SQ_j)$ Through repeated application of $\text{Assoc}$, and $+ \text{Ident}, S_1 \cdot S_2 \equiv^S 0 \mid [(\uparrow SQ_1 \cdot \uparrow TQ_1) \mid \ldots \mid (\uparrow SQ_n \cdot \uparrow TQ_m) \ldots]$. Furthermore, $0 \mid (\uparrow (SQ_1 \cdot (\uparrow TQ_1) \mid \ldots \mid (\uparrow SQ_n \cdot (\uparrow TQ_m) \ldots)) = \uparrow (\uparrow SQ_1 \cdot (\uparrow TQ_1 \mid \ldots \mid (\uparrow SQ_n \cdot (\uparrow TQ_m) \ldots)) = \uparrow (\uparrow SQ_1 \cdot (\uparrow TQ_1 \mid \ldots \mid (\uparrow SQ_n \cdot (\uparrow TQ_m) \ldots))$ as desired.

**Case 5** (Or). Let $(\uparrow \circ \downarrow)(S_1 \mid S_2) = (\uparrow(\downarrow S_1 \oplus \downarrow S_2)$. Let $\downarrow S_1 = \langle SQ_1 \mid \ldots \mid SQ_n \rangle$ and $\downarrow S_2 = \langle TQ_1 \mid \ldots \mid TQ_m \rangle$. So $(\uparrow \circ \downarrow)(S_1 \mid S_2) = (\uparrow SQ_1 \mid \ldots \mid SQ_n \mid TQ_1 \mid \ldots \mid TQ_m) = \emptyset \mid (\uparrow SQ_1 \mid (\ldots \mid (\uparrow SQ_n \ldots) \ldots))$. Through applying associativity a lot, and $+ \text{Ident}$ once, I get $(\uparrow \circ \downarrow)(S_1 \mid S_2) = (\emptyset \mid (\uparrow SQ_1 \mid (\ldots \mid (\uparrow SQ_n \ldots) \ldots)) = (\uparrow \circ \downarrow)S_1$ and $(\emptyset \mid (\uparrow SQ_1 \mid (\ldots \mid (\uparrow SQ_n \ldots) \ldots)) = (\uparrow \circ \downarrow)S_2$.

Through an application of structural $\text{Or}$ equality, $(\emptyset \mid (\uparrow SQ_1 \mid (\ldots \mid (\uparrow SQ_n \ldots) \ldots))) \equiv (\emptyset \mid (\uparrow TQ_1 \mid (\ldots \mid (\uparrow TQ_m \ldots) \ldots)) \equiv^S S_1 \mid S_2$, as desired.

\[ \square \]

**Lemma 54** (Equivalence of Preimage of $\downarrow$). If $\downarrow S \equiv \downarrow T$, then $S \equiv^S T$

**Proof.** $\downarrow S \equiv \downarrow T$, so $(\uparrow \circ \downarrow)S = (\uparrow \circ \downarrow)T$. By Lemma 53, $S \equiv^S (\uparrow \circ \downarrow)T \equiv^S T$

\[ \square \]

**Lemma 55** (Equivalence of Adjacent Swapping Permutation of $\text{Or}$). Let $S_1 \mid \ldots \mid S_n$. Let $\sigma_i$ be an adjacent swapping permutation. $S_1 \mid \ldots \mid S_n \equiv^S S_{\sigma_i(1)} \mid \ldots \mid S_{\sigma_i(n)}$

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Proof. \( S_1 \mid \ldots \mid S_n \equiv^* (S_1 \mid \ldots \mid S_{i-1}) \mid (S_i \mid S_{i+1}) \mid (S_{i+2} \mid \ldots \mid S_n) \) by repeated application of associativity.

\( S_1 \mid S_{i+1} = S_{i+1} \mid S_1 \) by \( Or \) commutativity, so by \( Or \) structural equality,

\( (S_1 \mid \ldots \mid S_{i-1}) \mid (S_i \mid S_{i+1}) \mid (S_{i+2} \mid \ldots \mid S_n) \equiv^* (S_1 \mid \ldots \mid S_{i-1}) \mid (S_{i+1} \mid S_i) \mid (S_{i+2} \mid \ldots \mid S_n) \)

\( (S_1 \mid \ldots \mid S_{i-1}) \mid (S_{i+1} \mid S_i) \mid (S_{i+2} \mid \ldots \mid S_n) \equiv^* S_{\sigma_1(n)} \mid \ldots \mid S_{\sigma_1(n)} \) by repeated application of associativity.

So, by the transitivity of equational theories, \( S_1 \mid \ldots \mid S_n \equiv^* S_{\sigma_1(1)} \mid \ldots \mid S_{\sigma_1(n)} \).

\[\square\]

Lemma 56 (Expressibility of \( \leftrightarrow_{\text{swap}} \) in \( \equiv^* \) Up To Preimage).

1. If \( \| S \leftrightarrow_{\text{swap}} \| T \), then \( S \equiv^* T \).
2. If \( \| S = \langle [A] \rangle \) and \( A \leftrightarrow_{\text{swap}} A \), then \( S \equiv^* T \).

Proof. By mutual induction on the derivation of \( \leftrightarrow_{\text{swap}} \) and \( \leftrightarrow_{\text{swap}} A \).

Case 1 (Atom UnrollStarL). Let \( \| S = \langle [A] \rangle \), and \( A \leftrightarrow_{\text{swap}} A \). This means that \( A = DS^* \) and \( \| T = \langle [e] \rangle \) (with \( DS = \langle DS^* \rangle \)).

Let \( S' = \| DS \). As \( \| S'' = \langle [DS^*] \rangle \), then from Lemma 54, \( S'' \equiv^* S \). Similarly, as \( \| (e \mid (S' \cdot S'')) = \langle [e] \rangle \) (with \( DS = \langle DS^* \rangle \)), then from Lemma 54, \( e \mid (S' \cdot S'') \equiv^* T \).

So, through an application of UnrollStarLeftRule, \( S \equiv^* S' \equiv^* e \mid (S' \cdot S'') \equiv^* T \), as desired.

Case 2 (Atom UnrollStarR). Let \( \| S = \langle [A] \rangle \), and \( A \leftrightarrow_{\text{swap}} A \). This means that \( A = DS^* \) and \( \| T = \langle [e] \rangle \) (with \( DS = \langle [DS^*] \rangle \)).

Let \( S' = \| DS \). As \( \| S'' = \langle [DS^*] \rangle \), then from Lemma 54, \( S'' \equiv^* S \). Similarly, as \( \| (e \mid (S' \cdot S'')) = \langle [e] \rangle \) (with \( DS = \langle [DS^*] \rangle \)), then from Lemma 54, \( e \mid (S' \cdot S'') \equiv^* T \).

So, through an application of UnrollStarRightRule, \( S \equiv^* S' \equiv^* e \mid (S' \cdot S'') \equiv^* T \), as desired.

Case 3 (Parallel Swap Atom Structural Rewrite). Let \( \| S = \langle [A] \rangle \), and \( A \leftrightarrow_{\text{swap}} A \). This means that \( A = DS^* \) and \( \| T = \langle [DT] \rangle \) where \( DS \leftrightarrow_{\text{swap}} DT \).

Let \( \| DS = S' \) and \( \| DT = T' \). By induction assumption, \( S' \equiv^* T' \). By structural equivalence, \( S'' = T'' \). As \( \| S'' = \langle [DS^*] \rangle \), from Lemma 54, \( S'' \equiv^* S \). As \( \| T'' = \langle [DT^*] \rangle \), from Lemma 54, \( T'' \equiv^* T \).

\( S \equiv^* S'' \equiv^* T'' \equiv^* T \), as desired.

Case 4 (DNF Reorder). Let \( \| S \leftrightarrow_{\text{swap}} T \), and the last step of the proof is an application of DNF Reorder. Let \( \| S = \langle SQ_1 | \ldots | SQ_n \rangle \). Then, for some \( \sigma \in S_n \), \( \| T = \langle SQ_{\sigma(1)} | \ldots | SQ_{\sigma(n)} \rangle \).

\( \| S = \| SQ_1 | \ldots | \| SQ_n \) and \( \| T = \| SQ_{\sigma(1)} | \ldots | \| SQ_{\sigma(n)} \).

\( \sigma \) can then be broken down into a number of adjacent swapping permutations, \( \sigma_1 \circ \ldots \circ \sigma_m = \sigma \).

By Lemma 55, each \( \sigma_i \) can be applied to a sequence of \( Ors \).

Consider the derivation

\[\begin{align*}
\| SQ_1 & | \ldots | \| SQ_n \equiv^* \| SQ_1 | \ldots | \| SQ_n \\
\| SQ_{\sigma_1(n)} & | \ldots | \| SQ_{\sigma_1(n)} \equiv^* \| SQ_1 | \ldots | \| SQ_n \\
& \vdots \\
\| SQ_{(\sigma_1 \circ \ldots \sigma_m)(n)} & | \ldots | \| SQ_{(\sigma_1 \circ \ldots \sigma_m)(n)} \equiv^* \| SQ_1 | \ldots | \| SQ_n
\end{align*}\]

So, by Lemma 54, \( S \equiv^* \| S \) and \( \| T \equiv^* T \). Furthermore, \( \| S \equiv^* \| T \). So by the transitivity of an equational theory, \( S \equiv^* T \).

Case 5 (Identity Rewrite). Let \( \| S \leftrightarrow_{\text{swap}} T \) by an application of Identity Rewrite.
By structural equality of $\text{Concat}$, $S_i \equiv^s T_i$.

Consider the regular expression $S' = S_1 \oplus \cdots \oplus S_n$ and the regular expression $T' = T_1 | \cdots | T_n$. By structural equality of $\text{Or}$, $S' \equiv^s T'$.

By induction on the typing derivation of $\equiv^s_{\text{swap}}$

**Case 1 (Reflexivity).** Let $\downarrow S \equiv^s_{\text{swap}} \downarrow T$, and the last step of the derivation is an application of Reflexivity.

This means $\downarrow S = \downarrow T$. That means $\upharpoonright S = \upharpoonright T$. By Lemma 54, $S \equiv^s \upharpoonright S$. By Lemma 54, $\upharpoonright T \equiv^s \upharpoonright T$. By the transitivity of equational theories, $S \equiv^s T$.

**Case 2 (Base).** Let $\downarrow S \equiv^s_{\text{swap}} \downarrow T$, and the last step of the derivation is an application of Base.

This means that $\downarrow S \leftrightarrow_{\text{swap}} \downarrow T$.

By Lemma 56, $S \equiv^s T$.

**Case 3 (Symmetry).** Let $\downarrow S \equiv^s_{\text{swap}} \downarrow T$, and the last step of the derivation is an application of Base.

This means that $\downarrow S \leftrightarrow_{\text{swap}} \downarrow T$.

By Lemma 56, $S \equiv^s T$.

**Lemma 58 (Propagation of $\leftrightarrow_{\text{swap}}$ through $\oplus$ on the left).** If $DS \leftrightarrow_{\text{swap}} DS'$, then $DS \oplus DT \leftrightarrow_{\text{swap}} DS' \oplus DT$.

**Proof.** This will be done by cases on the last step of the derivation of $\leftrightarrow_{\text{swap}}$.

**Case 1 (DNF Reorder).** Let $DS \leftrightarrow_{\text{swap}} DS'$ by an application of DNF Reorder. This means, for some $SQ_1, \ldots, SQ_n$, and some $\sigma \in S_n$, $DS = \langle SQ_1 | \cdots | SQ_n \rangle$ and $DS' = \langle SQ_{\sigma(1)} | \cdots | SQ_{\sigma(n)} \rangle$. The
DNF regular expression $DT = \langle TQ_1 \mid \ldots \mid TQ_m \rangle$ for some $TQ_1, \ldots, TQ_m$. Let $id_m$ be the identity permutation on $m$ elements. Define $\sigma' = \sigma \circ id_m$. Define $SQ'_i = \begin{cases} SQ_i \text{ if } i \leq n \\ TQ_{i-n} \text{ otherwise} \end{cases}$.

$\langle SQ'_1 \mid \ldots \mid SQ'_{n+m} \rangle = DS \oplus DT$.

$\langle SQ'_{\sigma(1)} \mid \ldots \mid SQ'_{\sigma(n+m)} \rangle = (SQ_{\sigma(1)} \mid \ldots \mid SQ_{\sigma(n)} \mid TQ_1 \mid \ldots \mid TQ_m) = DS' \oplus DT$.

Consider the derivation

$\langle SQ'_1 \mid \ldots \mid SQ'_{n+m} \rangle \leftrightarrow^{swap} \langle SQ'_{\sigma(1)} \mid \ldots \mid SQ'_{\sigma(n+m)} \rangle$ as desired.

**Case 2 (Parallel Swap Atom Structural Rewrite).** Let $DS \leftrightarrow^{swap} DS'$ by an application of Parallel Swap Atom Structural Rewrite.

$$DS = \langle SQ_1 \mid \ldots \mid SQ_n \rangle \quad \forall_i SQ_i = [s_{i,0} \cdot A_{i,1} \mid \ldots \mid A_{i,n_i} \cdot s_{i,n_i}],$$

$$\forall_{i,j} A_{i,j} \leftrightarrow^{swap} A \quad \forall_i DS_i = \langle \{s_{i,0}\} \circ DS_{i,1} \circ \ldots \circ DS_{i,n_i} \circ \{s_{i,n_i}\} \rangle$$

$$DS \leftrightarrow^{swap} DS_1 \oplus \ldots \oplus DS_n$$

Let $DT = \langle TQ_1 \mid \ldots \mid TQ_m \rangle$. Let $TQ_i = [i_0 \cdot B_{i,1} \mid \ldots \mid B_{i,m_i} \cdot t_{i,m_i}]$.

Let $k_i = \begin{cases} n_i \text{ if } i \leq n \\ m_i \text{ otherwise} \end{cases}$

Let $SQ''_i = \begin{cases} SQ_i \text{ if } i \leq n \\ TQ_{i-n} \text{ otherwise} \end{cases}$.

Let $DS'' = DS \oplus DT = \langle SQ''_1 \mid \ldots \mid SQ''_{n+m} \rangle$. Let $A''_{i,j} = \begin{cases} A_{i,j} \text{ if } i \leq n \\ B_{i-n,j} \text{ otherwise} \end{cases}$

Let $s''_{i,j} = \begin{cases} s_{i,j} \text{ if } i \leq n \\ t_{i-n,j} \text{ otherwise} \end{cases}$

Let $DS''_{i,j} = \begin{cases} DS_{i,j} \text{ if } i \leq n \\ D(B_{i-n,j}) \text{ otherwise} \end{cases}$

If $i \leq n$, by assumption $A''_{i,j} = A_{i,j} \leftrightarrow^{swap} A \quad DS_{i,j} = DS''_{i,j}$. If $i > n$, by Parallel Swap Atom Structural Rewrite, $A''_{i,j} = B_{i-n,j} \leftrightarrow^{swap} D(B_{i-n,j}) = DS''_{i,j}$.

Let $DS''_i = \begin{cases} DS_i \text{ if } i \leq n \\ \langle \{t_{i-n,0}\} \circ B_{i-n,1} \cdot \ldots \circ B_{i-n,m_i} \circ t_{i-n,m_i} \rangle \text{ otherwise} \end{cases}$

For $i > n$, $DS''_i = \langle \{t_{i-n,0}\} \circ B_{i-n,1} \cdot \ldots \circ B_{i-n,k_i} \circ t_{i-n,k_i} \rangle = \langle TQ_i \rangle$ through application of $\circ$ on many singleton DNF regular expressions.

$DS''_{n+1} \oplus \ldots \oplus DS''_{n+m} = \langle TQ_1 \rangle \oplus \ldots \oplus \langle TQ_m \rangle = \langle TQ_1 \mid \ldots \mid TQ_m \rangle$ through repeated application of $\oplus$ to singleton DNFs.

As $DS''_0 \oplus \ldots \oplus DS''_n = DS_0 \oplus \ldots \oplus DS_n = DS'$, and $DS''_{n+1} \oplus \ldots \oplus DS''_{n+m} = DT$, we get $DS''_0 \oplus \ldots \oplus DS''_{n+m} = DS' \oplus DT$.

Consider the derivation

$$DS'' = \langle SQ''_1 \mid \ldots \mid SQ''_{n+m} \rangle \quad \forall_i SQ''_i = [s''_{i,0} \cdot A''_{i,1} \mid \ldots \mid A''_{i,n_i} \cdot s''_{i,n_i}],$$

$$\forall_{i,j} A''_{i,j} \leftrightarrow^{swap} A \quad \forall_i DS''_i = \langle \{s''_{i,0}\} \circ DS''_{i,1} \circ \ldots \circ DS''_{i,n_i} \circ \{s''_{i,n_i}\} \rangle$$

$$DS'' \leftrightarrow^{swap} DS''_0 \oplus \ldots \oplus DS''_{n+m}$$

as desired.

**Lemma 59 (Propagation of $\leftrightarrow^{swap}$ through $\oplus$ on the right).** If $DS \leftrightarrow^{swap} DS'$, then $DS \oplus DT \leftrightarrow^{swap} DS' \oplus DT$

**Proof.** Proceeds as Lemma 58, but on the right. □
Lemma 60 (Propagation of $\equiv_{\oplus}$ through $\oplus$ on the left). If $DS \equiv_{\oplus} DS'$, then $DS \oplus DT \equiv_{\oplus} DS' \oplus DT$

Proof. By induction on the last step of the derivation of $DS \equiv_{\oplus} DS'$.

Case 1 (Reflexivity). If $DS \equiv_{\oplus} DS'$ through an application of Reflexivity, then $DS' = DS$. So, through reflexivity, $DS \oplus DT \equiv_{\oplus} DS' \oplus DT$.

Case 2 (Base). If $DS \equiv_{\oplus} DS'$ through an application of Reflexivity, then $DS' \leftrightarrow^{swap} DS$. From Lemma 58 $DS \oplus DT \leftrightarrow^{swap} DS' \oplus DT$, so $DS \oplus DT \equiv_{\oplus} DS' \oplus DT$.

Case 3 (Transitivity). If $DS \equiv_{\oplus} DS'$ through an application of Transitivity, then there exists a $DS''$ such that $DS \equiv_{\oplus} DS''$ and $DS'' \equiv_{\oplus} DS'$. By IH, $DS \oplus DT \equiv_{\oplus} DS'' \oplus DT$ and $DS'' \oplus DT \equiv_{\oplus} DS' \oplus DT$.

This gives us the derivation

$$DS \oplus DT \equiv_{\oplus} DS' \oplus DT \quad DS'' \oplus DT \equiv_{\oplus} DS' \oplus DT$$

$$\therefore DS \oplus DT \equiv_{\oplus} DS' \oplus DT$$

Lemma 61 (Propagation of $\equiv_{\oplus}$ through $\oplus$ on the right). If $DS \equiv_{\oplus} DS'$, then $DS \oplus DT \equiv_{\oplus} DS' \oplus DT$

Proof. Proceeds as Lemma 60, but on the right.

Lemma 62 (Propagation of $\equiv_{\oplus}$ through $\oplus$). If $DS \equiv_{\oplus} DS'$ and $DT \equiv_{\oplus} DT'$, then $DS \oplus DT \equiv_{\oplus} DS' \oplus DT'$.

Proof. By Lemma 60, $DS \oplus DT \equiv_{\oplus} DS' \oplus DT$. By Lemma 61, $DS' \oplus DT \equiv_{\oplus} DS' \oplus DT'$. Consider the derivation

$$DS \oplus DT \equiv_{\oplus} DS' \oplus DT \quad DS' \oplus DT \equiv_{\oplus} DS' \oplus DT'$$

$$\therefore DS \oplus DT \equiv_{\oplus} DS' \oplus DT'$$

Lemma 63 (Propagation of $\leftrightarrow^{swap}$ through $\ominus$ on the left). If $DS \leftrightarrow^{swap} DS'$, then $DS \ominus DT \leftrightarrow^{swap} DS' \ominus DT$

Proof. By induction on the derivation of $\leftrightarrow^{swap}$.

Case 1 (DNF Reorder). Let $DS \leftrightarrow^{swap} DS'$ by an application of DNF Reorder. This means, for some $SQ_1, \ldots, SQ_n$, and some $\sigma \in S_n$, $DS = \langle SQ_1 | \ldots | SQ_n \rangle$ and $DS' = \langle SQ_{\sigma(1)} | \ldots | SQ_{\sigma(n)} \rangle$. The DNF regular expression $DT = \langle TQ_1 | \ldots | TQ_m \rangle$ for some $TQ_1, \ldots, TQ_m$. Let $id_m$ be the identity permutation on $m$ elements. Define $\sigma' = \sigma \cdot id_m$. Define $SQ_{i,j} = SQ_i \cdot SQ_j$.

By definition of $\ominus$, $\langle SQ_{i,1} \cdot \ldots \cdot SQ_{n,m} \rangle = \langle SQ_{i} \ominus TQ_1 | \ldots | SQ_{n} \ominus TQ_m \rangle = DS \ominus DT$.

By the definition of $\ominus$ and $\ominus$, $\langle SQ_{\sigma'(1),1} \cdot \ldots \cdot SQ_{\sigma'(n),m} \rangle = \langle SQ_{\sigma(1),1} | \ldots | SQ_{\sigma(n),m} \rangle = \langle SQ_{\sigma(1)} \cdot TQ_1 | \ldots | SQ_{\sigma(n)} \cdot TQ_m \rangle = DS' \ominus DT$.

Case 2 (Parallel Swap DNF Structural Rewrite). Let $DS \leftrightarrow^{swap} DS'$ by an application of Parallel Swap DNF Structural Rewrite.

$$DS = \langle SQ_1 | \ldots | SQ_n \rangle \quad \forall i. (SQ_i) = [s_i,0 \cdot A_{i,1} \cdot \ldots \cdot A_{i,n_i} \cdot s_{i,n_i}]$$

$$\forall i,j.A_{i,j} \leftrightarrow^{A} DS_{i,j} \quad \forall i. DS_i = (\{s_{i,0}\} \ominus DS_{i,1} \ominus \ldots \ominus DS_{i,n_i} \ominus \{s_{i,n_i}\})$$

$$\therefore DS \leftrightarrow^{swap} DS_1 \ominus \ldots \ominus DS_n$$
Lemma 66

and

DS

Let DS'' = DS ⊗ DT = ⟨SQ''_1 | ... | SQ''_n,m⟩. Let A''_{i,j,k} = \{A_{i,j,k} if k ≤ n_i \}

\text{otherwise Let } s''_{i,j,k} = \begin{cases} s_{i,k} & \text{if } i < n_i \\ s_{i,n_i} \cdot t_{j,0} & \text{if } i = n_i \\ t_{j,k-n_i} & \text{otherwise} \end{cases}

Let DS''_{i,j,k} = DS_i \times DS_j \times \{\text{Propagate } A''_{i,j,k} = A_{i,j,k} \rightarrow parallel swap DS_{i,j,k} = DS''_{i,j,k}. If } k > n_i, \text{ by PARALLEL SWAP ATOM.}

\text{Structural Rewrite, } A''_{i,j,k} = B_{i,j-k-n_i} \rightarrow parallel swap D(B_{j,k-n_i}) = DS''_{i,j,k}.

\text{Through repeated application of } ○ \text{ on singletons, } DS_{i,j} = DS_i \circ \langle T_Q \rangle_j. \text{This means } DS_{i,j} = DS_0 \circ ... \circ DS_{n,m} = (DS_1 \circ ... \circ DS_n) \circ DT = DS' \circ DT.

Consider the derivation

\text{as desired.}

Lemma 64 (Propagation of } \rightarrow parallel swap\text{ through } ○ \text{ on the right). If } DT \rightarrow parallel swap DT', \text{ then } DS \circ DT \rightarrow parallel swap DS \circ DT'.

\text{Proof. Proceeds as Lemma 58, but on the right.}

Lemma 65 (Propagation of } \equiv parallel swap\text{ through } ○ \text{ on the left). If } DS \equiv parallel swap DS', \text{ then } DS \circ DT \equiv parallel swap DS' \circ DT.

\text{Proof. By induction on the last step of the derivation of } DS \equiv parallel swap DS'.

\text{Case 1 (Reflexivity). If } DS \equiv parallel swap DS' \text{ through an application of Reflexivity, then } DS' = DS. \text{So, through reflexivity, } DS \circ DT \equiv parallel swap DS \circ DT.

\text{Case 2 (Base). If } DS \equiv parallel swap DS' \text{ through an application of Reflexivity, then } DS' \rightarrow parallel swap DS. \text{From Lemma 63 } DS \circ DT \rightarrow parallel swap DS' \circ DT, \text{ so } DS \circ DT \equiv parallel swap DS' \circ DT.

\text{Case 3 (Transitivity). If } DS \equiv parallel swap DS' \text{ through an application of Transitivity, then there exists a } DS'' \text{ such that } DS \equiv parallel swap DS'' \text{ and } DS'' \equiv parallel swap DS'. \text{ By IH, } DS \circ DT \equiv parallel swap DS'' \circ DT \text{ and } DS'' \circ DT \equiv parallel swap DS' \circ DT.

\text{This gives us the derivation}

\text{as desired.}

Lemma 66 (Propagation of } \equiv parallel swap\text{ through } ○ \text{ on the right). If } DS \equiv parallel swap DS', \text{ then } DS \circ DT \equiv parallel swap DS' \circ DT.

\text{Proof. Proceeds as Lemma 65, but on the right.
Lemma 67 (Propagation of $\equiv_{\leftrightarrow\text{swap}}$ through $\circ$). If $DS \equiv_{\leftrightarrow\text{swap}} DS'$ and $DT \equiv_{\leftrightarrow\text{swap}} DT'$, then $DS \circ DT \equiv_{\leftrightarrow\text{swap}} DS' \circ DT'$.

Proof. By Lemma 65, $DS \circ DT \equiv_{\leftrightarrow\text{swap}} DS' \circ DT$. By Lemma 66, $DS' \circ DT \equiv_{\leftrightarrow\text{swap}} DS' \circ DT'$.

Consider the derivation

\[
\frac{DS \circ DT \equiv_{\leftrightarrow\text{swap}} DS' \circ DT \quad DS' \circ DT \equiv_{\leftrightarrow\text{swap}} DS' \circ DT'}{DS \circ DT \equiv_{\leftrightarrow\text{swap}} DS' \circ DT'}
\]

Lemma 68 (Propagation of $\equiv_{\leftrightarrow\text{swap}}$ through $\ast$). If $DS \equiv_{\leftrightarrow\text{swap}} DT$, then $D(DS) \equiv_{\leftrightarrow\text{swap}} D(DT)$.

Proof. By induction on the derivation of $\equiv_{\leftrightarrow\text{swap}}$.

Case 1 (Reflexivity). Let $DS \equiv_{\leftrightarrow\text{swap}} DT$, with the last step of the derivation being Reflexivity. This means $DT = DS$. Consider the derivation

\[
\frac{D(DS) \equiv_{\leftrightarrow\text{swap}} D(DS)}{D(DS) \equiv_{\leftrightarrow\text{swap}} D(DT)}
\]

Case 2 (Base). Let $DS \equiv_{\leftrightarrow\text{swap}} DT$, with the last step of the derivation being Base. That means $DS \leftrightarrow\text{swap} DT$. Consider the derivation

\[
\frac{DS \leftrightarrow\text{swap} DT}{DS \leftrightarrow\text{swap} D(DT)}
\]

\[
\frac{D(DS) \equiv_{\leftrightarrow\text{swap}} D(DT)}{D(DS) \equiv_{\leftrightarrow\text{swap}} D(DT)}
\]

Case 3 (Symmetry). Let $DS \equiv_{\leftrightarrow\text{swap}} DT$, with the last step of the derivation being Symmetry. That means $DT \leftrightarrow\text{swap} DS$. Consider the derivation

\[
\frac{DT \leftrightarrow\text{swap} DS}{DS \leftrightarrow\text{swap} D(DS)}
\]
\[
\frac{D(DT) \equiv_{\leftrightarrow\text{swap}} D(DS)}{D(DS) \equiv_{\leftrightarrow\text{swap}} D(DT)}
\]

Case 4 (Transitivity). Let $DS \equiv_{\leftrightarrow\text{swap}} DT$, with the last step of the derivation being Transitivity. That means that, for some $DS'$, the last step of the derivation is

\[
\frac{DS \equiv_{\leftrightarrow\text{swap}} DS' \quad DS' \equiv_{\leftrightarrow\text{swap}} DT}{DS \equiv_{\leftrightarrow\text{swap}} DT}
\]

By induction assumption, $D(DS) \equiv_{\leftrightarrow\text{swap}} D(DS')$ and $D(DS') \equiv_{\leftrightarrow\text{swap}} D(DT')$. Consider the derivation

\[
\frac{D(DS') \equiv_{\leftrightarrow\text{swap}} D(DS')} {D(DS') \equiv_{\leftrightarrow\text{swap}} D(DT')}
\]

\[
\frac{D(DS') \equiv_{\leftrightarrow\text{swap}} D(DT')} {D(DS) \equiv_{\leftrightarrow\text{swap}} D(DT)}
\]

Lemma 69 (Expressibility of $\equiv^S$ in $\equiv_{\leftrightarrow\text{swap}}$). If $S \equiv^S T$, then $\downarrow S \equiv_{\leftrightarrow\text{swap}} \downarrow T$.

Proof. Assume $S \equiv^S T$. Prove by induction on the deduction of $\equiv^S$.

Case 1 (Structural Equality Rule). Let $S \equiv^S T$, and the last step of the deduction is an application of structural equality rule. That means that $T = S$. Through reflexivity, $\downarrow S \equiv_{\leftrightarrow\text{swap}} \downarrow S = \downarrow T$. 


Case 2 (+ Ident). Let $S \equiv^s T$, and the last step of the deduction is an application of + Ident. Without loss of generality, from symmetry, $T = S \cdot \emptyset$.

\[ \downarrow S \cdot \emptyset = \downarrow S \oplus \downarrow \emptyset = \downarrow S \oplus \emptyset = \downarrow S. \]

Through reflexivity, $\downarrow S \equiv \downarrow \emptyset \downarrow S = \downarrow S \cdot \emptyset$. Without loss of generality, from symmetry, $T = S \cdot \emptyset$, and $T = \emptyset$.

\[ \downarrow S \cdot \emptyset = \downarrow S \oplus \downarrow \emptyset = \downarrow S \oplus \emptyset = \downarrow S. \]

Through reflexivity, $\downarrow S = \downarrow S \equiv \downarrow S \downarrow T = \downarrow T$.

Case 3 (0 ProjL). Let $S \equiv^s T$, and the last step of the deduction is an application of 0 ProjL. Without loss of generality, from symmetry, $S = S' \cdot \emptyset$, and $T = \emptyset$.

\[ \downarrow S' \cdot \emptyset = \downarrow S \oplus \downarrow \emptyset = \downarrow S \oplus \emptyset = \downarrow S. \]

Through reflexivity, $\downarrow S = \downarrow S \equiv \downarrow S \downarrow S = \downarrow S$.

Case 4 (0 ProjR). Let $S \equiv^s T$, and the last step of the deduction is an application of 0 ProjR. Without loss of generality, from symmetry, $S = \emptyset \cdot S'$, and $T = \emptyset$.

\[ \downarrow \emptyset \cdot S' = \downarrow \emptyset \oplus \downarrow S' = \downarrow \emptyset \oplus \emptyset = \downarrow \emptyset. \]

Through reflexivity, $\downarrow \emptyset = \downarrow \emptyset \equiv \downarrow \emptyset \downarrow T = \downarrow T$.

Case 5 (- Assoc). Let $S \equiv^s T$, and the last step of the deduction is an application of - Assoc. Without loss of generality, from symmetry, $S = S_1 \cdot (S_2 \cdot S_3)$, and $T = (S_1 \cdot S_2) \cdot S_3$.

\[ \downarrow (S_1 \cdot (S_2 \cdot S_3)) = \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) = (\downarrow S_1 \oplus \downarrow S_2 \oplus \downarrow S_3) = (S_1 \cdot S_2) \cdot S_3. \]

Through reflexivity, $\downarrow S = \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) \equiv \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) = \downarrow T$.

Case 6 (| Assoc). Let $S \equiv^s T$, and the last step of the deduction is an application of | Assoc. Without loss of generality, from symmetry, $S = S_1 \mid (S_2 \mid S_3)$, and $T = (S_1 \mid S_2) \mid S_3$.

\[ \downarrow (S_1 \mid (S_2 \mid S_3)) = \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) = (\downarrow S_1 \oplus \downarrow S_2 \oplus \downarrow S_3) = (S_1 \mid S_2) \mid S_3. \]

Through reflexivity, $\downarrow S = \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) \equiv \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) = \downarrow T$.

Case 7 (| Comm). Let $S \equiv^s T$, and the last step of the deduction is an application of | Comm. $S = S_1 \mid S_2$, and $T = S_2 \mid S_1$.

Let $\downarrow S_1 = \langle SQ \mid \ldots \mid SQ \rangle$ and $\downarrow S_2 = \langle TQ \mid \ldots \mid TQ \rangle$. $\langle SQ \mid \ldots \mid SQ \rangle \oplus \langle TQ \mid \ldots \mid TQ \rangle = \langle SQ \mid \ldots \mid SQ \rangle \oplus \langle TQ \mid \ldots \mid TQ \rangle$.

\[ \langle SQ \mid \ldots \mid SQ \rangle \oplus \langle TQ \mid \ldots \mid TQ \rangle = \downarrow S_1 \oplus \downarrow S_2 \]

Consider the deduction $\downarrow S_1 \oplus \downarrow S_2 \equiv \downarrow S_1 \oplus \downarrow S_2$.

$\downarrow S_1 \oplus \downarrow S_2 \equiv \downarrow S_1 \oplus \downarrow S_2$.

So $\downarrow S_1 \oplus \downarrow S_2 \equiv \downarrow S_1 \oplus \downarrow S_2$, which means $\downarrow S_1 \oplus \downarrow S_2 \equiv \downarrow S_1 \oplus \downarrow S_2$.

Case 8 (DistL). Let $S \equiv^s T$, and the last step of the deduction is an application of DistL. Without loss of generality, from symmetry, $S = S_1 \cdot (S_2 \mid S_3)$, and $T = (S_1 \cdot S_2) \mid (S_1 \cdot S_3)$.

Let $\downarrow S_1 = \langle SQ \mid \ldots \mid SQ \rangle$. Let $\downarrow S_2 = \langle SQ \mid \ldots \mid SQ \rangle$. Let $\downarrow S_3 = \langle SQ \mid \ldots \mid SQ \rangle$.

\[ \downarrow (S_1 \cdot (S_2 \mid S_3)) = \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) = \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) = \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) = \downarrow S_1 \oplus (\downarrow S_2 \oplus \downarrow S_3) = \downarrow T.

\]
Without loss of generality, from symmetry, $S_{16}$ (Structural Case 15 (Structural Case 14 (Structural Case 13 (Structural Case 12 (Structural Case 11 (Structural Case 10 (Structural Case 9 (DistL)) Let $S \equiv^s T$, and the last step of the deduction is an application of DistL. Without loss of generality, from symmetry, $S = (S_1 | S_2) \cdot S_3$, and $T = (S_1 \cdot S_3) | (S_2 \cdot S_3)$. 

$\downarrow (S_1 \cdot (S_2 | S_3)) \equiv \uplus \downarrow (S_1 \cdot S_2) (S_1 \cdot S_3) (S_2 \cdot S_3).$

Through reflexivity, $\downarrow S = \downarrow (S_1 \cdot (S_2 | S_3)) \equiv \uplus \downarrow (S_1 \cdot S_2) (S_1 \cdot S_3) (S_2 \cdot S_3) = \downarrow T$

Case 10 (\textit{Iden}t). Let $S \equiv^s T$, and the last step of the deduction is an application of DistL. Without loss of generality, from symmetry, $S = S' \cdot e$, and $T = S'$. 

$\downarrow (S' \cdot e) = \downarrow (S' \cdot e) = \downarrow (S' \cdot e) = \downarrow S' = \downarrow T$

Through reflexivity, $\downarrow S = \downarrow (S' \cdot e) \equiv \uplus \downarrow S' = \downarrow T$

Case 11 (\textit{Iden}t$_R$). Let $S \equiv^s T$, and the last step of the deduction is an application of DistL. Without loss of generality, from symmetry, $S = e \cdot S'$, and $T = S'$. 

$\downarrow (e \cdot S') = \downarrow (e \cdot S') = \downarrow (e \cdot S') = \downarrow S' = \downarrow T$

Through reflexivity, $\downarrow S = \downarrow (e \cdot S') \equiv \uplus \downarrow S' = \downarrow T$

Case 12 (Unrollstar$_L$). Let $S \equiv^s T$, and the last step of the deduction is an application of Unrollstar$_L$. Without loss of generality, from symmetry, $S = S''$, and $T = e \cdot (S' \cdot S'')$. 

$\downarrow S'' = D((\downarrow S'')^\star)^\star. \downarrow (e \cdot (S' \cdot S'')) = (\downarrow e) \oplus (\downarrow S' \cdot D((\downarrow S'')^\star)^\star)$

Through Atom Unrollstar$_L$, $\downarrow S = \downarrow S'' \equiv \uplus \downarrow (e \cdot (S' \cdot S'')) = \downarrow T$.

Case 13 (Unrollstar$_R$). Let $S \equiv^s T$, and the last step of the deduction is an application of Unrollstar$_R$. Without loss of generality, from symmetry, $S = S''$, and $T = e \cdot (S'' \cdot S')$. 

$\downarrow S'' = D((\downarrow S'')^\star)^\star. \downarrow (e \cdot (S'' \cdot S')) = (\downarrow e) \oplus (D((\downarrow S'')^\star)^\star \cdot \downarrow S'')^\star$.

Through Atom Unrollstar$_R$, $\downarrow S = \downarrow S'' \equiv \uplus \downarrow (e \cdot (S'' \cdot S')) = \downarrow T$.

Case 14 (Structural Or Equality). Let $S \equiv^s T$, through structural equality of Or. $S = S_1 | S_2$, and $T = T_1 | T_2$, $S_1 \equiv^s T_1$, and $S_2 \equiv^s T_2$.

By induction assumption, $\downarrow S_1 \equiv \uplus \downarrow \downarrow T_1$ and $\downarrow S_2 \equiv \uplus \downarrow \downarrow T_2$.

By Lemma 62, $\downarrow S_1 \oplus \downarrow S_2 \equiv \uplus \downarrow \downarrow T_1 \oplus \downarrow \downarrow T_2$. By the definition of $\downarrow \downarrow$, $\downarrow (S_1 | S_2) \equiv \uplus \downarrow \downarrow (T_1 \downarrow T_2)$, as desired.

Case 15 (Structural Concat Equality). Let $S \equiv^s T$, through structural equality of Concat. $S = S_1 \cdot S_2$, and $T = T_1 \cdot T_2$, $S_1 \equiv^s T_1$, and $S_2 \equiv^s T_2$.

By induction assumption, $\downarrow S_1 \equiv \uplus \downarrow \downarrow T_1$ and $\downarrow S_2 \equiv \uplus \downarrow \downarrow T_2$.

By Lemma 67, $\downarrow S_1 \oplus \downarrow S_2 \equiv \uplus \downarrow \downarrow T_1 \oplus \downarrow \downarrow T_2$. By the definition of $\downarrow \downarrow$, $\downarrow (S_1 \cdot S_2) \equiv \uplus \downarrow \downarrow (T_1 \cdot T_2)$, as desired.

Case 16 (Structural Star Equality). Let $S \equiv^s T$, through structural equality of Star. $S = S''$, $T = T''$, and $S' \equiv^s T'$. 

By induction assumption, $\downarrow S' \equiv \uplus \downarrow \downarrow T'$.

By Lemma 68, $D((\downarrow S'')^\star) \equiv \uplus \downarrow \downarrow D((\downarrow T'')^\star)$. By the definition of $\downarrow \downarrow$, $\downarrow S' \equiv \uplus \downarrow \downarrow T''$, as desired.
Case 17 (Transitivity of Equational Theory). Let $S \equiv^s T$ through the transitivity of an equational theory. This means there exists a $S'$ such that $S \equiv^s S'$ and $S' \equiv^s T$.

By induction assumption, $\downarrow S \equiv_{\downarrow \text{swap}} \downarrow S'$ and $\downarrow S' \equiv_{\downarrow \text{swap}} \downarrow T$.

Consider the derivation

\[
\begin{array}{c}
\downarrow S \equiv_{\downarrow \text{swap}} \downarrow S' \\
\downarrow S' \equiv_{\downarrow \text{swap}} \downarrow T
\end{array}
\]

\[\downarrow S \equiv_{\downarrow \text{swap}} \downarrow T\]

\[\text{□}
\]

Theorem 8 (Equivalence of $\equiv$ and $\equiv^s$). $S \equiv^s T$ if, and only if $\downarrow S \equiv_{\downarrow \text{swap}} \downarrow T$

Proof. The forward direction is proven by Lemma 69. The reverse direction is proven by Lemma 57

Lemma 70. $A \equiv_A^{\text{D}}(A)$

Proof. $A = DS^*$ for some DNF regular expression. Consider the derivation

\[
\begin{array}{c}
DS \rightarrow DS \\
DS^* \rightarrow_{\text{D}} (DS^*)
\end{array}
\]

as desired.

\[\text{□}
\]

Lemma 71.

- If $A \rightarrow DT$, then $A \equiv DT$.
- If $DS \rightarrow DT$, then $DS \equiv DT$.

Proof. By mutual induction on the derivation of $\rightarrow$ and $\rightarrow_A$

Case 1 (Atom UnrollstarL).

\[
DS^* \rightarrow_A ([\epsilon]) \oplus (DS \circ D(DS^*)
\]

Consider the derivation

\[
DS^* \equiv_A ([\epsilon]) \oplus (DS \circ D(DS^*)
\]

Case 2 (Atom UnrollstarR).

\[
DS^* \rightarrow_A ([\epsilon]) \oplus (D(DS^*) \circ DS)
\]

Consider the derivation

\[
DS^* \equiv_A ([\epsilon]) \oplus (D(DS^*) \circ DS)
\]

Case 3 (Atom Structural Rewrite).

\[
DS \rightarrow DT' \\
DS^* \rightarrow D(DT'^*)
\]

By IH, $DS \equiv DT'$, so consider the derivation

\[
DS \equiv DT' \\
DS^* \equiv D(DT'^*)
\]

\[\text{□}
\]
Case 4 (DNF Structural Rewrite).

\[ A_j \rightarrow_{A} DS \]

\[
\langle SQ_1 | \ldots | SQ_{i-1} \rangle \oplus \langle [s_0 \cdot A_1 \cdot \ldots \cdot s_j] \rangle \odot D(A_j) \odot \langle [s_j \ldots \cdot A_m \cdot s_m] \rangle \oplus \langle SQ_{i+1} | \ldots | SQ_n \rangle \rightarrow
\]

\[
\langle SQ_1 | \ldots | SQ_{i-1} \rangle \oplus \langle [s_0 \cdot A_1 \cdot \ldots \cdot s_j] \rangle \odot DS \,[\,|\,] \langle [s_j \ldots \cdot A_m \cdot s_m] \rangle \oplus \langle SQ_{i+1} | \ldots | SQ_n \rangle
\]

Define \( S_Q \) as \([s_0 \cdot A_1 \cdot \ldots \cdot A_m \cdot s_m] \). Through the definition of \( \oplus \) and \( \circledast \), \( DS = \langle SQ_1 | \ldots | SQ_n \rangle \).

Define \( S_Q_k = [s_k,0 \cdot A_k,1 \cdot \ldots \cdot A_k, n_k \cdot s_k, n_k] \). So, in particular, \( A_{i,j} = A_j \), and \( n_i = m \). Define \( DS_{k,l} = \)

\[
\text{if } (k, l) = (i, j), \text{ then by assumption } A_{i,j} \leftrightarrow_{A} DS, \text{ and otherwise, from Lemma 70, } A_{k,l} \leftrightarrow_{A} D(A_k, i).
\]

Define \( DS_k \) as \( \{ [s_k,0] \} \odot DS_{k,1} \odot \ldots \odot DS_{k,n_k} \odot \{ [s_k, n_k] \} \).

\[
DS = \langle SQ_1 | \ldots | SQ_n \rangle , \quad \forall i, SQ_i = [s_i,0 \cdot A_i,1 \cdot \ldots \cdot A_i, n_i \cdot s_i, n_i] \quad \forall i, DS_i = \{ [s_i,0] \} \odot DS_{i,1} \odot \ldots \odot DS_{i,n_i} \odot \{ [s_i, n_i] \}
\]

So \( DS_k \), for \( k \neq i \),

\[
DS_k = \{ [s_k,0] \} \odot D(A_k, 1) \odot \ldots \odot D(A_k, n_k) \odot \{ [s_k, n_k] \} = SQ_k
\]

\[
DS_i = \{ [s_i,0 \cdot A_i,1 \cdot \ldots \cdot s_j] \} \odot D(A_j) \odot \{ [s_j \ldots \cdot A_m \cdot s_m] \},
\]

so, through the definition of \( \odot \), \( DS_i \odot \ldots \odot DS_n \, = \langle SQ_1 | \ldots | SQ_{i-1} \rangle \odot \{ [s_i,0 \cdot A_i,1 \cdot \ldots \cdot s_j] \} \odot DS \odot \{ [s_j \ldots \cdot A_m \cdot s_m] \} \odot \langle SQ_{i+1} | \ldots | SQ_n \rangle \), so we get \( DS \leftrightarrow_{A} DS_1 \odot \ldots \odot DS_n = \langle SQ_1 | \ldots | SQ_{i-1} \rangle \odot \{ [s_i,0 \cdot A_i,1 \cdot \ldots \cdot s_j] \} \odot DS \odot \{ [s_j \ldots \cdot A_m \cdot s_m] \} \odot \langle SQ_{i+1} | \ldots | SQ_n \rangle \).

\[ \square \]

**Lemma 72.** If \( DS \rightarrow^{*} DT \), then \( DS \leftrightarrow^{*} DT \)

**Proof.** By induction on the derivation of \( \rightarrow^{*} 

Case 1 (Reflexivity).

\[
\begin{array}{c}
\text{Case 1 (Reflexivity).} \\
\hline
DS \rightarrow^{*} DS \\
\hline
\end{array}
\]

Consider the following derivation \( DS \leftrightarrow^{*} DS \)

Case 2 (Base).

\[
\begin{array}{c}
\text{Case 2 (Base).} \\
\hline
DS \rightarrow DT \\
\hline
\end{array}
\]

By Lemma 80, \( DS \leftrightarrow^{*} DT \).

Case 3 (Transitivity).

\[
\begin{array}{c}
\text{Case 3 (Transitivity).} \\
\hline
DS \rightarrow^{*} DS' \quad DS' \rightarrow^{*} DT \\
\hline
\end{array}
\]

By IH, \( DS \leftrightarrow^{*} DS' \) and \( DS' \leftrightarrow^{*} DT \).

Consider the following derivation

\[
\begin{array}{c}
\text{by IH, } DS \leftrightarrow^{*} DS' \text{ and } DS' \leftrightarrow^{*} DT. \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{Consider the following derivation} \\
\hline
DS \leftrightarrow^{*} DS' \quad DS' \leftrightarrow^{*} DT \\
\hline
\end{array}
\]

\[ \square \]

**Lemma 73.** If \( DS \rightarrow^{*} DT \), then \( D(DS^{*}) \rightarrow^{*} D(DT^{*}) \).
Proof. By induction on the derivation of $\rightarrow^*$

Case 1 (Reflexivity).

Consider the derivation

$$DS \rightarrow^* DS$$

By IH, there exists derivations of $D(DS)\rightarrow^* D(DS)$ and $D(DS')\rightarrow D(DS')$.

Case 2 (Base).

Consider the derivation

$$DS \rightarrow DT$$

so, by Reflexivity

$$DS_1 \rightarrow^* DS_1$$

Case 3 (Transitivity).

Consider the derivation

$$D(DS) \rightarrow^* D(DS)$$

$$D(DS) \rightarrow^* D(DS^*)$$

By IH, there exists derivations of $D(DS^*) \rightarrow^* D(DS^*)$ and $D(DS^*) \rightarrow D(DT^*)$.

$\square$

Lemma 74. If $DS_1 \rightarrow^* DS_2$, then for all $DT$, $DS_1 \oplus DT \rightarrow^* DS_2 \oplus DT$

Proof. By induction on the derivation of $\rightarrow^*$

Case 1 (Reflexivity).

Consider the derivation

$$DS \rightarrow^* DS$$

so, by Reflexivity

$$DS_1 \oplus DT \rightarrow^* DS_1 \oplus DT$$

Case 2 (Base).

Consider the derivation

$$DS \rightarrow DS_2$$

The only way to get a derivation of $\rightarrow$ is with an application of DNF STRUCTURAL REWRITE, so by inversion,
where $DS_1 = \langle SQ_1 \mid \ldots \mid SQ_{i-1} \rangle \oplus \langle [s_0 \cdot A_1 \cdot \ldots \cdot s_{j-1}] \rangle \odot \mathcal{D}(A_j) \odot \langle [s_j \cdot \ldots \cdot A_m \cdot s_m] \rangle \oplus \langle SQ_{i+1} \mid \ldots \mid SQ_n \rangle$
and where $DS_2 = \langle SQ_1 \mid \ldots \mid SQ_{i-1} \rangle \oplus \langle [s_0 \cdot A_1 \cdot \ldots \cdot s_{j-1}] \rangle \odot DS \circ [s_j \cdot \ldots \cdot A_m \cdot s_m] \oplus \langle SQ_{i+1} \mid \ldots \mid SQ_n \rangle$.
So, let $DT = \langle TQ_1 \mid \ldots \mid TQ_n \rangle$.

Consider the derivation

$$A_j \rightarrow DS$$

$$\langle SQ_1 \mid \ldots \mid SQ_{i-1} \rangle \oplus \langle [s_0 \cdot A_1 \cdot \ldots \cdot s_{j-1}] \rangle \odot \mathcal{D}(A_j) \odot \langle [s_j \cdot \ldots \cdot A_m \cdot s_m] \rangle \oplus \langle SQ_{i+1} \mid \ldots \mid SQ_n \rangle \odot TQ_1 \mid \ldots \mid TQ_n \rangle.$$}

$$\langle [s_j \cdot \ldots \cdot A_m \cdot s_m] \rangle \oplus \langle SQ_{i+1} \mid \ldots \mid SQ_n \rangle \odot TQ_1 \mid \ldots \mid TQ_n \rangle = DS_1 \oplus DT.$$}

$$\langle SQ_1 \mid \ldots \mid SQ_{i-1} \rangle \oplus \langle [s_0 \cdot A_1 \cdot \ldots \cdot s_{j-1}] \rangle \odot DS \circ \langle [s_j \cdot \ldots \cdot A_m \cdot s_m] \rangle \oplus \langle SQ_{i+1} \mid \ldots \mid SQ_n \rangle \odot TQ_1 \mid \ldots \mid TQ_n \rangle = DS_2 \oplus DT.$$}

Case 3 (Transitivity).

$$DS_1 \rightarrow DS_3 \quad DS_3 \rightarrow DS_2$$

$$DS_1 \rightarrow DS_2$$

By IH, $DS_1 \oplus DT \rightarrow DS_1 \oplus DT$. By IH, $DS_3 \oplus DT \rightarrow DS_2 \oplus DT$.

Consider the derivation

$$DS_1 \oplus DT \rightarrow DS_3 \oplus DT \quad DS_3 \oplus DT \rightarrow DS_2 \oplus DT$$

$$DS_1 \oplus DT \rightarrow DS_2 \oplus DT$$

**Lemma 75.** If $DS_1 \rightarrow DS_2$, then for all $DT, DT \oplus DS_1 \rightarrow DT \oplus DS_2$

**Proof.** Proven symmetrically to Lemma 74. □

**Lemma 76.** If $DS_1 \rightarrow DS_2$, and $DT_1 \rightarrow DT_2$, then $DS_1 \oplus DT_1 \rightarrow DS_2 \oplus DT_2$

**Proof.** By Lemma 74, $DS_1 \oplus DS_2 \rightarrow DT_1 \oplus DS_2$. By Lemma 75, $DT_1 \oplus DS_2 \rightarrow DT_1 \oplus DT_2$.

Consider the derivation

$$DS_1 \oplus DS_2 \rightarrow DT_1 \oplus DS_2 \quad DT_1 \oplus DS_2 \rightarrow DT_1 \oplus DT_2$$

$$DS_1 \oplus DS_2 \rightarrow DT_1 \oplus DT_2$$

□

**Lemma 77.** If $DS_1 \rightarrow DS_2$, then for all $SQ, DS_1 \circ SQ \rightarrow DS_2 \circ SQ$

**Proof.** By induction on the derivation of $\rightarrow$.

**Case 1 (Reflexivity).**

$$DS_1 \rightarrow DS_1$$

so, by Reflexivity
\[ DS_1 \odot (SQ) \rightarrow^* DS_1 \odot (SQ) \]

**Case 2 (Base).**

\[ DS_1 \rightarrow DS_2 \]

\[ DS_1 \rightarrow^* DS_2 \]

The only way to get a derivation of \( \rightarrow^* \) is with an application of DNF STRUCTURAL REWRITE, so by inversion,

\[ A_j \rightarrow DS \]

\[ \langle SQ_1 \mid \ldots \mid SQ_{i+1} \rangle \odot (\langle \ldots \rangle \odot DS(A_j) \odot (\langle \ldots \rangle \odot DS_3) \odot \langle \ldots \rangle \odot DS_2) \odot \langle \ldots \rangle \odot DS_1 \rightarrow^* \langle \ldots \rangle \odot DS_1 \odot (SQ) \odot DS_2 \odot (SQ) \]

Consider the derivation

\[ A_j \rightarrow DS \]

\[ DS_1 \odot (SQ) \rightarrow DS_2 \odot (SQ) \]

\[ DS_1 \odot (SQ) \rightarrow^* DS_2 \odot (SQ) \]

**Case 3 (Transitivity).**

\[ DS_1 \rightarrow^* DS_3 \]

\[ DS_3 \rightarrow^* DS_2 \]

By IH, \( DS_1 \odot (SQ) \rightarrow^* DS_3 \odot (SQ) \). By IH, \( (SQ) \odot DS_3 \rightarrow^* (SQ) \odot DS_2 \).

Consider the derivation

\[ (SQ) \odot DS_3 \rightarrow^* (SQ) \odot DS_2 \]

\[ (SQ) \odot DS_2 \rightarrow^* (SQ) \odot DS_2 \]

\[ (SQ) \odot DS_1 \rightarrow^* (SQ) \odot DS_2 \]

**Lemma 78.** If \( DS_1 \rightarrow^* DS_2 \), then for all \( DT, DT \odot DS_1 \rightarrow^* DT \odot DS_2 \)

**Proof.** By induction on the derivation of \( \rightarrow^* \)

**Case 1 (Reflexivity).**

\[ DS_1 \rightarrow^* DS_1 \]

so, by Reflexivity

\[ DT \odot DS_1 \rightarrow^* DT \odot DS_1 \]

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Case 2 (Base).

\[
\frac{DS_1 \rightarrow DS_2}{DS_1 \rightarrow^* DS_2}
\]

The only way to get a derivation of \(\rightarrow\) is with an application of DNF STRUCTURAL REWRITE, so by inversion,

\[
A_j \rightarrow DS
\]

\[
(SQ_1 | \ldots | SQ_{i-1}) \oplus \langle [s_0 \cdot A_1 \cdot \ldots \cdot s_{j-1}] \rangle \circ D(A_j) \oplus \langle [s_j \cdot \ldots \cdot A_m \cdot s_m] \rangle \oplus \langle SQ_{i+1} | \ldots | SQ_n \rangle
\]

Let \(DT = (SQ'_1 | \ldots | SQ'_{n'})\).

By Lemma 78, \((SQ'_1) \circ DS_1 \rightarrow (SQ'_1) \circ DS_2\). So, through repeated application of Lemma 76, \((SQ'_1) \circ DS_1 \oplus \ldots \oplus (SQ'_{n'}) \circ DS_1 \rightarrow (SQ'_1) \circ DS_2 \oplus \ldots \oplus (SQ'_{n'}) \circ DS_2\).

From Lemma 35, \((SQ'_1) \circ DS_1 \oplus \ldots \oplus (SQ'_{n'}) \circ DS_1 = (SQ'_1 \oplus \ldots \oplus SQ'_{n'}) \circ DS_1 = DT \circ DS_1\) and \((SQ'_1) \circ DS_2 \oplus \ldots \oplus (SQ'_{n'}) \circ DS_2 = (SQ'_1 \oplus \ldots \oplus SQ'_{n'}) \circ DS_2 = DT \circ DS_2\).

So we have \(DT \circ DS_1 \rightarrow^* DT \circ DS_2\).

Case 3 (Transitivity).

\[
\frac{DS_1 \rightarrow^* DS_2 DS_3 \rightarrow^* DS_2}{DS_1 \rightarrow^* DS_2}
\]

By IH, \(DS_1 \circ DT \rightarrow^* DS_2 \circ DT\). By IH, \(DS_3 \circ DT \rightarrow^* DS_2 \circ DT\).

Consider the derivation

\[
\frac{DS_1 \circ DT \rightarrow^* DS_3 \circ DT}{DS_1 \circ DT \rightarrow^* DS_2 \circ DT}
\]

\[
\frac{DS_3 \circ DT \rightarrow^* DS_2 \circ DT}{DS_1 \circ DT \rightarrow^* DS_2 \circ DT}
\]

\(\Box\)

**Lemma 79.** Let \(\mathcal{D}(A_i) \rightarrow^* DS_i\). \(\langle [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \rangle \rightarrow^* \langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [s_n] \rangle\)

**Proof.** By induction on \(n\).

*Case 1 \((n = 0)\).* Through use of Reflexivity

\[
\langle [s_0] \rangle \rightarrow^* \langle [s_0] \rangle
\]

*Case 2 \((n > 0)\).* \(\langle [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \rangle = \langle [s_0 \cdot A_1 \cdot \ldots \cdot A_{n-1} \cdot s_{n-1}] \rangle \circ \langle [\varepsilon \cdot A_n \cdot s_n] \rangle\) by the definition of \(\odot\).

From IH, \(\langle [s_0 \cdot A_1 \cdot \ldots \cdot A_{n-1} \cdot s_{n-1}] \rangle \rightarrow^* \langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [s_0] \rangle\). From Lemma 77, \(\langle [s_0 \cdot A_1 \cdot \ldots \cdot A_{n-1} \cdot s_{n-1}] \rangle \circ \langle [\varepsilon \cdot A_n \cdot s_n] \rangle \rightarrow^* \langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [\varepsilon \cdot A_n \cdot s_n] \rangle\).

\(\mathcal{D}(A_n) = \langle [s_n] \rangle\). From Lemma 77, as \(\mathcal{D}(A_n) \rightarrow^* DS_n \circ \langle [\varepsilon \cdot A_n \cdot s_n] \rangle\).

As \(\langle [\varepsilon \cdot A_n \cdot s_n] \rangle \rightarrow^* DS_n \circ \langle [s_n] \rangle\), from Lemma 78, \(\langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [s_n] \rangle \rightarrow^* \langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [s_n] \rangle\).

Consider the derivation

\[
\langle [s_0 \cdot A_1 \cdot \ldots \cdot A_{n-1} \cdot s_{n-1}] \rangle \circ \langle [\varepsilon \cdot A_n \cdot s_n] \rangle \rightarrow^* \langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [s_0] \rangle
\]

\[
\langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [s_0] \rangle \rightarrow^* \langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [s_0] \rangle\]

So, \(\langle [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \rangle \rightarrow^* \langle [s_0] \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle [s_n] \rangle\), as desired.
Lemma 80.

• If $A \parallel DT$, then $D(DS) \rightarrow^* DT$.
• If $DS \parallel DT$, then $D(DS) \rightarrow^* DT$.

Proof. By mutual induction on the derivation of $\parallel A$

Case 1 (Atom UnrollStarL).

\[
DS^* \parallel_A ([\varepsilon]) \oplus (DS \odot D(DS^*))
\]

Consider the derivation

\[
DS^* \rightarrow_A ([\varepsilon]) \oplus (DS \odot D(DS^*))
\]

\[
D(DS^*) \rightarrow ([\varepsilon]) \oplus (DS \odot D(DS^*))
\]

\[
D(DS^*) \rightarrow^* ([\varepsilon]) \oplus (DS \odot D(DS^*))
\]

Case 2 (Atom UnrollStarR).

\[
DS^* \parallel_A ([\varepsilon]) \oplus (D(DS^*) \odot DS)
\]

Consider the derivation

\[
DS^* \rightarrow_A ([\varepsilon]) \oplus (D(DS^*) \odot DS)
\]

\[
D(DS^*) \rightarrow ([\varepsilon]) \oplus (DS \odot D(DS^*))
\]

\[
D(DS^*) \rightarrow^* ([\varepsilon]) \oplus (DS \odot D(DS^*))
\]

Case 3 (Parallel Atom Structural Rewrite).

\[
DS \parallel DT'
\]

\[
DS^* \parallel_D(D(T)^*)
\]

By IH, $DS \rightarrow^* DT'$, so by Lemma 73, $D(DS^*) \rightarrow^* D(DT^*)$.

Case 4 (ParallelDNFStructuralRewriteRule).

\[
DS = \langle SQ_1, \ldots, SQ_n \rangle \quad \forall i. SQ_i = [s_{i,0} \cdot A_{i,1} \cdot \ldots \cdot A_{i,n_i} \cdot s_{i,n_i}]
\]

\[
\forall i, j. A_{i,j} \parallel_A DS_{i,j} \quad \forall i. DS_i = ([s_{i,0}] \odot DS_{i,1} \odot \ldots \odot DS_{i,n_i} \odot ([s_{i,n_i}])
\]

\[
DS \parallel_D DS_1 \oplus \ldots \oplus DS_n
\]

From the definition of $\oplus$, $DS = \langle SQ_1 \rangle \oplus \ldots \oplus \langle SQ_n \rangle$. From IH, $D(A_{i,1}) \rightarrow^* DS_{i,1}$. From Lemma 79, $\langle SQ_i \rangle = ([s_{i,0} \cdot A_{i,1} \cdot \ldots \cdot A_{i,n_i} \cdot s_{i,n_i}]) \rightarrow^* ([s_{i,0}]) \odot DS_{i,1} \odot \ldots \odot DS_{i,n_i} \odot ([s_{i,n_i}]) = DS_i$, so $\langle SQ_i \rangle \rightarrow^* DS_i$.

From Lemma 76, $DS = \langle SQ_1 \rangle \oplus \ldots \oplus \langle SQ_n \rangle \rightarrow^* DS_1 \oplus \ldots \oplus DS_n$.

Case 5 (Identity Rewrite).

\[
DS \parallel_D DS
\]

Through application of Reflexivity

\[
DS \rightarrow^* DS
\]
**Lemma 81.** If \(DS \Rightarrow ^* DT\), then \(DS \rightarrow ^* DT\)

**Proof.** By induction on the derivation of \(\Rightarrow ^*\)

**Case 1 (Reflexivity).**

\[
\begin{align*}
DS &\Rightarrow ^* DS \\
\hline
DS \rightarrow ^* DS
\end{align*}
\]

Consider the following derivation

\[
DS \rightarrow ^* DS
\]

**Case 2 (Base).**

\[
\begin{align*}
DS &\Rightarrow DT \\
\hline
DS \Rightarrow ^* DT
\end{align*}
\]

By Lemma 80, \(DS \rightarrow ^* DT\).

**Case 3 (Transitivity).**

\[
\begin{align*}
DS &\Rightarrow ^* DS' & DS' &\Rightarrow ^* DT \\
\hline
DS &\Rightarrow ^* DT
\end{align*}
\]

By IH, \(DS \rightarrow ^* DS'\) and \(DS' \rightarrow ^* DT\).

Consider the following derivation

\[
\begin{align*}
DS &\rightarrow ^* DS' & DS' &\rightarrow ^* DT \\
\hline
DS &\rightarrow ^* DT
\end{align*}
\]

□

**Theorem 9.** \(DS \Rightarrow ^* DS'\), if, and only if \(DS \rightarrow ^* DS'\)

**Proof.** Forward direction is proven by Lemma 72. Reverse direction is proven by Lemma 81. □

**Corollary 1 (\(\Rightarrow ^*\) Maintained Under Iteration).** If \(DS \rightarrow ^* DT\), then \(\langle [DS^*] \rangle \rightarrow ^* \langle [DT^*] \rangle\).

**Proof.** From Theorem 9 applied to Lemma 52. □

**Lemma 82 (\(\Rightarrow\) can be expressed in \(\Rightarrow ^{swap}\)).** If \(DS \Rightarrow DT\) then \(DS \Rightarrow ^{swap} DT\)

**Proof.** \(\Rightarrow ^{swap}\) has all of the inference rules of \(\Rightarrow\), so a straightforward induction using those rules can prove this. □

**Lemma 83 (\(\Rightarrow\) can be expressed in \(\equiv^s\)).** If \(\downarrow S \Rightarrow \downarrow T\), then \(S \equiv^s T\).

**Proof.** By Lemma 82, \(\downarrow S \Rightarrow ^{swap} \downarrow T\), then by Lemma 56, \(S \equiv^s T\). □

**Lemma 84 (\(\Rightarrow ^*\) can be expressed in \(\equiv^s\)).** If \(\downarrow S \Rightarrow ^* \downarrow T\), then \(S \equiv^s T\).

**Proof.** By straightforward induction, using for base rule, Lemma 83, for transitivity the transitivity of equational theories, and for reflexivity the reflexivity of equational theories. □

**Lemma 85 (\(\Rightarrow ^*\) can be expressed in \(\equiv^s\)).** If \(\downarrow S \rightarrow ^* \downarrow T\), then \(S \equiv^s T\).

**Proof.** By Lemma 84 and Theorem 9. □
B.7 Lens Soundness

Using the above machinery, we prove the soundness of DNF lenses. The unambiguity is guaranteed through prior unambiguity proofs. The rewrite portion of DNF lenses are proven to be correct through the above subsection. The bulk of this is showing that the lenses can be built up from their subcomponents, and that arbitrary permutations can be expressed.

**Lemma 86** (Expressibility of Safe Boilerplate Alterations). Suppose

\[ \text{(1)} \quad \lambda l \in \text{const} \cdot \text{A}_1 \cdot \ldots \cdot \text{A}_n \cdot \text{s}_n \]
\[ \text{(2)} \quad \lambda t \in \text{const} \cdot \text{A}_1 \cdot \ldots \cdot \text{A}_n \cdot \text{t}_n \]

Then there exists a lens \( l : S \leftrightarrow T \) such that

\[ \text{(1)} \quad S = \llbracket (\text{const} \cdot \text{A}_1 \cdot \ldots \cdot \text{A}_n \cdot \text{s}_n) \rrbracket \]
\[ \text{(2)} \quad T = \llbracket (\text{const} \cdot \text{A}_1 \cdot \ldots \cdot \text{A}_n \cdot \text{t}_n) \rrbracket \]
\[ \text{(3)} \quad \llbracket I \rrbracket = \{ (s, t) \mid s = s_0 \cdot s_1' \cdot \ldots \cdot s_n' \cdot \text{s}_n \wedge t = t_0 \cdot s_1' \cdot \ldots \cdot s_n' \cdot \text{t}_n \wedge s_i \in \mathcal{L}(A_i) \} \]

**Proof.** By induction on \( n \).

Let \( n = 0 \). Consider the Lens

\[ \text{const}(s_0, t_0) : s_0 \leftrightarrow t_0 \]

By inspection, this satisfies the desired properties.

Let \( n > 0 \). By induction, there exists a lens \( l : S \leftrightarrow T \) satisfying the desired properties. Consider the lens

\[ \text{D} \quad l : S \leftrightarrow T \quad \text{const}(s_n, t_n) : s_n \leftrightarrow t_n \]
\[ \text{const}(\llbracket l, \text{const}(s_n, t_n) \rrbracket) : S \cdot s_n \leftrightarrow T \cdot t_n \]
\[ \text{id}_{\llbracket (A_n) \rrbracket} : (A_n) \leftrightarrow \llbracket (A_n) \rrbracket \]
\[ \text{const}(\text{concat}(\llbracket l, \text{const}(s_n, t_n) \rrbracket), \text{id}_{\llbracket (A_n) \rrbracket}) : S \cdot s_n \cdot \llbracket (A_n) \rrbracket \leftrightarrow T \cdot t_n \cdot \llbracket (A_n) \rrbracket \]

By inspection, this satisfies the desired properties. \( \square \)

**Lemma 87** (Creation of Lens from Identity Perm Sequence Lens). Suppose

\[ \text{(1)} \quad \text{SQ} = [s_0 \cdot \text{A}_1 \cdot \ldots \cdot \text{A}_n \cdot \text{s}_n] \]
\[ \text{(2)} \quad \text{TQ} = [t_0 \cdot \text{B}_1 \cdot \ldots \cdot \text{B}_n \cdot \text{t}_n] \]
\[ \text{(3)} \quad \llbracket (\text{SQ}) \rrbracket = \text{SQ} \leftrightarrow \text{TQ} \]
\[ \text{(4)} \quad \text{For each } \text{al}_i \vdash \text{A}_i \leftrightarrow \text{B}_i, \text{there exists a } l_i \vdash \text{Q} \text{ such that } \llbracket l_i \rrbracket = \llbracket \text{al}_i \rrbracket \]

then there exists a \( l \vdash \text{Q} \leftrightarrow \llbracket (\text{TQ}) \rrbracket \) such that \( \llbracket I \rrbracket = \llbracket ((s_0, t_0) \cdot \text{al}_1 \cdot \ldots \cdot \text{al}_n \cdot (s_n, t_n), \text{id}) \rrbracket \).

**Proof.** By induction on \( n \).

Let \( n = 0 \). \( \llbracket (s_0, t_0) \rrbracket, \text{id}_{\llbracket (s_0, t_0) \rrbracket} \) \( \llbracket \text{const}(s_0, t_0) \rrbracket = \{ s_0, t_0 \} = \llbracket ((s_0, t_0), \text{id}) \rrbracket \).

Let \( n > 0 \). Let \( \text{SQ}' = [s_0 \cdot \text{A}_1 \cdot \ldots \cdot \text{A}_{n-1} \cdot \text{s}_{n-1}] \), and \( \text{TQ}' = [t_0 \cdot \text{B}_1 \cdot \ldots \cdot \text{B}_{n-1} \cdot \text{t}_{n-1}] \) By induction assumption, there exists a typing derivation

\[ l : \llbracket (\text{SQ}) \rrbracket \leftrightarrow \llbracket (\text{TQ}) \rrbracket \]
Lemma 88 (Unambiguity of $\Sigma$). Let $\Sigma$ be an alphabet. Let $\Sigma_0 = \Sigma \cup \{\$\}$, where $\$ is a character not in $\Sigma$. If $L_1, \ldots, L_n$, are languages in $\Sigma^*$, then $^1[\mathcal{L}(\$); L_1; \mathcal{L}(\$); \ldots; \mathcal{L}(\$); L_n; \mathcal{L}(\$)]$.

Proof. We prove this by induction on $n$.
Let $n = 0$. $^1[\mathcal{L}(\$)]$, as $^1[L]$, for any language $L$.
Let $n > 0$. Let $s_i, t_i \in L_i$ for all $i \in [1, n]$, and let $s_1\$s_2\ldots s_n\$ = t_1\$t_2\ldots t_n\$. We want to show that $s_1\$ = t_1\$. If they were not equal, then one string is strictly contained in the other, say without loss of generality $s_n\$ is strictly contained in $t_n\$. Because of that $s_n\$ is contained in $t_n\$, so $\$ is contained in $t_n \in \Sigma^*$. This is a contradiction, as $\$ \notin \Sigma$, so we know $s_n\$ = $t_n\$, and so $s_n = t_n$. This means that $s_1\$s_2\ldots s_{n-1}\$ = $t_1\$t_2\ldots t_{n-1}$, so by induction, I know $s_i = t_i$ for all $i$.

Definition 15 (Adjacent Swapping Permutation). Let $\sigma_i \in S_n$ be the permutation where $\sigma_i(i) = i+1$, $\sigma_i(i+1) = i$, $\sigma_i(k) = k$ when $k \neq i$, and $k \neq i+1$.

Lemma 89 (Expressibility of Adjacent Swapping Permutation Lens). Suppose

1. $\sigma_i$ is an adjacent element swapping permutation
2. $\{\$ \cdot A_1 \cdot \ldots \cdot A_n \cdot \$\}$ is a sequence with all base strings equal to $\$.

Then there exists a typing of a lens $l : S \leftrightarrow T$ such that

1. $\mathcal{L}(S) = \mathcal{L}(\{\$ \cdot A_1 \cdot \ldots \cdot A_n \cdot \$\})$
2. $\mathcal{L}(T) = \mathcal{L}(\{\$ \cdot A_{\sigma_1(1)} \cdot \ldots \cdot A_{\sigma_1(n)} \cdot \$\})$
3. $\mathcal{L}(l) = \{(s, t) \mid s = \$ \cdot s_1 \cdot \$ \ldots \cdot \$ \cdot s_n \cdot \$ \land t = \$ \cdot s_{\sigma_1(1)} \cdot \$ \ldots \cdot \$ \cdot s_{\sigma_1(n)} \$ \land s_i \in \mathcal{L}(A_i)\}$
Proof. By the soundness of regular expressions, define regular expressions $S_1, S_2, S_3, S_4$ as

$S_1 = \{ [\$ \cdot A_1 \cdot \ldots \cdot A_{n-1} \cdot \$] \}$,
$S_2 = \{ (A_i) \}$,
$S_3 = \{ (A_{i+1}) \}$,
and $S_4 = \{ [\$ \cdot A_{i+1} \cdot \ldots \cdot A_n \cdot \$] \}$.

Consider the following deduction

\[ \frac{D}{\text{swap}(id_{S_2}, id_{S_3}) : S_2 \Rightarrow S_3 \\ id_{S_1} : S_1 \Rightarrow S_3 \cdot s_i} \quad \frac{D'}{\text{swap}(id_{S_2}, s(id_{S_3}, id_{S_4})) : S_2 \cdot S_3 \Rightarrow S_3 \cdot S_2} \quad \frac{D}{\text{swap}(id_{S_2}, id_{S_3}) : S_2 \Rightarrow S_3} \]

By inspection, the final lens $\text{concat}(c(id_{S_1}, s(id_{S_2}, id_{S_4})), id_{S_3}) : S_1 \cdot S_2 \cdot S_3 \cdot S_4 \Rightarrow S_1 \cdot S_3 \cdot S_2 \cdot S_4$ satisfies $L(S_1 \cdot S_2 \cdot S_3 \cdot S_4) = L([\$ \cdot A_1 \cdot \ldots \cdot A_n \cdot \$])$ and has the desired semantics of swapping the strings at spots $i$ and $i+1$. \hfill \Box

Lemma 90 (Expressibility of Adjacent Swapping Permutation Composition). Suppose

(1) $\sigma = \sigma_{l_1} \circ \ldots \circ \sigma_{l_m}$
(2) $[\$ \cdot A_1 \cdot \ldots \cdot \$ \cdot A_n \cdot \$]$ is a sequence with all base strings equal to $\$.$

Then there exists a typing of a lens $l : S \Rightarrow T$ such that

(1) $L(S) = L([\$ \cdot A_1 \cdot \ldots \cdot A_n \cdot \$])$,
(2) $L(T) = L([\$ \cdot A_{\sigma(1)} \cdot \ldots \cdot A_{\sigma(n)} \cdot \$])$,
(3) $[I] = \{(s, t) \mid s = \$ \cdot s_1 \cdot \$ \cdot \ldots \cdot \$ \cdot s_n \cdot \$ \wedge \quad t = \$ \cdot s_{\sigma(1)} \cdot \$ \cdot \ldots \cdot \$ \cdot s_{\sigma(n)} \$ \wedge \quad s_i \in L(A_i)\}$

Proof. By induction on $m$.

Let $m = 0$. Then $\sigma = id$. Consider the lens $id_{\{[\$ \cdot A_1 \cdot \ldots \cdot A_n \cdot \$]\}} : \{[\$ \cdot A_1 \cdot \ldots \cdot A_n \cdot \$]\} \Rightarrow \{[\$ \cdot A_1 \cdot \ldots \cdot A_n \cdot \$]\}$. By inspection, this lens satisfies the requirements.

Let $m > 0$. Let $\sigma' = \sigma_{l_1} \circ \ldots \circ \sigma_{l_{m-1}}$. Let $l : S \Rightarrow T$ be the lens obtained by an application of the induction assumption on $\sigma'$. Let $l_{m} : T' \Rightarrow T''$ be the lens obtained by an application of Lemma 89 to the permutation $\sigma_m$ and the sequence $[\$ \cdot A_{i_{\sigma(1)}} \cdot \ldots \cdot A_{i_{\sigma(n)}} \cdot \$]$. From the induction assumption and the previous lemmas, we know $L(T) = L([\$ \cdot A_{\sigma(1)} \cdot \ldots \cdot A_{\sigma(n)} \cdot \$]) = L(T')$. Consider the following Lens typing

\[ \frac{l : S \Rightarrow T \quad L(T) = L(T')} {l : S \Rightarrow T'} \quad \frac{l_{m} : T' \Rightarrow T''} {l_{m} \cdot l : S \Rightarrow T''} \]

The language of $S$ is already as desired, and $L(T'') = L([\$ \cdot A_{i_{\sigma(1)}} \cdot \ldots \cdot A_{i_{\sigma(n)}} \cdot \$]) = L([\$ \cdot A_{i_{\sigma(1)}} \cdot \ldots \cdot A_{i_{\sigma(n)}} \cdot \$])$, as desired. Moreover, the composition of the lenses composes the permutations of strings, giving the semantics as desired. \hfill \Box

Lemma 91 (Expressibility of Permutation). Suppose

(1) $\sigma$ is a permutation in $S_n$
(2) $[\$ \cdot A_1 \cdot \ldots \cdot \$ \cdot A_n \cdot \$]$ is a sequence with all base strings equal to $\$. 

\[ \text{Vol. 1, No. 1, Article 1. Publication date: January 2018.} \]
Then there exists a typing of a lens $l : S \iff T$ such that

1. $\mathcal{L}(S) = \mathcal{L}([S \cdot A_1 \cdot \ldots \cdot A_n \cdot S])$
2. $\mathcal{L}(T) = \mathcal{L}([S \cdot A_{\sigma(1)} \cdot \ldots \cdot A_{\sigma(n)} \cdot S])$
3. $\{l : S \iff T | s = S \cdot s_1 \cdot \ldots \cdot s_n \cdot S \land t = S \cdot s_{\sigma(1)} \cdot \ldots \cdot s_{\sigma(n)} \cdot S \land s_i \in \mathcal{L}(A_i)\}$

**Proof.** By algebra, any permutation can be expressed as the composition of adjacent swapping permutations. As such, $\sigma = \sigma_i \circ \ldots \circ \sigma_j$, for some adjacency swapping permutations $\sigma_i$. By Lemma 90, we obtain a lens with the properties desired. □

**Lemma 92 (Creation of Lens from Identity Perm DNF Lens).** Suppose

1. $DS = \langle SQ_1 | \ldots | SQ_n \rangle$
2. $DT = \langle TQ_1 | \ldots | TQ_n \rangle$
3. $\langle (s_{ql_1} | \ldots | s_{ql_n}, id) \rangle : DS \iff DT$
4. For each $s_{ql_i} : SQ_i \iff TQ_i$, there exists an $l_i$ such that $\llbracket l_i \rrbracket = \llbracket s_{ql_i} \rrbracket$.

then there exists a $l \vdash \upharpoonright (DS) \iff \upharpoonright (DT)$ such that $\llbracket l \rrbracket = \llbracket (s_{ql_1} | \ldots | s_{ql_n}, id) \rrbracket$.

**Proof.** By induction on $n$

Let $n = 0$. $\langle \rangle \vdash \langle \rangle \iff \langle \rangle$. Then consider

$$\llbracket id_{\llbracket \langle \rangle \rrbracket} \rrbracket \vdash \upharpoonright (\langle \rangle) \iff \upharpoonright (\langle \rangle)$$

This has the desired typing, and $\llbracket id_{\llbracket \langle \rangle \rrbracket} \rrbracket = \llbracket id_0 \rrbracket = \{\} = \llbracket \langle \rangle \rrbracket$.

Let $n > 0$. Define $DS' = \langle SQ_1 | \ldots | SQ_{n-1} \rangle$, and $DT' = \langle TQ_1 | \ldots | TQ_{n-1} \rangle$. By induction assumption, there exists a derivation of $l : \upharpoonright (DS') \iff \upharpoonright (DT')$. By problem statement, there exists a typing derivation $l_n : \upharpoonright (SQ_n) \iff \upharpoonright (TQ_n)$ Consider the following derivation

$$l : \upharpoonright (DS') \iff \upharpoonright (DT') \quad l_n : \upharpoonright (SQ_n) \iff \upharpoonright (TQ_n)$$

$$l \vdash \upharpoonright (DS') \iff \upharpoonright (DT') \quad \Rightarrow \quad \llbracket (l_n, l) \rrbracket : \upharpoonright (DS') \iff \upharpoonright (DT') \iff \upharpoonright (SQ_n) \iff \upharpoonright (TQ_n)$$

$\llbracket (l, l_n) \rrbracket = \{(s, t) | (s, t) \in l \lor (s, t) \in l_n\}$

$\llbracket (s, t) | (s, t) \in \langle sq_{l_1} | \ldots | sq_{l_n} \rangle \}$

$\llbracket (s, t) | (s, t) \in \langle sq_{l_1} | \ldots | sq_{l_n} \rangle \}$

$\llbracket (s, t) | (s, t) \in \langle sq_{l_1} | \ldots | sq_{l_n} \rangle \}$

**Lemma 93 (Ineffectiveness of Permutation on DNF Regex Semantics).** Let $\sigma \in S_n$, and $\langle SQ_1 \ldots SQ_n \rangle$ be a DNF regex. $\mathcal{L}(\langle SQ_1 \ldots SQ_n \rangle) = \mathcal{L}(\langle SQ_{\sigma(1)} \ldots SQ_{\sigma(n)} \rangle)$.

**Proof.** By inspection. □

**Lemma 94 (Ineffectiveness of Permutation on DNF Lens Semantics).** Let $\sigma \in S_n$, and

$\langle (s_{ql_1} | \ldots | s_{ql_n}, id) \rangle : \langle SQ_1 | \ldots | SQ_n \rangle \iff \langle TQ_1 | \ldots | TQ_n \rangle$ be a typing of a DNF lens with an identity permutation. $\llbracket \langle (s_{ql_1} | \ldots | s_{ql_n}, id) \rangle \rrbracket = \llbracket \langle (s_{ql_1} | \ldots | s_{ql_n}, \sigma) \rangle \rrbracket$

**Proof.** By inspection. □

**Lemma 95 (Soundness of DNF, Sequence, and Atom Lenses).**

1. Let $DS$ and $DT$ be two dnf regular expressions, and $dl : DS \iff DT$. Then there exists a $l$ such that $l : \upharpoonright (DS) \iff \upharpoonright (DT)$, $\llbracket l \rrbracket = \llbracket dl \rrbracket$
2. Let $SQ$ and $TQ$ be two clauses, and $sql : SQ \iff TQ$. Then there exists a $l$ such that $l : \upharpoonright (SQ) \iff \upharpoonright (TQ)$, $\llbracket l \rrbracket = \llbracket sql \rrbracket$. 
(3) Let \( A \) and \( B \) be two atoms, and \( al \vdash A \leftrightarrow B \). Then there exists a \( l \), such that \( l \vdash (A) \leftrightarrow (B) \), 
\[
\llbracket l \rrbracket = \llbracket al \rrbracket.
\]

**Proof.** By mutual induction on the structure of the DNF Regex, Sequence, and Atom lenses typing.

Let \( dl \vdash DS \leftrightarrow DT \) be formed from an application of 

Rewrite DNF Regex Lens.

\[
\begin{array}{c}
dl \vdash DS' \leftrightarrow DT' \\
DS' \rightarrow DS \\
DT' \rightarrow DT
\end{array}
\]

By induction assumption, there exists a \( l \vdash (DS') \leftrightarrow (DT') \), and from Lemma 42, we know 
\( \mathcal{L}(DS) = \mathcal{L}(DS') \), and \( \mathcal{L}(DT) = \mathcal{L}(DT') \). Consider the derivation

\[
l \vdash (DS') \leftrightarrow (DT') \\
\mathcal{L}(\llbracket (DS') \rrbracket) = \mathcal{L}(\llbracket (DS) \rrbracket) \\
\mathcal{L}(\llbracket (DT') \rrbracket) = \mathcal{L}(\llbracket (DT) \rrbracket)
\]

This has the desired typing, and by induction assumption, has the desired semantics.

Let \( \langle sq_1 \mid \ldots \mid sq_n \rangle, \sigma \rangle \vdash \langle SQ_1 \mid \ldots \mid SQ_n \rangle \leftrightarrow \langle TQ_{\sigma(1)} \mid \ldots \mid TQ_{\sigma(n)} \rangle \) be formed from an application of DNF Lens. By Induction assumption, for each \( sq_i \vdash SQ_i \leftrightarrow TQ_i \) there exists a \( l_i \vdash (SQ_i) \leftrightarrow (TQ_i) \).

By Lemma 92 there exists a \( l \vdash (\langle SQ_1 \mid \ldots \mid SQ_n \rangle) \leftrightarrow (\langle TQ_1 \mid \ldots \mid TQ_n \rangle) \) such that \( \llbracket l \rrbracket = \llbracket (\langle sq_1 \mid \ldots \mid sq_n \rangle, \sigma) \rrbracket \), By Lemma 94, \( \llbracket ([DNFO] sq_1 \mid \ldots \mid sq_n, id) \rrbracket = \llbracket (\langle sq_1 \mid \ldots \mid sq_n \rangle, \sigma) \rrbracket \).

By Lemma 93, \( \mathcal{L}(\langle TQ_1 \mid \ldots \mid TQ_n \rangle) = \mathcal{L}(\langle TQ_{\sigma(1)} \mid \ldots \mid TQ_{\sigma(n)} \rangle) \). Consider the following typing

\[
l \vdash (\langle SQ_1 \mid \ldots \mid SQ_n \rangle) \leftrightarrow (\langle TQ_1 \mid \ldots \mid TQ_n \rangle) \\
\mathcal{L}(\langle TQ_{\sigma(1)} \mid \ldots \mid TQ_{\sigma(n)} \rangle) = \mathcal{L}(\langle TQ_{\sigma(1)} \mid \ldots \mid TQ_{\sigma(n)} \rangle)
\]

This has the typing and semantics as desired.

Let \( \{(s_0 \cdot t_0) \cdot a_1 \cdot \ldots \cdot a_n \cdot (s_n \cdot t_n)\}, \sigma \in S_n \rangle \vdash [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \leftrightarrow [t_0 \cdot B_{\sigma(1)} \cdot \ldots \cdot B_{\sigma(n)} \cdot t_n] \) be formed from an application of 

Sequence Lens. By induction assumption, for each \( al_i \vdash A_i \leftrightarrow B_i \), there exists a \( l_i \vdash (s_i) \leftrightarrow (T_i) \).

By Lemma 87, there exists a \( l \vdash S \leftrightarrow T \) such that \( \llbracket l \rrbracket = \llbracket \langle (s_0 \cdot t_0) \cdot a_1 \cdot \ldots \cdot a_n \cdot (s_n \cdot t_n), id \rangle \llbracket \), \( \mathcal{S} \vdash (\langle (s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n) \rangle \), \( \mathcal{S} = \llbracket (\langle (s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n) \rangle \), \( \mathcal{T} = \llbracket (\langle t_0 \cdot B_1 \cdot \ldots \cdot B_n \cdot t_n) \rangle \). Define \( T_S \) as \( \llbracket (\langle \mathcal{S} \cdot B_1 \cdot \ldots \cdot B_n \cdot \mathcal{T} \rangle \)

By Lemma 86, there exists a \( l' \vdash T \leftrightarrow T_S \), with semantics of merely changing the boilerplate. By Lemma 91, there exists a \( l'' : T'_S \leftrightarrow T'' \) where \( \llbracket T'' \rrbracket = \llbracket T_S \rrbracket \) and \( \llbracket T'' \rrbracket = \llbracket \langle \mathcal{S} \cdot B_1 \cdot \ldots \cdot B_n \cdot \mathcal{T} \rangle \rrbracket \). Lastly, with Lemma 86, there exists a \( l''' : T'' \leftrightarrow T' \), where \( T = \llbracket (\langle t_0 \cdot B_{\sigma(1)} \cdot \ldots \cdot B_{\sigma(n)} \cdot t_n) \rangle \). Through composition of all these lenses, we finally get a lens with the desired type and semantics.

Let \( iterate(dl) \vdash DS \leftrightarrow DT \) be introduced through an application of Atom Lens. From induction assumption, I know that there exists \( l \vdash S \leftrightarrow T \), such that \( \llbracket l \rrbracket = \llbracket D \rrbracket, S = \llbracket DS \rrbracket, and T = \llbracket DT \rrbracket \).

Consider \( iterate(l) \vdash S' \leftrightarrow T' \).

By definition, \( S' \) and \( T' \) are \( \llbracket (DS) \rrbracket \) and \( \llbracket (S) \rrbracket \), respectively.

\[
\llbracket iterate(l) \rrbracket = \{s_0 \cdots s_n, l_0 \cdots t_n \mid (s_j, t_j) \in \llbracket l \rrbracket\}
\]

\[
= \{s_0 \cdots s_n, l_0 \cdots t_n \mid (s_j, t_j) \in \llbracket dl \rrbracket\}
\]

\[
= \llbracket iterate(dl) \rrbracket
\]
Theorem 10. If there exists a derivation of $dl : DS \iff DT$, then there exist a lens, $\uparrow dl$, and regular expressions, $S$ and $T$, such that $\uparrow dl : S \iff T$ and $\downarrow S = DS$ and $\downarrow T = DT$ and $\llbracket \uparrow dl \rrbracket = \llbracket dl \rrbracket$.

Proof. Let $dl : DS \iff DT$.

By inversion, the last step is

$$
\begin{array}{ccc}
  \uparrow dl : DS' \iff DT' & DS \rightarrow DS' & DT \rightarrow DT' \\
  dl : DS \iff DT
\end{array}
$$

From Lemma 95, there exists $l : \uparrow DS' \iff \uparrow DT'$. So, as $\uparrow DS \rightarrow \uparrow DS'$ and $\uparrow DT \rightarrow \uparrow DT'$, from Lemma 85, $\uparrow DS \equiv \uparrow DS'$, and $\uparrow DT \equiv \uparrow DT'$.

$$
\begin{array}{ccc}
  l : \uparrow DS' \iff \uparrow DT' & \llbracket \uparrow DS' \rrbracket = \llbracket \uparrow DS \rrbracket & \llbracket \uparrow DT' \rrbracket = \llbracket \uparrow DT \rrbracket \\
  l : \uparrow DS \iff \uparrow DT
\end{array}
$$

We call this lens $\uparrow dl$ (constructive proof).

Furthermore, we also know that $\llbracket DS = DS \rrbracket$, and similarly for $DT$. □

B.8 DNF Lens Operators

DNF lens operators are defined to give DNF lenses similar capabilities to lenses. This allows the proof of many of the cases of completeness to be trivial, leaving only the complications of proving statements about rewrites, proving closure under composition, and proving the ability to use rewrites to express lens retyping.

Definition 16 (Permutation Functions).

\( \ominus : S_n \rightarrow S_m \rightarrow S_{n+m} \)

\( (\sigma_1 \ominus \sigma_2)(i) = \begin{cases} 
\sigma_1(i) & \text{if } i \leq n \\
\sigma_2(i-n) + n & \text{otherwise}
\end{cases} \)


\( \ominus \uplus : S_n \rightarrow S_m \rightarrow S_{n \times m} \)

\( (\sigma_1 \ominus \sigma_2)(i,j) = (\sigma_1(i,j), \sigma_2(j)) \)

\( \ominus^\flat : S_n \rightarrow S_m \rightarrow S_{n \times m} \)

\( (\sigma_1 \ominus \sigma_2)(i,j) = (\sigma_2(i), \sigma_1(j)) \)

Definition 17 (DNF Lens Functions).

\( \ominus_{sql} : \text{SequenceLens} \rightarrow \text{SequenceLens} \rightarrow \text{SequenceLens} \)

\( (\{(s_0, l_0) \cdot a_{l_1} \cdot \ldots \cdot a_{l_n} \cdot (s_n, t_n)\}, \sigma_1) \ominus_{sql} (\{(s'_0, t'_0) \cdot a'_{l'_1} \cdot \ldots \cdot a'_{l'_m} \cdot (s'_m, t'_m)\}, \sigma_2) = \\
(\{(s_0, l_0) \cdot a_{l_1} \cdot \ldots \cdot a_{l_n} \cdot (s_n, s'_0, t_n, t'_0) \cdot a'_{l'_1} \cdot \ldots \cdot a'_{l'_m} \cdot (s'_m, t'_m)\}, \sigma_1 \ominus \sigma_2) \)

\( \ominus_{sql} : \text{SequenceLens} \rightarrow \text{SequenceLens} \rightarrow \text{SequenceLens} \)

Let \( s_i' = \begin{cases} 
  s_i & \text{for } i \in [0, n-1] \\
  s_n \cdot s_0 & \text{for } i = n \\
  s'_i & \text{for } i \in [n+1, n+m]
\end{cases} \)

Let \( t'_i = \begin{cases} 
  t'_i & \text{for } i \in [0, n-1] \\
  t'_m \cdot t_0 & \text{for } i = m \\
  t_i & \text{for } i \in [n+1, n+m]
\end{cases} \)
Lemma 96. \((\langle (s'_0, t'_0) \cdot al_1 \cdot \ldots \cdot al_m \cdot (s_n, t_n), \sigma_1 \rangle \circ sql \langle (s'_0, t'_0) \cdot al_1 \cdot \ldots \cdot al_m \cdot (s'_n, t'_n), \sigma_2 \rangle) = \langle (s'_0, t'_0) \cdot al_1 \cdot \ldots \cdot al_n \cdot (s'_n, t'_n), \sigma_1 \circ \sigma_2 \rangle\)

\(\circ : DNFLens \rightarrow DNFLens \rightarrow DNFLens\)

\(\langle (sql_1 | \ldots | sql_n), \sigma_1 \rangle \circ \langle (sql'_1 | \ldots | sql'_n), \sigma_2 \rangle = \langle sql_1 \circ_{sql} sql'_1 | \ldots | sql_n \circ_{sql} sql'_n \rangle, \sigma_1 \otimes \sigma_2 \rangle\)

\(\odot : DNFLens \rightarrow DNFLens \rightarrow DNFLens\)

\(\langle (sql_1 | \ldots | sql_n), \sigma_1 \rangle \odot \langle (sql'_1 | \ldots | sql'_n), \sigma_2 \rangle = \langle sql_1 \odot_{sql} sql'_1 | \ldots | sql_n \odot_{sql} sql'_n \rangle, \sigma_1 \otimes \sigma_2 \rangle\)

\(\oplus : DNFLens \rightarrow DNFLens \rightarrow DNFLens\)

\(\langle (sql_1 | \ldots | sql_n), \sigma_1 \rangle \oplus \langle (sql'_1 | \ldots | SQL'_n), \sigma_2 \rangle = \langle (sql_1 | \ldots | SQL_n | sql'_1 | \ldots | SQL'_n), \sigma_1 \otimes \sigma_2 \rangle\)

\(D : AtomLens \rightarrow DNFLens\)

\(D(al) = \langle \langle (id, e) \cdot id \cdot (e, e) \rangle, id \rangle\)

**Lemma 96.** \(\langle \langle (id, e), id_0 \rangle \rangle, id_1 \rangle \odot dl = dl\), where \(id_0\) is the identity permutation on 0 elements, and \(id_1\) is the identity permutation on 1 element.

**Proof.** Let \(dl = \langle \langle sql_1 | \ldots | sql_n \rangle, \sigma \rangle\). By definition, \((id_1 \otimes \sigma)(1, i) = (1, \sigma(i))\). By definition, \((id_0 \circ \sigma) = \sigma\). Let \(sql_1 = \langle (s_i, t_i, n) \cdot al_1 \cdot \ldots \cdot al\rangle\), \(sql_2 = \langle (s_i, t_i, n) \cdot (s_i, t_i, n) \rangle\), \(\sigma_i = \langle s_i \cdot (s_i, t_i, n) \rangle\). So \((\langle (id, e), id_0 \rangle \circ_{sql} sql_1 = \langle (id, e), id_0 \rangle \circ_{sql} \langle sql_1 | \ldots | sql_n \rangle = \langle (\langle id, e \rangle \cdot id_0 \cdot id_1 \cdot \ldots \cdot id_n \rangle, \sigma_1 \otimes \sigma_2 \rangle\rangle\langle (id, e), id_0 \rangle \circ_{sql} sql_1 | \ldots | SQL_n, \sigma_1 \otimes \sigma_2 \rangle\rangle. \)

**Lemma 97.** \(dl \odot \langle \langle (id, e) \rangle \rangle = dl\)

**Proof.** Done similarly to Lemma 96. \(\square\)

**Lemma 98 (Typing and Semantics of \(\circ_{sql}\)).** Let \(sql_1 : SQ_1 \iff TQ_1\) and \(sql_2 : SQ_2 \iff TQ_2\) be the typing of two sequence lenses, where \(L(SQ_1) \vdash L(SQ_2)\) and \(L(TQ_1) \vdash L(TQ_2)\). Then \(sql_1 \circ_{sql} sql_2 : SQ_1 \circ_{sql} SQ_2 \iff TQ_1 \circ_{sql} TQ_2\) and \(\|sql_1 \circ_{sql} sql_2\| = \{(s_1 \cdot s_2, t_1 \cdot t_2) | (s_1, t_1) \in \|sql_1\| \land (s_2, t_2) \in \|sql_2\|\}\)

**Proof.** By assumption, there exists typing derivations

\(sql_1 : SQ_1 \iff TQ_1\)

and

\(sql_2 : SQ_2 \iff TQ_2\)

By inversion, we know that the last rule application on each side was DNF LENS, giving

\[
\begin{align*}
al_i & : A_i \iff B_i \quad \sigma_i \in S_n \quad \vdash (s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n) \quad \vdash (t_0 \cdot B_{\sigma_1(1)} \ldots \cdot B_{\sigma_1(n)} \cdot t_n) \\
\vdash ([(s_0, t_0) \cdot al_1 \ldots \cdot al_n, \sigma_1] : [s_0 \cdot A_1 \ldots \cdot A_n \cdot s_n] \iff [t_0 \cdot B_{\sigma_1(1)} \ldots \cdot B_{\sigma_1(n)} \cdot t_n])
\end{align*}
\]

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
and

\[ \begin{align*}
\sigma_2 \in S_m & \quad \vdash (s_0' \cdot A_1' \cdot \ldots \cdot A_m' \cdot s_m') \quad \vdash (t_0' \cdot B_{\sigma(1)}' \cdot \ldots \cdot B_{\sigma(n)}' \cdot t_n')
\end{align*} \]

\[ \begin{align*}
([s_0', t_0'] \cdot a_1' \cdot \ldots \cdot a_m' \cdot (s_m', t_m')] \cdot \sigma_2) & \quad \vdash [s_0' \cdot A_1' \cdot \ldots \cdot A_m' \cdot s_m'] \quad \vdash [t_0' \cdot B_{\sigma(1)}' \cdot \ldots \cdot B_{\sigma(m)}' \cdot t_m']
\end{align*} \]

where

\[ \begin{align*}
sq_1' & = ([s_0, t_0] \cdot a_1 \cdot \ldots \cdot a_n], \sigma_1) \\
SQ_1 & = [s_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s_n] \\
TQ_1 & = [t_0 \cdot B_{\sigma(1)} \cdot \ldots \cdot B_{\sigma(n)} \cdot t_n] \\
sq_2' & = ([s_0', t_0'] \cdot a_1' \cdot \ldots \cdot a_m' \cdot (s_m', t_m')], \sigma_2) \\
SQ_2 & = [s_0' \cdot A_1' \cdot \ldots \cdot A_m' \cdot s_m'] \\
TQ_2 & = [t_0' \cdot B_{\sigma(1)}' \cdot \ldots \cdot B_{\sigma(m)}' \cdot t_m']
\end{align*} \]

Define \( s''_i \) as \( s_i \) for \( i \in [1, n-1] \), and as \( s'_{i-n} \) for \( i \in [n+1, n+m] \), and as \( s_n \cdot s_i' \) for \( i = n \).

Define \( t''_i \) as \( t_i \) for \( i \in [1, n-1] \), and as \( t_{i-n} \) for \( i \in [1, n-1, n+m] \), and as \( t_n \cdot t_0 \) for \( i = n \).

Define \( A''_i \) as \( A_i \) for \( i \in [1, n] \), and as \( A'_{i-n} \) for \( i \in [1, n+1, n+m] \).

Define \( B''_i \) as \( A_i \) for \( i \in [1, n] \), and as \( B'_{i-n} \) for \( i \in [1, n+1, n+m] \).

Define \( a_i \) as \( a_l \) for \( i \in [1, n] \), and as \( a_{l-n} \) for \( i \in [1, n+1, n+m] \).

From Lemma 8, as \( \vdash (s_0; A_1; \ldots; A_n; s_n \cdot s_0'; A_1'; \ldots; A_m'; s_m') \), so \( \vdash (s_0' \cdot A_1' \cdot \ldots; A_m' \cdot s_m', s''_n) \).

From Lemma 8, as \( \vdash (t_0; B_{\sigma(1)}; \ldots; B_{\sigma(n)}; t_n), (t_0'; B_{\sigma(1)}'; \ldots; B_{\sigma(n)}'; t_n'), (t_0 \cdot B_{\sigma(1)} \cdot \ldots \cdot B_{\sigma(n)} \cdot t_n), (t_0' \cdot B_{\sigma(1)}' \cdot \ldots \cdot B_{\sigma(m)}' \cdot t_m'), (t_0 \cdot t_0') \), then \( \vdash (t_0; B_{\sigma(1)}; \ldots; B_{\sigma(n)}; t_n; t_0' \cdot B_{\sigma(1)}'; \ldots; B_{\sigma(m)}' \cdot t_m') \), so \( \vdash (t_0; B_{\sigma(1)} \cdot \ldots \cdot B_{\sigma(m)}' \cdot t_m') \).

Consider the derivation

\[ \begin{align*}
\sigma_1 \odot \sigma_2 \in S_{n+m} & \quad \vdash a_l : A_i \iff B_i \\
([s_0'; t_0'] \cdot a_1' \cdot \ldots \cdot a_m' \cdot (s_m', t_m')] \cdot \sigma_1 \odot \sigma_2) & \quad \vdash [s_0' \cdot A_1' \cdot \ldots \cdot A_m' \cdot s_m'] \cdot \sigma_1 \odot \sigma_2
\end{align*} \]

\[ \begin{align*}
([s_0'; t_0'] \cdot a_1' \cdot \ldots \cdot a_m' \cdot (s_m', t_m')] \cdot \sigma_1 \odot \sigma_2) & \quad \vdash [s_0' \cdot A_1' \cdot \ldots \cdot A_m' \cdot s_m'] \cdot \sigma_1 \odot \sigma_2
\end{align*} \]

We wish to show that this is a derivation of \( \vdash \sigma_1 \odot SQ_1 \odot SQ_2 \iff TQ_1 \odot SQ TQ_2 \).

\[ \begin{align*}
([s_0'; t_0'] \cdot a_1' \cdot \ldots \cdot a_m' \cdot (s_m', t_m')] \cdot \sigma_1 \odot \sigma_2) & \quad \vdash [s_0' \cdot A_1' \cdot \ldots \cdot A_m' \cdot s_m'] \cdot \sigma_1 \odot \sigma_2
\end{align*} \]
So we have a derivation of $sql_1 \circ_{sql} sql_2 : SQ_1 \circ_{SQ} SQ_2 \Leftrightarrow TQ_1 \circ_{SQ} TQ_2$

We also wish to have the desired semantics.

$$
\begin{align*}
&[(s'_1 \cdot \cdot \cdot \cdot t'_0) \cdot a l'_1 \cdot \cdot \cdot \cdot a l'_{n+m} \cdot (s''_{n+m} \cdot t''_{n+m})], \sigma_1 \circ \sigma_2] \\
= & [(s'_0 \cdot s'_{1} \cdot \cdot \cdot \cdot s'_{n+m} \cdot s''_{n+m}, t'_0 \cdot t_{\sigma_1(1)} \cdot \cdot \cdot \cdot t_{\sigma_1(n+m)} \cdot t''_{n+m})] \\
| & \forall i \in [1, n + m], (\overline{s_i}, \overline{t_i}) \in sql'_0 \\
= & [(s_0 \cdot s_{1} \cdot \cdot \cdot \cdot s_n \cdot s'_m \cdot s''_m, t_0 \cdot t_{1(1)} \cdot \cdot \cdot \cdot t_{1(n)} \cdot t_n \cdot t'_0 \cdot t'_{\sigma_2(0)} \cdot \cdot \cdot \cdot t'_{\sigma_2(m)} \cdot t'_{m})] \\
| & \forall i \in [1, n], (\overline{s'_i}, \overline{t'_i}) \in sql'_1 \\
= & [(s \cdot s', t \cdot t') | (s, t) \in [sql_1] \wedge (s', t') \in [sql_2]] 
\end{align*}
$$

\[ \Box \]

**Lemma 99** (Typing and Semantics of $\circ_{sql}$). Let $sql_1 : SQ_1 \Leftrightarrow TQ_1$ and $sql_2 : SQ_2 \Leftrightarrow TQ_2$ be the typing of two sequence lenses, where $L(SQ_1)$ $\downarrow L(SQ_2)$ and $L(TQ_1)$ $\downarrow L(TQ_2)$ Then $sql_1 \circ_{sql} sql_2 : SQ_1 \circ_{SQ} SQ_2 \Leftrightarrow TQ_1 \circ_{SQ} TQ_2$ and $[sql_1 \circ_{sql} sql_2] = \{(s_1 \cdot s_2, t_1 \cdot t_2) | (s_1, t_1) \in [sql_1] \wedge (s_2, t_2) \in [sql_2]\}$

**Proof.** By assumption, there exists typing derivations

$$
sql_1 : SQ_1 \Leftrightarrow TQ_1
$$

and

$$
sql_2 : SQ_2 \Leftrightarrow TQ_2
$$

By inversion, we know that the last rule application on each side was DNF LENS, giving

$$
\begin{align*}
al_i : A_i & \Leftrightarrow B_i \quad \sigma_i \in S_n \quad i \cdot (s_0 \cdot A_1 \cdot \cdot \cdot \cdot A_n \cdot s_n) \quad \Leftrightarrow (t_0 \cdot B_{\sigma_1(1)} \cdot \cdot \cdot \cdot B_{\sigma_1(n)} \cdot t_n) \\
& \quad \left(\left[(s_0, t_0) \cdot al_1 \cdot \cdot \cdot \cdot al_n\right], \sigma_1\right) \Leftrightarrow \left[t_0 \cdot B_{\sigma_1(1)} \cdot \cdot \cdot \cdot B_{\sigma_1(n)} \cdot t_n\right] \\
\end{align*}
$$

and

$$
\begin{align*}
al'_i : A'_i & \Leftrightarrow B'_i \quad \sigma_2 \in S_m \quad i' \cdot (s'_0 \cdot A'_1 \cdot \cdot \cdot \cdot A'_m \cdot s'_m) \quad \Leftrightarrow \left(t'_0 \cdot B'_{\sigma_2(1)} \cdot \cdot \cdot \cdot B'_{\sigma_2(n)} \cdot t'_n\right) \\
& \quad \left(\left[(s'_0, t'_0) \cdot al'_1 \cdot \cdot \cdot \cdot al'_m\right], \sigma_2\right) \Leftrightarrow \left[t'_0 \cdot B'_{\sigma_2(1)} \cdot \cdot \cdot \cdot B'_{\sigma_2(n)} \cdot t'_n\right]
\end{align*}
$$

where

$$
\begin{align*}
sql_1 & = \left(\left[(s_0, t_0) \cdot al_1 \cdot \cdot \cdot \cdot al_n\right], \sigma_1\right), \\
SQ_1 & = \left[[s_0 \cdot A_1 \cdot \cdot \cdot \cdot A_n \cdot s_n]\right], \\
TQ_1 & = \left[[t_0 \cdot B_{\sigma_1(1)} \cdot \cdot \cdot \cdot B_{\sigma_1(n)} \cdot t_n]\right], \\
sql_2 & = \left(\left[(s'_0, t'_0) \cdot al'_1 \cdot \cdot \cdot \cdot al'_m\right], \sigma_2\right), \\
SQ_2 & = \left[[s'_0 \cdot A'_1 \cdot \cdot \cdot \cdot A'_m \cdot s'_m]\right], \\
TQ_2 & = \left[[t'_0 \cdot B'_{\sigma_2(1)} \cdot \cdot \cdot \cdot B'_{\sigma_2(n)} \cdot t'_n]\right]
\end{align*}
$$

Define $s''_i$ as $s_i$ for $i \in [1, n - 1]$, and as $s'_{i-n}$ for $i \in [n + 1, n + m]$, and as $s_n \cdot s'_m$ for $i = n$.

Define $t''_i$ as $t'_i$ for $i \in [1, m - 1]$, and as $t_{i-m}$ for $i \in [m + 1, m + n]$, and as $t'_n$ $\cdot$ $t_0$ for $i = m$.

Define $A''_i$ as $A_i$ for $i \in [1, n]$, and as $A'_{i-n}$ for $i \in [n + 1, n + m]$. Define $B''_i$ for $i \in [1, m]$, and as $B_{i-m}$ for $i \in [m + 1, m + n]$. Define $al_i$ for $i \in [1, n]$, and as $al'_{i-n}$ for $i \in [n + 1, n + m]$.

From Lemma 8, as $\cdot (s_0; A_1; \cdot \cdot \cdot A_n; s_n)$, $\cdot (s'_0; A'_1; \cdot \cdot \cdot A'_m; s'_m)$, and $\cdot (s_0; A_1; \cdot \cdot \cdot A_n; s_n)$, $\cdot (s'_0; A'_1; \cdot \cdot \cdot A'_m; s'_m)$, then $\cdot (s''_0; A''_1; \cdot \cdot \cdot A''_{n+m}; s''_{n+m})$.

From Lemma 8, as $\cdot (t'_0; B'_{\sigma_1(1)}; \cdot \cdot \cdot B'_{\sigma_1(m); t'_m})$, $\cdot (t_0; B'_{\sigma_1(1)}; \cdot \cdot \cdot B_{\sigma_1(n); t_n})$, and $\cdot (t'_0; B'_{\sigma_2(1)}; \cdot \cdot \cdot B'_{\sigma_2(m); t'_m} \cdot \cdot \cdot B'_{\sigma_2(n); t'_n})$, then $\cdot (t'_0; B'_{\sigma_1(1)}; \cdot \cdot \cdot B'_{\sigma_1(m); t'_m})$, $\cdot (t_0; B'_{\sigma_1(1)}; \cdot \cdot \cdot B_{\sigma_1(n); t_n})$, and $\cdot (t'_0; B'_{\sigma_2(1)}; \cdot \cdot \cdot B'_{\sigma_2(m); t'_m} \cdot \cdot \cdot B'_{\sigma_2(n); t'_n})$.

Consider the derivation
We wish to show that this is a derivation of \( \text{sql}_1 \otimes \text{sql}_2 : \text{SQ}_1 \circledast \text{SQ}_2 \Rightarrow \text{TQ}_1 \circledast \text{TQ}_2 \).

By the definition of \( \otimes \), we have:

\[
\begin{align*}
[s_0' \cdot A_1' \cdot \ldots \cdot A_n' \cdot s_{n+m}'] &= [s_0' \cdot A_1' \cdot \ldots \cdot A_n' \cdot s_{n+1}' \cdot A_{n+1}' \cdot \ldots \cdot A_{n+m}' \cdot s_{n+m}'] \\
[t_0' \cdot B_1'' \cdot \ldots \cdot B_2'' \cdot s_{n+m}'] &= [t_0' \cdot B_1'' \cdot \ldots \cdot B_2'' \cdot s_{n+1}'' \cdot B_{n+1}'' \cdot \ldots \cdot B_{n+m}'' \cdot t_{n+m}'] \\
&= \text{SQ}_1 \circledast \text{SQ}_2 \\
&= \text{TQ}_1 \circledast \text{TQ}_2
\end{align*}
\]

We also have a derivation of \( \text{sql}_1 \circledast \text{sql}_2 : \text{SQ}_1 \circledast \text{SQ}_2 \Rightarrow \text{TQ}_1 \circledast \text{TQ}_2 \).

\[
[\langle (s_0', t_0'), \ldots, s_{n+m}', t_{n+m}' \rangle, \sigma_1 \otimes \sigma_2(1) \cdot \ldots \cdot \sigma_1 \otimes \sigma_2(n+m) \cdot t_{n+m}']
\]

\[
= \langle (s_0', t_0'), \ldots, s_{n+m}', t_{n+m}' \rangle \\
\text{sql}_1 \circledast \text{sql}_2
\]

\[
\left[ (s', t') \mid (s, t) \in [\text{sql}_1] \cup (s', t') \in [\text{sql}_2] \right]
\]

\[
\square
\]

**Lemma 100** (Typing and Semantics of \( \otimes \)). Let \( d_1 : \text{DS}_1 \Rightarrow \text{DT}_1 \) and \( d_2 : \text{DS}_2 \Rightarrow \text{DT}_2 \) be the typing of two DNF lenses, where \( \mathcal{L}(\text{DS}_j) \cdot \mathcal{L}(\text{DS}_j) \) and \( \mathcal{L}(\text{DT}_j) \cdot \mathcal{L}(\text{DT}_j) \). Then \( \text{ds}_1 \otimes \text{ds}_2 : \text{ds}_1 \circ \text{ds}_2 \Rightarrow \text{dt}_1 \circ \text{dt}_2 \) and \( \langle \text{ds}_1 \circ \text{ds}_2 \rangle = \{(s_1, t_1, t_2) \mid (s_1, t_1) \in \langle \text{ds}_1 \rangle \cup (s_2, t_2) \in \langle \text{ds}_2 \rangle \} \)

**Proof.** By assumption, there exists typing derivations

\[
d_1 : \text{DS}_1 \Rightarrow \text{DT}_1
\]

and

\[
d_2 : \text{DS}_2 \Rightarrow \text{DT}_2
\]

By inversion, we know that the last rule application on each side was DNF Lens, giving

\[
\left[ \langle \text{sql}_1 \rangle \ldots \langle \text{sql}_n \rangle, \sigma_1 \rangle \right] \Rightarrow \langle \text{SQ}_1 \rangle \ldots \langle \text{SQ}_n \rangle \Rightarrow \langle \text{TQ}_{\sigma_1(1)} \rangle \ldots \langle \text{TQ}_{\sigma_1(n)} \rangle
\]

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Lemma 101

Let $d_1 : D_{S_1} \Leftrightarrow D_{T_1}$ and $d_2 : D_{S_2} \Leftrightarrow D_{T_2}$ be the typing of two DNF lenses, where $\mathcal{L}(D_{S_1}) \downarrow \mathcal{L}(D_{S_2})$ and $\mathcal{L}(D_{T_1}) \downarrow \mathcal{L}(D_{T_2})$. Then $d_1 \circ d_2 : D_{S_1} \circ D_{S_2} \Leftrightarrow D_{T_1} \circ D_{T_2}$ and $\llbracket d_1 \circ d_2 \rrbracket = \{(s_1 \cdot s_2, t_1 \cdot t_2) | (s_1, t_1) \in \llbracket d_1 \rrbracket \land (s_2, t_2) \in \llbracket d_2 \rrbracket\}$

Proof. By assumption, there exists typing derivations

$d_1 : D_{S_1} \Leftrightarrow D_{T_1}$
and

$$dl_2 : DS_2 \leftrightarrow DT_2$$

By inversion, we know that the last rule application on each side was DNF LENS, giving

$$\sigma_1 \in S_n \quad i \neq j \Rightarrow L(SQ_i) \cap L(SQ_j) = \emptyset \quad i \neq j \Rightarrow L(TQ_i) \cap L(TQ_j) = \emptyset$$

$$\langle \langle sq_1 | \ldots | sq_n \rangle, \sigma_1 \rangle : \langle SQ_1 | \ldots | SQ_n \rangle \leftrightarrow \langle TQ_{\sigma_1(1)} | \ldots | TQ_{\sigma_1(n)} \rangle$$

and

$$\sigma_2 \in S_m \quad i \neq j \Rightarrow L(SQ'_i) \cap L(SQ'_j) = \emptyset \quad i \neq j \Rightarrow L(TQ'_i) \cap L(TQ'_j) = \emptyset$$

$$\langle \langle sq'_1 | \ldots | sq'_n \rangle, \sigma_2 \rangle : \langle SQ'_1 | \ldots | SQ'_n \rangle \leftrightarrow \langle TQ'_{\sigma_2(1)} | \ldots | TQ'_{\sigma_2(m)} \rangle$$

where

$$dl_1 = \langle \langle sq_1 | \ldots | sq_n \rangle, \sigma_1 \rangle$$

$$DS_1 = \langle SQ_1 | \ldots | SQ_n \rangle$$

$$DT_1 = \langle TQ_{\sigma_1(1)} | \ldots | TQ_{\sigma_1(n)} \rangle$$

$$dl_2 = \langle \langle sq'_1 | \ldots | sq'_n \rangle, \sigma_2 \rangle$$

$$DS_2 = \langle SQ'_1 | \ldots | SQ'_n \rangle$$

$$DT_2 = \langle TQ'_{\sigma_2(1)} | \ldots | TQ'_{\sigma_2(m)} \rangle$$

Define \( SQ_{i,j} \) as \( SQ_i \circ SQ' \).

Define \( TQ_{i,j} \) as \( TQ'_i \circ SQ TQ_l \).

From Lemma 9, as \( i \neq j \Rightarrow L(SQ_j) \cap L(SQ_j) = \emptyset \), \( i \neq j \Rightarrow L(SQ'_j) \cap L(SQ'_j) = \emptyset \), and

$$\langle \langle sq_1 | \ldots | sq_n \rangle, \sigma_1 \rangle : \langle SQ_1 | \ldots | SQ_n \rangle \leftrightarrow \langle TQ_{\sigma_1(1)} | \ldots | TQ_{\sigma_1(n)} \rangle$$

and

$$\langle \langle sq'_1 | \ldots | sq'_n \rangle, \sigma_2 \rangle : \langle SQ'_1 | \ldots | SQ'_n \rangle \leftrightarrow \langle TQ'_{\sigma_2(1)} | \ldots | TQ'_{\sigma_2(m)} \rangle$$

Consider the derivation

$$\langle \langle sq_1 | \ldots | sq_n \rangle, \sigma_1 \circ \sigma_2 \rangle : \langle SQ_1 | \ldots | SQ_n \rangle \leftrightarrow \langle TQ_{\sigma_1(1)} | \ldots | TQ_{\sigma_1(n)} \rangle$$

We wish to show that this is a derivation of \( dl_1 \circ dl_2 : DS_1 \circ DS_2 \leftrightarrow DT_2 \circ DT_1 \).

\( \langle \langle sq_1 | \ldots | sq_n \rangle, \sigma_1 \circ \sigma_2 \rangle = \langle \langle sq_1 \circ SQ \sqcap sq_1' | \ldots | sq_n \circ SQ \sqcap sq_n' \rangle, \sigma_1 \circ \sigma_2 \rangle = dl_1 \circ dl_2 \).

\( \langle SQ_1 | \ldots | SQ_n \rangle = \langle SQ_1 \circ SQ \sqcap SQ_1' | \ldots | SQ_n \circ SQ \sqcap SQ_n' \rangle = DS_1 \circ DS_2 \).

\( \langle TQ_{\sigma_1(1)} | \ldots | TQ_{\sigma_1(n)} \rangle = \langle TQ'_1 \circ SQ TQ_l | \ldots | SQ TQ_n' \rangle = DT_2 \circ DT_1 \).

So we have a derivation of \( \langle \langle sq_1 \circ \sqcap sq \rangle, \sigma_1 \circ \sigma_2 \rangle : \langle SQ_1 \circ SQ \sqcap SQ_1' | \ldots | SQ_n \circ SQ \sqcap SQ_n' \rangle \leftrightarrow \langle TQ_{\sigma_1(1)} | \ldots | TQ_{\sigma_1(n)} \rangle \).

We also wish to have the desired semantics.

\[
\ll \langle \langle sq_1 | \ldots | sq_n \rangle, \sigma_1 \circ \sigma_2 \rangle \rr = \{(s_1 \cdot s_2, t_2 \cdot t_1) | \exists i, j, (s_1, t_1) \in \ll sq_1 \rr \land (s_2, t_2) \in \ll sq_j \rr \}
\]

\[
= \{(s_1 \cdot s_2, t_2 \cdot t_1) | (s_1, t_1) \in \ll dl \rr \land (s_2, t_2) \in \ll dl' \rr \}
\]

\[
\square
\]
Lemma 102 (Typing and Semantics of $\oplus$). Let $dl_1 : DS_1 \iff DT_1$ and $dl_2 : DS_2 \iff DT_2$ be the typing of two DNF lenses, where $L(DS_1) \cap L(DS_2) = \emptyset$ Then $dl_1 \oplus dl_2 : DS_1 \oplus DS_2 \iff DT_1 \oplus DT_2$ and $[dl_1 \oplus dl_2] = \{(s, t) \mid (s, t) \in [dl_1] \lor (s, t) \in [dl_2]\}$

Proof. By assumption, there exists typing derivations

$$dl_1 : DS_1 \iff DT_1$$

and

$$dl_2 : DS_2 \iff DT_2$$

By inversion, we know that the last rule application on each side was DNF LENS, giving

$$sql_i : SQ_i \iff TQ_i \quad \sigma_i \in S_n \quad i \neq j \Rightarrow SQ_i \cap SQ_j = \emptyset \quad i \neq j \Rightarrow TQ_i \cap TQ_j = \emptyset$$

$$i \neq j \Rightarrow SQ_i \cap SQ_j = \emptyset \quad i \neq j \Rightarrow TQ_i \cap TQ_j = \emptyset$$

where $dl_1 = (\langle sql_1 | \ldots | sql_n \rangle, \sigma_1), DS_1 = \langle SQ_1 | \ldots | SQ_n \rangle$, $DT_1 = \langle TQ_{\sigma_1(1)} | \ldots | TQ_{\sigma_1(n)} \rangle$, $dl_2 = (\langle sql_1' | \ldots | sql_m' \rangle, \sigma_2), DS_2 = \langle SQ_1' | \ldots | SQ_m' \rangle$, and $DT_2 = \langle TQ_{\sigma_2(1)} | \ldots | TQ_{\sigma_2(m)} \rangle$.

Define $SQ_i$ as $SQ_i \cap \emptyset$ for $i \in [n + 1, n + m]$. Define $TQ_i$ as $TQ_i \cap \emptyset$ for $i \in [n + 1, n + m]$. Define $sql_i$ as $sql_i \cap \emptyset$ for $i \in [n + 1, n + m]$.

If $i \neq j$, then $TQ_i \cap TQ_j = \emptyset$, for all $i, j \in [n + 1, n + m]$.

Consider the derivation

$$sql_i : SQ_i \iff TQ_i \quad \sigma_1 \cap \sigma_2 \in S_n \quad i \neq j \Rightarrow SQ_i \cap SQ_j = \emptyset \quad i \neq j \Rightarrow TQ_i \cap TQ_j = \emptyset$$

$$i \neq j \Rightarrow SQ_i \cap SQ_j = \emptyset \quad i \neq j \Rightarrow TQ_i \cap TQ_j = \emptyset$$

We wish to show that this is a derivation of $dl_1 \oplus dl_2 : DS_1 \oplus DS_2 \iff DT_1 \oplus DT_2$.

$$\langle sql_1 | \ldots | sql_n \rangle, \sigma_1 \cap \sigma_2 \rangle = \langle sql_1' | \ldots | sql_m' \rangle, \sigma_1 \cap \sigma_2 \rangle = dl_1 \oplus dl_2$$

$$\langle SQ_1 | \ldots | SQ_n \rangle = \langle SQ_1' | \ldots | SQ_m' \rangle = DS_1 \oplus DS_2$$

$$\langle TQ_{\sigma_1(1)} | \ldots | TQ_{\sigma_1(n + m)} \rangle = \langle TQ_{\sigma_2(1)} | \ldots | TQ_{\sigma_2(n + m)} \rangle = DT_1 \oplus DT_2$$

So we have a derivation of $dl_1 \oplus dl_2 : DS_1 \oplus DS_2 \iff DT_1 \oplus DT_2$.

We also wish to have the desired semantics.

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Lemma 103 (Typing and Semantic of $\mathcal{D}$). If $al : A \Leftrightarrow B$ is the typing of a rewriteless Atom lens, then $\mathcal{D}(al) : \mathcal{D}(A) \Leftrightarrow \mathcal{D}(B)$, and $\{\mathcal{D}(al)\} = \{al\}$.


$\vdash ((\epsilon; A; \epsilon))$ \because $\mathcal{L}(\epsilon) = \{\epsilon\}$. $\vdash ((\epsilon; B; \epsilon))$ \because $\mathcal{L}(\epsilon) = \{\epsilon\}$.

As there is only one sequence, the pointwise disjoint condition for DNF lenses are true vacuously. Consider the typing derivation

$$al : A \Leftrightarrow B \quad \vdash ((\epsilon; A; \epsilon)) \quad \vdash ((\epsilon; B; \epsilon))$$

$$\frac{\{((\epsilon, e)\cdot al\cdot(\epsilon, e)), id\} \vdash [e \cdot A \cdot e] \Leftrightarrow [e \cdot B \cdot e] \quad i \neq j \Rightarrow \mathcal{L}(SQ_i) \cap \mathcal{L}(SQ_j) = \emptyset}{\{([[(\epsilon, e)\cdot al\cdot(\epsilon, e)], id]), id\} \vdash ([e \cdot A \cdot e]) \Leftrightarrow ([e \cdot B \cdot e])}$$

$\mathcal{D}(al) = ((([(\epsilon, e)\cdot al\cdot(\epsilon, e)], id)), id))$.

$\{[(\epsilon, e)\cdot al\cdot(\epsilon, e)]\} = \{(\epsilon, s, e, s, t, e) \mid (s, t) \in \mathcal{L}(al)\} = \{al\}$.

$\{([(\epsilon, e)\cdot al\cdot(\epsilon, e)], id), id\} = \{(s, t) \mid (s, t) \in \{[(\epsilon, e)\cdot al\cdot(\epsilon, e)]\}\} = \{(s, t) \mid (s, t) \in \{al\}\} = \{al\}$

Lemma 104 (Typing and Semantics of $\mathcal{D}(iterate(\cdot))$). Let $dl : DS \Leftrightarrow DT$ be the typing of a rewriteless DNF lens, where $DS^\dagger$ and $DT^\dagger$. $\langle iterate(dl)\rangle : \langle DS^\dagger\rangle \Leftrightarrow \langle DT^\dagger\rangle$ and $\langle iterate(dl)\rangle = \{(s_1 \cdot \ldots \cdot s_n, t_1 \cdot \ldots \cdot t_n) \mid (s_i, t_i) \in \{dl\}\}$

Proof. By assumption, there exists a typing derivation

$$dl : DS \Leftrightarrow DT$$

Consider the typing derivation

$$\frac{dl : DS \Leftrightarrow DT \quad DS^\dagger \quad DT^\dagger}{iterate(dl) : DS^\dagger \Leftrightarrow DT^\dagger}$$

$\langle iterate(dl)\rangle : \langle DS^\dagger\rangle \Leftrightarrow \langle DT^\dagger\rangle$ and the semantics are shown to be equal to the desired semantics.

$$\{[(iterate(dl))]\} = \{(s, t) \mid (s, t) \in \{iterate(dl)\}\}$$

$\{([e \cdot s \cdot e, e \cdot t \cdot e] \mid (s, t) \in \{iterate(dl)\}\}$

$\{([s_1 \cdot \ldots \cdot s_n, t_1 \cdot \ldots \cdot t_n]) \mid (s_i, t_i) \in \{dl\}\}$

$\Box$
B.9 Complex Lens Operator Properties

The previous properties of lens operators were merely about the operators, and how they could be typed. This portion writes about how lens operators have the same semantics as lenses with very complex properties, up to the existence of an identity lens. Much of this complication comes from the fact that the DNF regular expression operators don’t have right distributivity. An analogue to the commutativity of regular expression Or to be expressed using these properties.

Lemma 105 (Commutativity of $\oplus$). If there exists a lens $dl : DS_1 \oplus DS_2 \Leftrightarrow DT_1 \oplus DT_2$, then there exists a lens $dl : DS_1 \oplus DS_2 \Leftrightarrow DT_2 \oplus DT_1$.

**Proof.** Let $DS_1 = \langle SQ_{1,1} \mid \ldots \mid SQ_{1,n} \rangle$.
Let $DS_2 = \langle SQ_{2,1} \mid \ldots \mid SQ_{2,n} \rangle$.
Let $DT_1 = \langle TQ_{1,1} \mid \ldots \mid TQ_{1,m} \rangle$.
Let $DT_2 = \langle TQ_{2,1} \mid \ldots \mid TQ_{2,m} \rangle$.
Let $dl = \langle (sq_i | \ldots | sq_{n+m}), \sigma \rangle$

So $DS_1 \oplus DS_2 = \langle SQ_1 | \ldots | SQ_{n+m} \rangle$, where $SQ_i = \begin{cases} SQ_{1,i} & \text{if } i \leq n \\ SQ_{2,i-n} & \text{if } i > n \end{cases}$

So $DT_1 \oplus DT_2 = \langle TQ_{\sigma(1)} | \ldots \mid TQ_{\sigma(m+m')} \rangle$, where $TQ_{\sigma(i)} = \begin{cases} TQ_{1,i} & \text{if } i \leq m \\ TQ_{2,i-m} & \text{if } i > m \end{cases}$

By inversion

$$
\begin{array}{c}
\sigma \in S_n \\
i \neq j \Rightarrow L(SQ_i) \cap L(SQ_j) = \emptyset \\
i \neq j \Rightarrow L(TQ_i) \cap L(TQ_j) = \emptyset \\
\end{array}
$$

Consider the permutation $\sigma'(i) = \begin{cases} \sigma(i+m) & \text{if } \sigma(i) \leq m' \\ \sigma(i-m) & \text{if } \sigma(i) > m' \end{cases}$

Consider the lens

$$
\begin{array}{c}
\sigma' \in S_n \\
i \neq j \Rightarrow L(SQ_i) \cap L(SQ_j) = \emptyset \\
i \neq j \Rightarrow L(TQ_i) \cap L(TQ_j) = \emptyset \\
\end{array}
$$

So $TQ_{\sigma'(i)} = \begin{cases} TQ_{\sigma(i+m)} = TQ_{2,i} & \text{if } \sigma(i) \leq m' \\ TQ_{\sigma(i-m)} = TQ_{1,i} & \text{if } \sigma(i) > m \end{cases}$

So $\langle TQ_{\sigma'(1)} \mid \ldots \mid TQ_{\sigma'(m+m')} \rangle = DT_2 \oplus DT_1$.

Furthermore, The semantics are the same, as permutation has no impact on the semantics of DNF lenses. $\square$

Lemma 106 (Left Unrolling of $iterate$). If $iterate(dl) : DS^* \Leftrightarrow DT^*$ is an atom lens, then $dl' = \langle ([e, e]) \oplus (dl \circ D(iterate(dl))) \rangle : \langle [e] \rangle \oplus (DS \circ D(DS^*)) \Leftrightarrow \langle [e] \rangle \oplus (DT \circ D(DT^*))$ is a DNF Lens with $\langle iterate(dl) \rangle = \langle dl' \rangle$

**Proof.** So $DS^*$ and $DT^*$ are strongly unambiguous atoms.

As such, this means $DS^{\ast} \circ D^{\ast}$ is strongly unambiguous, and $DT$ is strongly unambiguous.

Want to show: because $DS^{\ast} \circ D^{\ast}$ is strongly unambiguous, and $DT$ is strongly unambiguous.

This is $s_1 \cdot l_{1,1} \cdot \ldots \cdot l_{1,n}$ and $s_2 \cdot l_{2,1} \cdot \ldots \cdot l_{2,m}$, where each substring is in $L(DS)$. By unambiguous iteration, $n = n'$, and $s_1 = s_2$, $s_{1,1} = l_{1,1}$, so $s_1 = s_2$ and $l_{1,1} = l_{2,1}$. As such Lemma 100 applies, so $\langle dl \circ D(iterate(dl)) \rangle : (DS \circ D(DS^*)) \Leftrightarrow (DT \circ D(DT^*))$.
Want to show: because $DS^{\sigma}, \mathcal{L}(\langle[e] \rangle) \cap \mathcal{L}(DS \odot D(\mathcal{D}(DS^*))) = \emptyset$. $\epsilon$ is the only element of $\mathcal{L}(\langle[e] \rangle)$. $\epsilon \not\in L(DS \odot D(\mathcal{D}(DS^*))$, as it cannot be in $\mathcal{L}(DS)$. Otherwise if $\epsilon \in L(\mathcal{D}(DS))$, then for all $s_1 \cdots s_n = e \cdot s_1 \cdots s_n$, betraying unambiguous iteration.

As such Lemma 102 applies, so $\langle[(\epsilon, e)]\rangle \odot (dl \odot D(\text{iterate}(dl))) : \langle[(\epsilon, e)]\rangle \odot (D(\mathcal{D}(DS^*)) \Leftrightarrow \langle[(\epsilon, e)]\rangle \odot (DT \odot D(DT^*))$.

If $\langle[(\epsilon, e)]\rangle \odot (dl \odot D(\text{iterate}(dl))) : \langle[(\epsilon, e)]\rangle \odot (D(\mathcal{D}(DS^*))$.

**Lemma 107** (Right Unrolling of **iterate**). If $\text{iterate}(dl) : DS^* \Leftrightarrow DT^*$ is an atom lens, then $dl' = \langle[(\epsilon, e)]\rangle \odot (D(\text{iterate}(dl)) \odot dl) : \langle[e]\rangle \odot (D(\mathcal{D}(DS^*)) \odot DT) \Leftrightarrow \langle[e]\rangle \odot (D(DT^*) \odot DT)$ is a DNF Lens with $\text{iterate}(dl) = \|dl'\|$

**Proof.** This is proven symmetrically to Lemma 106.

**Lemma 108** (Left Unrolling of **iterate** DNF). If $D(\text{iterate}(dl))$ is a DNF lens, then $dl' = \langle[(\epsilon, e)]\rangle \odot (dl \odot D(\text{iterate}(dl)))$ is a DNF Lens with $\text{iterate}(dl) = \|dl'\|$

**Proof.** This is through a combination of Lemma 106 and Lemma 103.

**Lemma 109** (Right Unrolling of **iterate** DNF). If $D(\text{iterate}(dl))$ is a DNF lens, then $dl' = \langle[(\epsilon, e)]\rangle \odot (D(\text{iterate}(dl)) \odot dl)$ is a DNF Lens with $\text{iterate}(dl) = \|dl'\|$

**Proof.** This is through a combination of Lemma 107, and Lemma 103.

**Lemma 110** (Expressibility of Adjacency Swapping Permutation of Separated Concat List). Let for all $\sigma \in [1, n]$, $dl_i : DS_i \Leftrightarrow DT_i$. Let $\sigma_i$ be an adjacency swapping permutation, where $1 \leq i < n$. There exists a DNF lens $dl : \langle\langle]\rangle \odot (\text{DS}_1 \odot \ldots \odot (\text{DS}_n \odot \langle\langle]\rangle) \Leftrightarrow \langle\langle]\rangle \odot DT_{\sigma_1(1)} \odot \ldots \odot DT_{\sigma_1(n)}$, where $\|DS\| = \{(s \cdot s_1 \cdot s_\ldots \cdot s_{n-1} \cdot s_{n} \cdot \sigma_{\alpha(1)} \cdot s_{\ldots} \cdot s_{\sigma_{\alpha(n)} \cdot s} \mid (s_i, t_i) \in \|dl\|\}$

**Proof.** As $DS_i$ and $DT_i$ are strongly unambiguous, by Lemma 17, and from Lemma 88, we have that $\cdot (\langle\langle]\rangle \odot (\text{DS}_1 \odot \ldots \odot (\text{DS}_n \odot \langle\langle]\rangle)$ and $\cdot (\langle\langle]\rangle \odot DT_{\sigma(1)} \odot \ldots \odot DT_{\sigma(n)}$.\(\langle\langle]\rangle\)$

Consider the lens $\langle\langle]\rangle \odot dl_1 \odot \ldots \odot ((\langle\langle]\rangle \odot dl_1) \odot dl_1) \odot \ldots \odot dl_{n-1} \odot \langle\langle]\rangle)$, which by Lemma 100 and Lemma 101. $\langle\langle]\rangle \odot dl_1 \odot \ldots \odot ((\langle\langle]\rangle \odot dl_1) \odot dl_1) \odot \ldots \odot dl_{n-1} \odot \langle\langle]\rangle \Leftrightarrow \langle\langle]\rangle \odot DS_1 \odot \langle\langle]\rangle \odot \ldots \odot \langle\langle]\rangle \odot DS_n \odot \langle\langle]\rangle \Leftrightarrow \langle\langle]\rangle \odot DS_1 \odot \langle\langle]\rangle \odot \ldots \odot \langle\langle]\rangle \odot DS_n \odot \langle\langle]\rangle \odot \langle\langle]\rangle$ as desired.

Also by Lemma 101, the semantics are as desired.

**Lemma 111** (Expressibility of Permutation of Separated Concat List). Let for all $\sigma \in [1, n]$, $dl_i : DS_i \Leftrightarrow DT_i$. Let $\sigma$ be a permutation, where $1 \leq i < n$. There exists a DNF lens $dl : \langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle)$, where $\|DS\| = \{(s \cdot s_1 \cdot s_{\ldots} \cdot s_{n} \cdot s_{\sigma_{\alpha(1)} \cdot s_{\ldots} \cdot s_{\sigma_{\alpha(n)} \cdot s} \mid (s_i, t_i) \in \|dl\|\}$

**Proof.** As $DS_i$ and $DT_i$ are strongly unambiguous, by Lemma 17, and as $\langle\langle]\rangle \cdot L$, and $L \cdot (\langle\langle]\rangle)$, for all $L$, we have that $\langle\langle]\rangle \odot DS_1 \odot \langle\langle]\rangle \odot \ldots \odot \langle\langle]\rangle \odot (\text{DS}_n \langle\langle]\rangle)$ is strongly unambiguous.

From algebra, $\sigma$ can be decomposed into a series of $\sigma_{ij} \ldots \sigma_{ij}$. We proceed by induction.

**Case 1** ($j = 0$). $\sigma = id$.

Through repeated application of $\odot$, $\langle\langle]\rangle \odot dl_1 \odot \ldots \odot dl_n \odot \langle\langle]\rangle$.

**Case 2** ($j < 0$). $\odot (\langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle) \odot (\text{DS}_1 \odot \langle\langle]\rangle)$, where $\|DS\| = \{(s \cdot s_1 \cdot s_{\ldots} \cdot s_{n} \cdot s_{\sigma_{\alpha(1)} \cdot s_{\ldots} \cdot s_{\sigma_{\alpha(n)} \cdot s} \mid (s_i, t_i) \in \|dl\|\}$

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Case 2 \( (j > 0) \). \( \sigma = \sigma_{1j} \circ \ldots \circ \sigma_{1i}. \sigma' = \sigma_{ij-1} \circ \ldots \circ \sigma_{1i}. \sigma = \sigma_{ij} \circ \sigma' \).

By IH there exists a DNF lens \( dl : \langle \langle s \rangle \rangle DS_1 \circ \langle \langle s \rangle \rangle \ldots \circ \langle \langle s \rangle \rangle DS_n \langle \langle s \rangle \rangle \leftrightarrow \langle \langle s \rangle \rangle \circ DT_{\sigma_{1i}} \circ \ldots \circ DT_{\sigma_{1i}(n)} \circ \langle \langle s \rangle \rangle \), where \( \langle \{DS_i \} = \{\langle s \cdot s_1 \cdot \ldots \cdot s_{n-1} \cdot s_n \cdot t \cdot s'_{\sigma_{1i}(1)} \cdot \ldots \cdot t'_{\sigma_{1i}(n)} \cdot s \rangle \mid (s_i, t_i) \in [dl_i] \} \)

As \( DT_i \) are strongly unambiguous, by Lemma 15, there exists an identity lens \( dl_i' : DT_i \leftrightarrow DT_i \), for each \( DT_i \).

By Lemma 110, there exists a DNF lens \( dl' : \langle \langle s \rangle \rangle DT_{\sigma_{1i}(1)} \circ \langle \langle s \rangle \rangle \ldots \circ \langle \langle s \rangle \rangle DT_{\sigma_{1i}(n)} \langle \langle s \rangle \rangle \leftrightarrow \langle \langle s \rangle \rangle \circ DT_{\sigma_{1i}(1)} \circ \ldots \circ DT_{\sigma_{1i}(n)} \langle \langle s \rangle \rangle \), where \( \langle \{DS \} = \{\langle s \cdot t_{\sigma_{1i}(1)} \cdot \ldots \cdot t_{\sigma_{1i}(n)} \cdot s \rangle \mid (s_i, t_i) \in [dl_i'] \} \}

So, by Lemma 14, there exists a DNF lens \( dl : \langle \langle s \rangle \rangle DS_1 \circ \langle \langle s \rangle \rangle \ldots \circ \langle \langle s \rangle \rangle DS_n \langle \langle s \rangle \rangle \leftrightarrow \langle \langle s \rangle \rangle \circ DT_{\sigma_{1i}(1)} \circ \ldots \circ DT_{\sigma_{1i}(n)} \langle \langle s \rangle \rangle \), where \( \langle \{DS \} = \{\langle s \cdot s_1 \cdot \ldots \cdot s_n \cdot t \cdot s'_{\sigma_{1i}(1)} \cdot \ldots \cdot t'_{\sigma_{1i}(n)} \cdot s \rangle \mid (s_i, t_i) \in [dl_i'] \} \}

\( \square \)

Lemma 112 (Expressibility of Concat Permutation). Let for all \( i \in [1, n], dl_i : DS_i \leftrightarrow DT_i. \) Let \( \sigma \) be a permutation. Let \( \cdot \langle (s_0, DS_1, \ldots, DT_{n}, s_n) \rangle. \) There exists a DNF lens \( dl : \langle \langle s_0 \rangle \rangle \langle \langle s_1 \rangle \rangle \ldots \circ \langle \langle s_{n-1} \rangle \rangle DS_n \langle \langle s_n \rangle \rangle \leftrightarrow \langle \langle s_0 \rangle \rangle \circ DS_n \langle \langle s_n \rangle \rangle \), where \( \langle \{DS \} = \{\langle s_0 \cdot s_1 \cdot \ldots \cdot s_n \cdot s_0 \cdot t \cdot s'_{\sigma(1)} \cdot \ldots \cdot t'_{\sigma(n)} \cdot t_n \rangle \mid (s_i, t_i) \in [dl_i] \} \}

Proof. By Lemma 17, \( DS_i \) and \( DT_i \) are strongly unambiguous.

By Lemma 15, there exists \( dl_i' : DS_i \leftrightarrow DT_i \), which are the identity transformations.

By Lemma 15, there exists \( dl_i' : DT_i \leftrightarrow DT_i \), which are the identity transformations.

Consider the lenses

\[
\begin{align*}
\langle \langle s_1, s \rangle \rangle &\circ \langle \langle s \rangle \rangle : [s_1] \leftrightarrow [s] \\
\langle \langle s, s \rangle \rangle &\circ \langle \langle s \rangle \rangle : [s] \leftrightarrow [s] \\
\langle \langle s, t \rangle \rangle &\circ \langle \langle t \rangle \rangle : [s] \leftrightarrow [t] \\
\langle \langle s, t \rangle \rangle &\circ \langle \langle s \rangle \rangle : [s] \leftrightarrow [t]
\end{align*}
\]

Because \( \cdot \langle (s, DS_1, \ldots, DS_n, s) \rangle \), through repeated application of Lemma 100, \( \langle \langle s, DS_1 \rangle \rangle \circ \ldots \circ \langle \langle s \rangle \rangle DS_n \circ \langle \langle s \rangle \rangle \), with semantics

\( \langle \{s_0 \rangle \langle \langle s_1 \rangle \rangle \ldots \circ \langle \langle s_n \rangle \rangle \rangle : \langle \langle s_0 \rangle \rangle \circ DS_n \langle \langle s_n \rangle \rangle \),

where \( \langle \{DS \} = \{\langle s_0 \cdot s_1 \cdot \ldots \cdot s_n \cdot s_0 \cdot t \cdot s'_{\sigma(1)} \cdot \ldots \cdot t'_{\sigma(n)} \cdot t_n \rangle \mid (s_i, t_i) \in L(DS_i) \} \}

Because \( \cdot \langle (s, DS_1, \ldots, DS_n, s) \rangle \), through repeated application of Lemma 100, \( \langle \langle s \rangle \rangle \circ \ldots \circ \langle \langle s_n \rangle \rangle \rangle : \langle \langle s \rangle \rangle \circ DS_n \langle \langle s_n \rangle \rangle \), with semantics

\( \langle \{s_0 \rangle \langle \langle s_1 \rangle \rangle \ldots \circ \langle \langle s_n \rangle \rangle \rangle : \langle \langle s_0 \rangle \rangle \circ DS_n \langle \langle s_n \rangle \rangle \),

where \( \langle \{DS \} = \{\langle s_0 \cdot s_1 \cdot \ldots \cdot s_n \cdot s_0 \cdot t \cdot s'_{\sigma(1)} \cdot \ldots \cdot t'_{\sigma(n)} \cdot t_n \rangle \mid (s_i, t_i) \in L(DS_i) \} \}

By Lemma 111, there exists a lens \( dl : \langle \langle s \rangle \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle \langle s \rangle \rangle : \langle \langle s \rangle \rangle \circ DS_1 \circ \ldots \circ \langle \langle s \rangle \rangle DS_n \circ \langle \langle s \rangle \rangle \), with semantics

\( \langle \{ds \} = \{\langle s_0 \cdot s_1 \cdot \ldots \cdot s_n \cdot s_0 \cdot t \cdot s'_{\sigma(1)} \cdot \ldots \cdot t'_{\sigma(n)} \cdot t_n \rangle \mid (s_i, t_i) \in L(DS_i) \} \}

By Lemma 14, there exists a lens \( dl' : \langle \langle s \rangle \rangle \circ DS_1 \circ \ldots \circ DS_n \circ \langle \langle s \rangle \rangle : \langle \langle s \rangle \rangle \circ DS_1 \circ \ldots \circ \langle \langle s \rangle \rangle DS_n \circ \langle \langle t \rangle \rangle \rangle \), where \( \langle \{DS \} = \{\langle s_0 \cdot s_1 \cdot \ldots \cdot s_n \cdot s_0 \cdot t \cdot s'_{\sigma(1)} \cdot \ldots \cdot t'_{\sigma(n)} \cdot t_n \rangle | (t', t) \in [dl_i'] \} \}

\( \square \)

Lemma 113 (Identity Transformation on Adjacent Swapping Or). Let \( DS_1, \ldots, DS_n \) be strongly unambiguous DNF regular expressions, where \( j \neq k \Rightarrow L(DS_j) = L(DS_j) \).

Let \( \sigma_i \) be an adjacent swapping permutation.

There exists a lens \( dl : DS_1 \circ \ldots \circ DS_n : DS_{\sigma(1)} \circ \ldots \circ DS_{\sigma(n)} \), such that \( \langle dl \rangle = \{s, s \mid s \in L(DS_1 \circ \ldots \circ DS_n) \} \).
Proof. As each DNF regular expression is strongly unambiguous, there exists a DNF lens
\( dl_j : DS_j \iff DS_j \) such that \( \| dl_j \| = \{(s, s) \mid s \in \mathcal{L}(DS_j)\} \). By assumption, \( \mathcal{L}(DS_j) \cap \mathcal{L}(DS_{j+1}) = \emptyset \), so
by Lemma 105, there exists \( dl' : DS_i \oplus DT_i \iff DT_i \oplus DS_i \). By repeated application of Lemma 102,
\( dl_0 \oplus \ldots \oplus dl_{i-1} \oplus dl' \oplus dl_{i+2} \oplus \ldots \oplus dl_n : DS_1 \oplus \ldots \oplus DS_n \iff DS_1 \oplus \ldots \oplus DS_{i+1} \oplus DS_i \oplus \ldots \oplus DS_n \).

Because each of the lenses included in this is the identity lens, the overall lens is the identity lens. \( \square \)

Lemma 114 (Identity Transformation on Or Permutations). Let \( DS_1, \ldots, DS_n \) be strongly unambiguous DNF regular expressions, where \( j \neq k \Rightarrow L(DS_j) = L(DS_k) \).

Let \( \sigma \) be a permutation.

There exists a lens \( dl : DS_1 \oplus \ldots \oplus DS_n \iff DS_{\sigma(1)} \oplus \ldots \oplus DS_{\sigma(n)} \), such that \( \| dl \| = \{(s, s) \mid s \in L(DS_1 \oplus \ldots \oplus DS_n)\} \).

Proof. From algebra, there exists a decomposition of \( \sigma \) into adjacency switching permutations \( \sigma = \sigma_{i_1} \circ \ldots \circ \sigma_{i_l} \).

We prove this by induction on \( n \! . \!

Case 1 (\( n = 0 \)). \( \sigma = id \)

As each \( DS_i \) there exists an identity transformation \( dl_i : DS_i \iff DS_i \).

By repeated application of Lemma 102, \( dl_0 \oplus \ldots \oplus dl_n : DS_1 \oplus \ldots \oplus DS_n \iff DS_1 \oplus \ldots \oplus DS_n \)
with semantics, as each of the lenses that built it up have identity semantics.

Case 2 (\( n > 0 \)). \( \sigma = \sigma_{i_1} \circ \ldots \circ \sigma_{i_l} \) Define \( \sigma' = \sigma_{i_{l-1}} \circ \ldots \circ \sigma_{i_l} \) By IH, there exists a DNF lens
\( dl : DS_1 \oplus \ldots \oplus DS_n \iff DS_{\sigma'(1)} \oplus \ldots \oplus DS_{\sigma'(n)} \).

By Lemma 113, there exists a lens \( dl' : DS_{\sigma'(1)} \oplus \ldots \oplus DS_{\sigma'(n)} \iff DS_{\sigma_{\sigma'(1)}(1)} \oplus \ldots \oplus DS_{\sigma_{\sigma'(n)}(n)} \), so \( dl' : DS_{\sigma'(1)} \oplus \ldots \oplus DS_{\sigma'(n)} \iff DS_{\sigma(1)} \oplus \ldots \oplus DS_{\sigma(n)} \), where \( dl \) has the identity semantics.

By Lemma 14, there exists \( dl'' : DS_1 \oplus \ldots \oplus DS_n \iff DS_{\sigma(1)} \oplus \ldots \oplus DS_{\sigma(n)} \). As each of its component transformations has identity semantics, it too has identity semantics. \( \square \)

Lemma 115 (Or Permutating Lenses). Let \( n \) a natural number, and for all \( i \in [1, n] \), \( dl_i : DS_i \iff DT_i \).

Let \( i \neq j \Rightarrow DS_i \cap DS_j = \emptyset \) and \( i \neq j \Rightarrow DT_i \cap DT_j = \emptyset \). Let \( \sigma \) be a permutation. There exists a lens
\( dl : DS_1 \oplus \ldots \oplus DS_n \iff DT_{\sigma(1)} \oplus \ldots \oplus DT_{\sigma(n)} \) such that \( \| dl \| = \{(s, t) \mid \exists i. (s, t) \in \| dl_i \|\} \).

Proof. By Lemma 102, there exists \( dl_1 \oplus \ldots \oplus dl_n : DS_1 \oplus \ldots \oplus DS_n \iff DT_1 \oplus \ldots \oplus DT_n \) with
\( \| dl_1 \oplus \ldots \oplus dl_n \| = \{(s, t) \mid \exists i. (s, t) \in \| dl_i \|\} \). By Lemma 114, there exists a lens \( dl' : DT_1 \oplus \ldots \oplus DT_n \iff DT_{\sigma(1)} \oplus \ldots \oplus DT_{\sigma(n)} \), with \( \| dl' \| = \{(s, s) \mid s \in L(DT_1 \oplus \ldots \oplus DT_n)\} \). By Lemma 14, there exists \( dl'' : DS_1 \oplus \ldots \oplus DS_n \iff DT_{\sigma(1)} \oplus \ldots \oplus DT_{\sigma(n)} \) with semantics \( \{(s, t) \mid \exists s'. (s, s') \in \| dl_1 \oplus \ldots \oplus dl_n \| \land (s', t) \in \| dl' \|\} \). As \( dl' \) is merely the identity, this has the desired semantics. \( \square \)

Lemma 116 (Propagation of Unambiguity to Subcomponents \( \oplus \)). If \( DS \oplus DT \) is strongly unambiguous, then \( DS \) is strongly unambiguous, \( DT \) is strongly unambiguous, and \( L(DS) \cap L(DT) = \emptyset \).

Proof. Let \( DS = \langle SQ_1 | \ldots | SQ_m \rangle \).

Let \( DT = \langle TQ_1 | \ldots | TQ_m \rangle \).

\( DS \oplus DT = \langle SQ_1 | \ldots | SQ_m | TQ_1 | \ldots | TQ_m \rangle \).

This means that, as it is strongly unambiguous, all of the sequences are pairwise disjoint, and each sequence is strongly unambiguous. By Lemma 10, this means that all of the sequences in \( DS \) are pairwise disjoint, all the sequences in \( DT \) are pairwise disjoint, and \( L(DS) \cap L(DT) = \emptyset \). \( \square \)
Lemma 117 (Reordering of $\odot$ Right). If there exists a DNF lens $dl \uparrow DS_1 \odot \ldots \odot DS_n \iff DT_1 \odot \ldots \odot DT_n$, then for all permutations $\sigma \in S_n$, there exists a DNF lens $dl' \uparrow DS_1 \odot \ldots \odot DS_n \iff DT_{\sigma(1)} \odot \ldots \odot DT_{\sigma(n)}$ where $\|dl'\| = \|dl\|$. 

Proof. From Lemma 17, $DT_1 \odot \ldots \odot DT_n$ is strongly unambiguous. By repeated application of Lemma 116, $i \neq j \Rightarrow DT_i \cap DT_j = \{\}$, and each $DT_i$ is strongly unambiguous.

This means Lemma 114 applies, so there exists a DNF lens $dl' \uparrow DS_1 \odot \ldots \odot DS_n \iff DT_{\sigma(1)} \odot \ldots \odot DT_{\sigma(n)}$ such that $\|dl'\|$ is the identity semantics.

So, by composing $dl$ with $dl'$ from Lemma 14, we get $dl'' \uparrow DS_1 \odot \ldots \odot DS_n \iff DT_{\sigma(1)} \odot \ldots \odot DT_{\sigma(n)}$, which has $\|dl''\| = \|dl\|$ as $dl'$ has identity semantics. 

Lemma 118. If $DS_1 \odot (DS_2 \odot DS_3)$ is strongly unambiguous, then there exists a lens $dl \uparrow DS_1 \odot (DS_2 \odot DS_3) \iff (DS_1 \odot DS_2) \odot (DS_1 \odot DS_3)$. 

Proof. If $L(DS_1 \odot (DS_2 \odot DS_3)) = \{\}$, then this is trivial, as $DS_1 \odot (DS_2 \odot DS_3) = \{\}$.

Assume the language is nonempty. Let $DS_1 = \langle SQ_{1,1}, \ldots, SQ_{1,n_1} \rangle$.

Let $DS_2 = \langle SQ_{2,1}, \ldots, SQ_{2,n_2} \rangle$.

Let $DS_3 = \langle SQ_{3,1}, \ldots, SQ_{3,n_3} \rangle$.

$DS_1 \odot DS_2 \odot DS_3 = \langle SQ_{1,1}, \ldots, SQ_{2,n_2}, SQ_{3,1}, \ldots, SQ_{3,n_3} \rangle$.

$DS_1 \odot (DS_2 \odot DS_3) = \langle SQ_{1,1} \odot SQ_{2,1}, \ldots, SQ_{1,1} \odot SQ_{2,n_2}, SQ_{1,1} \odot SQ_{3,1}, \ldots, SQ_{1,1} \odot SQ_{3,n_3} \rangle$.

As this is strongly unambiguous, $SQ_{1,i} \odot SQ_{j,k}$ is strongly unambiguous for all $i, j, k$.

Furthermore, by strong unambiguity, if $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$, then $SQ_{1,i_1} \odot SQ_{j_1,k_1} \cap SQ_{1,i_2} \odot SQ_{j_2,k_2} = \{\}$.

$DS_1 \odot DS_2 = \langle SQ_{1,1}, \ldots, SQ_{2,n_2}, SQ_{3,1}, \ldots, SQ_{3,n_3} \rangle$.

From before, if $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$, then $SQ_{1,i_1} \odot SQ_{j_1,k_1} \cap SQ_{1,i_2} \odot SQ_{j_2,k_2} = \{\}$.

As $SQ_{1,i} \odot SQ_{j,k}$ is strongly unambiguous, there exists $sql_{i,j,k} \uparrow SQ_{1,i} \odot SQ_{j,k} \iff SQ_{1,i} \odot SQ_{j,k}$, from Lemma 15.

There exists a unique permutation $\sigma$ that sends $SQ_{1,i} \odot SQ_{j,k}$ in $DS_1 \odot (DS_2 \odot DS_3)$ to $SQ_{1,i} \odot SQ_{j,k}$ in $(DS_1 \odot DS_2) \odot (DS_1 \odot DS_3)$. As a permutation is merely a bijective between a finite number of elements. Note, this permutation is not necessarily the identity permutation. In particular, the sequence at position $n_1 + 1$, if such a sequence exists, in $DS_1 \odot (DS_2 \odot DS_3)$ is $SQ_{1,1} \odot SQ_{3,1}$. However, the sequence at position $n_1 + 1$, if such a sequence exists, in $(DS_1 \odot DS_2) \odot (DS_1 \odot DS_3)$, is $SQ_{1,2} \odot SQ_{2,1}$.

Consider the derivation

$sql \uparrow SQ_{1,i} \odot SQ_{j,k} \iff SQ_{1,i} \odot SQ_{j,k}$

$(i_1, j_1, k_1) \neq (i_2, j_2, k_2) \Rightarrow (SQ_{1,i_1} \odot SQ_{j_1,k_1}) \cap (SQ_{1,i_2} \odot SQ_{j_2,k_2}) = \{\}$

$(i_1, j_1, k_1) \neq (i_2, j_2, k_2) \Rightarrow (SQ_{1,i_1} \odot SQ_{j_1,k_1}) \cap (SQ_{1,i_2} \odot SQ_{j_2,k_2}) = \{\} \quad \sigma \in S_{n_xn_yn_z}$

Furthermore, as each $sql$ has the identity transformation, then as $\sigma$ has no impact on semantics, the total DNF lens has the identity transformation. 

\[\square\]
Lemma 119. If \((DS_1 \odot DS_2) \oplus (DS_1 \odot DS_3)\) is strongly unambiguous, then there exists a lens \(dl : (DS_1 \odot DS_2) \oplus (DS_1 \odot DS_3) \iff DS_1 \odot (DS_2 \odot DS_3)\).

Proof. By Lemma 40, \(DS_1 \odot (DS_2 \odot DS_3)\) is strongly unambiguous. So by Lemma 118, there exists an identity lens \(dl : (DS_1 \odot DS_2) \oplus (DS_1 \odot DS_3) \iff DS_1 \odot (DS_2 \odot DS_3)\). As rewriteless DNF lenses are closed under inversion, there exists a lens \(dl^{-1} : DS_1 \odot (DS_2 \odot DS_3) \iff (DS_1 \odot DS_2) \oplus (DS_1 \odot DS_3)\). □

Lemma 120. If \((DS_1 \odot DS_2) \odot DS_3\) is strongly unambiguous, then there exists a lens \(dl : (DS_1 \odot DS_2) \odot DS_3 \iff (DS_1 \odot DS_3) \odot (DS_2 \odot DS_3)\).

Proof. By Lemma 35, \((DS_1 \odot DS_2) \odot DS_3 = (DS_1 \odot DS_3) \odot (DS_2 \odot DS_3)\), so by Lemma 15, there is an identity lens between them. □

Lemma 121. If \((DS_1 \odot DS_2) \odot DS_3\) is strongly unambiguous, then there exists a lens \(dl : (DS_1 \odot DS_3) \odot (DS_2 \odot DS_3) \iff (DS_1 \odot DS_2) \odot DS_3\).

Proof. By Lemma 35, \((DS_1 \odot DS_2) \odot DS_3 = (DS_1 \odot DS_3) \odot (DS_2 \odot DS_3)\), so by Lemma 15, there is an identity lens between them. □

B.10 Rewrite Property Maintenance

Here the proof of bisimilarity and confluence on parallel rewrites with respect to the property of having a lens’s semantics is presented. First a proof must be presented on Parallel Rewrites’ ability to be built up from smaller parts through concatenation. Because of the lack of a distributivity rule, this is only maintained up to an identity lens, we cannot merely concatenate the two rewritten parts. With this, bisimilarity is proven, as is confluence.

Lemma 122 (\(\iff\) Maintained Under \(\odot\) up to \(id\)). Let \(DS\) be strongly unambiguous. Let \(DT\) be strongly unambiguous. Let \(\mathcal{L}(DS)\) \(\iff\) \(\mathcal{L}(DT)\). If \(DS \iff DS'\), \(DT \iff DT'\), and \(DS \odot DT \iff DS''\) such that there exists a rewriteless DNF lens \(dl : DS' \odot DT' \iff DS''\), and \([dl] = \{(s, s) \mid s \in \mathcal{L}(DS\odot DT)\}\).

Proof. Because \(\mathcal{L}(DS) \iff \mathcal{L}(DT)\), \(DS \odot DT\) is strongly unambiguous.

By induction on the derivation of \(\iff\)

\[DS = \langle SQ_1 | \ldots | SQ_n \rangle \quad \forall i. SQ_i = [s_{i,0} \cdot A_{i,1} \cdots A_{i,n_i} \cdot s_{i,n_i}]
\]

\[\forall i, j. A_{i,j} \iff_A DS_{i,j} \quad \forall i. DS_i = \langle [s_{i,0}] \rangle \odot DS_{i,1} \odot \ldots \odot DS_{i,n_i} \odot \langle [s_{i,n_i}] \rangle\]

\[DS \iff DS_1 \odot \ldots \odot DS_n\]

\[DT = \langle TQ_1 | \ldots | SQ_m \rangle \quad \forall i. TQ_i = [t_{i,0} \cdot B_{i,1} \cdots B_{i,m_i} \cdot t_{i,m_i}]
\]

\[\forall i, j. B_{i,j} \iff_B DT_{i,j} \quad \forall i. DT_i = \langle [t_{i,0}] \rangle \odot DT_{i,1} \odot \ldots \odot DT_{i,n_i} \odot \langle [t_{i,n_i}] \rangle\]

\[DT \iff DT_1 \odot \ldots \odot DT_n\]

Define \(A_{i,j,k}'\) as:

\[
A_{i,j,k}' = \begin{cases} 
A_{i,k} & \text{if } k \leq n_i \\
B_{j,k-n_i} & \text{if } i > n_i
\end{cases}
\]

Define \(s_{i,j,k}'\) as:

\[
s_{i,j,k}' = \begin{cases} 
s_{i,n_i} \cdot t_{i,0} & \text{if } k = n_i \\
t_{j,k-n_i} & \text{if } i > n
\end{cases}
\]

Define \(n_{i,j} = n_i + m_j\).

Define \(SQ''_{i,j} = [s_{i,j,0} \cdot A_{i,j,1}' \cdots A_{i,j,n_i} \cdot s_{i,j,n_i}']\). By inspection, \(SQ''_{i,j} = SQ_i \odot SQ_j\).

Define \(DS'' = \langle SQ''_{1,1} | \ldots | SQ''_{n,m} \rangle\). By inspection, \(DS'' = DS \odot DT\).

Define \(DS''_{i,j,k} =\)

\[
\begin{cases} 
DS_{i,k} & \text{if } k \leq n_i \\
DT_{j,k-n_i} & \text{if } i > n_i
\end{cases}
\]

By inspection \(A_{i,j,k}' \iff_A DS''_{i,j,k}\).
Define $DS^\prime\prime_{i,j}$ as $\langle \{s^\prime_{i,j,0}\} \rangle \odot DS^\prime_{i,j,1} \odot \ldots \odot DS_{i,j,n_{i,j}} \odot \langle \{s^\prime_{i,j,n_{i,j}}\} \rangle$. By inspection, $DS^\prime\prime_{i,j} = DS_i \odot DS_j$.

This means that $DS^\prime\prime_{i,j,0} \odot \ldots \odot DS^\prime\prime_{i,j,m} \odot \ldots \odot DS^\prime\prime_{n_{i,j},n_{i,j}} \odot (DS_{i} \odot DT) \odot \ldots \odot (DS_{n_{i,j}} \odot DT)$. By repeated application of Lemma 119 and Lemma 121, there exists a DNF lens $dl : DS^\prime\prime_{i,j,0} \odot \ldots \odot DS^\prime\prime_{n_{i,j},n_{i,j}} \Rightarrow (DS_i \odot \ldots \odot DS_{n_{i,j}}) \odot (DT \odot \ldots \odot DT)$. So $dl : DS^\prime\prime_{i,j,1} \odot \ldots \odot DS^\prime\prime_{n_{i,j},n_{i,j}} \Leftrightarrow DS' \odot DT'$.

Consider the derivation

$$DS'' = \langle SQ'_{1,1} \mid \ldots \mid SQ'_{n_{i,j},n_{i,j}} \rangle$$

$$\forall i, j. A_{i,j} \Leftrightarrow_A DS'_{i,j}$$

$$\forall i, j. DS_{i,j} = \langle \{s^\prime_{i,j,0}\} \rangle \odot DS_{i,j,1} \odot \ldots \odot DS_{i,j,n_{i,j}} \odot \langle \{s^\prime_{i,j,n_{i,j}}\} \rangle$$

$DS \odot DT \Leftrightarrow DS'_{1,1} \odot \ldots \odot DS'_{n_{i,j},n_{i,j}} \odot DS''_{i,j,1} \odot \ldots \odot DS''_{n_{i,j},n_{i,j}}$

If $DS \Leftrightarrow DS'$ and $DT \Leftrightarrow DT'$, then $DS \odot DS \Leftrightarrow DS''$ such that there exists a rewriteless DNF lens $dl : DS \odot DT \Leftrightarrow DS''$, and $[dl] = \{(s, s) \mid s \in L(DS \odot DT)\}$, as desired.

**Lemma 123** (Swap’s Unimportance For Identity).

1. If $DS$ is strongly unambiguous and $DS \Leftrightarrow_A swap DS_1$, then there exists a $DS_2$ such that $DS \Leftrightarrow DS_2$ and there exists a lens $l : DS_1 \Leftrightarrow DS_2$ such that $[l] = \{(s, s) \mid s \in L(DS)\}$.

2. If $A$ is strongly unambiguous and $A \Leftrightarrow_A swap DS_1$, then there exists a $DS_2$ such that $A \Leftrightarrow_A DS_2$ and there exists a lens $l : DS_1 \Leftrightarrow DS_2$ such that $[l] = \{(s, s) \mid s \in L(A)\}$.

**Proof.** By mutual induction on the derivation of $\Leftrightarrow_A swap$.

**Case 1** (AtomUnrollStarL). Let $A \Leftrightarrow_A swap DS_1$, and the last step of the derivation is an application of AtomUnrollStarL. That means $A = DS^*_{i,j}$ and $DS_1 = \langle [e] \rangle \odot (DS \odot D(DS'))$.

Consider an application of $\Leftrightarrow_A$’s AtomUnrollStarL. $A \Leftrightarrow DS_1$. By Lemma 15, there exists a DNF lens $dl : DS_1 \Leftrightarrow DS_1$ and $[dl] = \{(s, s) \mid s \in L(A)\}$.

**Case 2** (AtomUnrollStarRightRule). Let $A \Leftrightarrow_A swap DS_1$, and the last step of the derivation is an application of AtomUnrollStarR. That means $A = DS^*_{i,j}$ and $DS_1 = D(DS')$, where

Consider an application of $\Leftrightarrow_A$’s AtomUnrollStarR. $A \Leftrightarrow DS_1$. By Lemma 15, there exists a DNF lens $dl : DS_1 \Leftrightarrow DS_1$ and $[dl] = \{(s, s) \mid s \in L(A)\}$.

**Case 3** (Parallel Swap Atom Structural Rewrite). Let $A \Leftrightarrow_A swap DS_1$, and the last step of the derivation is an application of Parallel Swap Atom Structural Rewrite. That means $A = DS^*_{i,j}$ and $DS_1 = D(DS')$, and $DS \Leftrightarrow_A swap DS_{2}'$.

By IH, there exists $DS_{2}'$ such that $DS \Leftrightarrow DS_{2}'$, and there exists a rewriteless DNF lens $dl : DS_1 \Leftrightarrow DS_{2}'$.

By Parallel Swap Atom Structural Rewrite, $A \Leftrightarrow_A D(DS_{2}')$.

By Lemma 104, $\mathcal{D}(iterate(dl)) \odot \mathcal{D}(DS_{2}') \Leftrightarrow \mathcal{D}(DS_{2}')$, with $ \mathcal{D}(iterate(dl)) = \{(s_1, \ldots, s_n, t_1, \ldots, t_n) \mid (s_j, t_j) \in [dl]\}$ = $\{(s_1, \ldots, s_n) \mid (s_i, s_i) \in L(DS)\}$ = $\{(s, s) \mid s \in L(A)\}$

**Case 4** (Parallel Swap DNF Structural Rewrite). Let $DS \Leftrightarrow_A swap DS'$, and the last step of the derivation is an application of Parallel Swap DNF Structural Rewrite.

$$DS = \langle SQ_1 \mid \ldots \mid SQ_n \rangle$$

$$\forall i. SQ_i = [s_{i,0}, A_{i,1}, \cdots, A_{i,n_i}, s_{i,n_i}]$$

$$\forall i, j. A_{i,j} \Leftrightarrow_A swap DS_{i,j}$$

$$\forall i, j. DS_i = \langle \{s_{i,0}\} \rangle \odot DS_{i,1} \odot \ldots \odot DS_{i,n_i} \odot \langle \{s_{i,n_i}\} \rangle$$

$DS \Leftrightarrow_A swap DS_1 \odot \ldots \odot DS_n$, and $DS' = DS_1 \odot \ldots \odot DS_n$. 

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
There exists lenses By IH, there exist \( DT_{i,j} \) and \( dl_{i,j} \), such that \( A_{i,j} \vdash DT_{i,j}, dl_{i,j} : DS_{i,j} \Leftarrow DT_{i,j} \), and \( \| dl_{i,j} \| = \{(s, s) \mid s \in L(A_{i,j}) \} \).

Define \( DT_i = (\langle s_{i,0} \rangle \circ DT_{i,1} \circ \ldots \circ DT_{i,n_i} \circ (\langle s_{i,n_i} \rangle) \).

Define \( DT = DT_0 \oplus \ldots \oplus DT_n \)

By repeated application of Lemma 100, there exists a lens \( dl_i = (\langle (\langle s_{i,0}, s_{i,0} \rangle), id \rangle \circ dl_{i,1} \circ \ldots \circ dl_{i,n_i} \circ (\langle (s_{i,n_i}, s_{i,n_i}) \rangle)) \).

By Lemma 124, there exists a DNF lens

\[
\begin{align*}
 & \lor_{\mathcal{P}} (\langle s_{i,0} \rangle) \\
\forall i, j, A_{i,j} \vdash & \Rightarrow DT_{i,j} \\
\forall i, j, & DT_i = (\langle s_{i,0} \rangle) \circ DT_{i,1} \circ \ldots \circ DT_{i,n_i} \circ (\langle s_{i,n_i} \rangle)
\end{align*}
\]

By pushing around the definitions of \( \circ \), this becomes \( dl_i : DS_i \Leftarrow DT_i \) and \( \| dl_i \| = \{(s, s) \mid s \in L(DS_i) \} \).

By repeated applications of Lemma 102, there exists a lens \( dl = dl_0 \oplus \ldots \oplus dl_n : DS_1 \oplus \ldots \oplus DS_n \Leftarrow DT_1 \oplus \ldots \oplus DT_n \).

By pushing around the definitions of \( \oplus \), this becomes \( dl : DS' \Leftarrow DT \), and \( \| dl \| = \{(s, s) \mid s \in L(DS) \} \).

Furthermore,

\[
\begin{align*}
& DS = \langle SQ_1 \mid \ldots \mid SQ_n \rangle \\
& \forall i, SQ_i = [s_{i,0} \cdot A_{i,1} \cdot \ldots \cdot A_{i,n_i} \cdot s_{i,n_i}] \\
& \forall i, j, A_{i,j} \vdash \Rightarrow DT_{i,j} \\
& \forall i, & DT_i = (\langle s_{i,0} \rangle) \circ DT_{i,1} \circ \ldots \circ DT_{i,n_i} \circ (\langle s_{i,n_i} \rangle)
\end{align*}
\]

Case 5 (Identity Rewrite). Let \( DS \vdash_{\text{swap}} DS_1 \), and the last step of the derivation is an application of Identity Rewrite.

This means \( DS \vdash_{\text{swap}} DS \)

Consider the application of \( \vdash_{\text{swap}} \)’s Identity Rewrite, \( DS \vdash_{\text{swap}} DS \).

By Lemma 15, there exists a DNF lens \( dl : DS \Leftarrow DS \) and \( \| dl \| = \{(s, s) \mid s \in L(DS) \} \).

Case 6 (DNF Reorder). Let \( DS \vdash_{\text{swap}} DS_1 \), and the last step of the derivation is an application of DNF Reorder.

Let \( DS = \langle SQ_1 \mid \ldots \mid SQ_n \rangle \). This means that there exists a \( \sigma \) such that \( DS_1 = \langle SQ_{\sigma(1)} \mid \ldots \mid SQ_{\sigma(n)} \rangle \).

Consider \( DS \vdash_{\text{swap}} DS \). By Lemma 15, there exists sequence lenses \( sql_i \) such that \( sql_i : SQ_i \Leftarrow SQ_i \) and \( \| sql_i \| = \{(s, s) \mid s \in L(SQ_i) \} \).

Consider \( (\langle sql_1 \mid \ldots \mid sql_n \rangle, \sigma) : \langle SQ_1 \mid \ldots \mid SQ_n \rangle \Leftarrow \langle SQ_{\sigma(1)} \mid \ldots \mid SQ_{\sigma(n)} \rangle \), which is typed as desired. \((\| (\langle sql_1 \mid \ldots \mid sql_n \rangle, \sigma) \| = \{(s, t) \mid \exists i. (s, t) \in \| sql_i \| \} = \{(s, s) \mid \exists i. s \in L(SQ_i) \} = \{(s, s) \mid s \in L(DS) \}\)

\[
\Box
\]

Definition 18. Let \( l \) be a lens. Define the binary relation, \( \Leftarrow \subseteq \text{DNF} \times \text{DNF} \), as \( DS \vdash_{\text{if}} DT \) if, and only if there exists a DNF Lens \( dl \) such that \( dl : DS \Leftarrow DT \), and \( \| dl \| = \| l \| \). \( DS \Leftarrow_{\text{id}} DT \) is shorthand for \( DS \Leftarrow_{\text{id}}^{\text{if}} DT \).

Lemma 124.

(a) Let \( dl : DS \Leftarrow DT \) and \( DS \vdash_{\text{swap}} DS' \). There exists some \( DT' \), \( dl' \) such that \( DT \vdash_{\text{swap}} DT', dl' : DS' \Leftarrow DT' \), and \( \| dl' \| = \| dl \| \).

(b) Let \( a \vdash A \Leftarrow B \) and \( A \vdash_{\text{swap}} DS \). There exists some \( DT, dl \) such that \( B \vdash_{\text{swap}} A DT, dl' : DS \Leftarrow DT \), and \( \| dl \| = \| dl' \| \).

Proof. By mutual induction on the derivation of \( \vdash_{\text{swap}} \) and \( \vdash_{\text{swap}} \).
Case 1 (Atom UnrollStarL). Let \( al : A \leftrightarrow B \), and \( A \vdash_A DS \) through an application of Atom UnrollStarL. By inversion, there exists a derivation of 

\[
\begin{array}{ccc}
dl : DS' \Leftrightarrow DT' & DS^*: & DT^*! \\
\text{iterate}(dl) : DS^* \Leftrightarrow DT^*
\end{array}
\]

Where \( \text{iterate}(dl) = al, DS^* = A \), and \( DT^* = B \).

As Atom UnrollStarL was applied, \( DS = \langle [e] \rangle \odot (DS' \odot (DS^*)) \).

Consider applying Atom UnrollStarL to \( DT^* \). \( DT^* \vdash_A [e] \odot (DT' \odot ([DT^*])) \)

Consider the lenses

\[
\begin{array}{c}
\langle [e, e] \rangle \\
\langle [e, e] \rangle
\end{array}
\]

As \( DS^* \), \( \mathcal{L}(DS)^:\mathcal{L}([DS^*]) \). As \( DS^* \), \( e \notin \mathcal{L}(DS) \). This means \( e \notin \mathcal{L}(DS \odot [DS^*]) \), so \( \mathcal{L}([e]) \cap \mathcal{L}(DS \odot [DS^*]) = \emptyset \)

Because of this, by Lemma 100 and Lemma 102, there exists the typing for the lens \( \langle [e, e] \rangle \odot (dl \odot \langle \text{iterate}(dl) \rangle) : \langle [e] \odot (DS \odot [DS^*]) \rangle \leftrightarrow \langle [e] \odot (DT \odot [DT^*]) \rangle \), which is the desired typing.

\[
\begin{array}{c}
\langle [e, e] \rangle \odot (dl \odot \langle \text{iterate}(dl) \rangle) = \langle \{s, t \mid (s, t) \in \{ \langle [e, e] \rangle \} \} \\
\mathcal{V}(s, t) \in \{ dl \odot \langle \text{iterate}(dl) \rangle \} \}
\end{array}
\]

\[
\begin{array}{c}
\langle [e, e] \rangle \odot (dl \odot \langle \text{iterate}(dl) \rangle) = \langle \{s, t \mid (s, t) \in \{ dl \odot \langle \text{iterate}(dl) \rangle \} \} \\
\mathcal{V}(s, t) \in \{ dl \odot \langle \text{iterate}(dl) \rangle \} \}
\end{array}
\]

Case 2 (Atom UnrollStarR). Let \( al : A \leftrightarrow B \), and \( A \vdash_A DS \) through an application of Atom UnrollStarR.

Case 3 (Parallel Atom Structural Rewrite). Let \( al : A \leftrightarrow B \), and \( A \vdash_A DS \) through an application of Parallel Atom Structural Rewrite. By inversion, there exists a derivation of 

\[
\begin{array}{ccc}
dl : DS' \Leftrightarrow DT' & DS^*: & DT^*! \\
\text{iterate}(dl) : DS^* \Leftrightarrow DT^*
\end{array}
\]

Where \( \text{iterate}(dl) = al, DS^* = A \), and \( DT^* = B \).

As Parallel Atom Structural Rewrite was applied, \( DS' \vdash DS'', \) and \( DS = \langle [DS'''] \rangle \).
By induction hypothesis, there exists some $dl', DT''$, such that $dl' : DS' \leftrightarrow DS''$, and $\| dl' \| = \| dl \|$. Because $L(DS') = L(DS')$ and $L(DT'') = L(DT')$, $DS''$ and $DT''$.

Consider the typing

$$dl' : DS'' \leftrightarrow DT'' \quad DS'' \leftrightarrow DT''$$

This is the desired typing. The semantics are as desired as well.

$$\| \langle \text{iterate}(dl') \rangle \| = \{ (s_1, \ldots, s_n, t_1, \ldots, t_n) \mid n \geq 0 \land \forall i \in [1, n](s_i, t_i) \in \| dl' \| \}$$

$$= \{ (s_1, \ldots, s_n, t_1, \ldots, t_n) \mid n \geq 0 \land \forall i \in [1, n](s_i, t_i) \in \| dl \| \}$$

$$= \| al \|$$

Case 4 (DNF REORDER). Let $dl : DS \leftrightarrow DT$, and $DS \leftrightarrow DS'$ through an application of DNF REORDER.

Case 5 (PARALLEL DNF STRUCTURAL REWRITE). Let $dl : DS \leftrightarrow DT$, and $DS \leftrightarrow DS'$ through an application of PARALLEL DNF STRUCTURAL REWRITE.

Case 6 (IDENTITY REWRITE). Let $dl : DS \leftrightarrow DT$, and $DS \leftrightarrow DS'$ through an application of IDENTITY REWRITE.

□

**Lemma 125.**

(1) Let $dl : DS \leftrightarrow DT$ and $DT \leftrightarrow DT'$.

There exists some $DS'$, $dl'$ such that $DS \leftrightarrow DS'$, $dl' : DS' \leftrightarrow DT'$, and $\| dl \| = \| dl' \|$.

(2) Let $al : A \leftrightarrow B$ and $B \leftrightarrow A DT$.

There exists some $DS, dl$, such that $A \leftrightarrow A DS$, $dl' : DS \leftrightarrow DT$, and $\| dl \| = \| al \|$.

**Proof.** This can be proven symmetrically to Lemma 124. □

**Lemma 126.** For all lenses $l : S \leftrightarrow T$, bisimilar $\leftrightarrow (\leftrightarrow)$, over the set of strongly unambiguous DNF regular expressions.

**Proof.** Let $DS, DT$ be strongly unambiguous DNF regular expressions, with $DS \xleftarrow{\rightarrow} DT$. So there exists a rewriteless DNF lens $dl : DS \leftrightarrow DT$ where $\| dl \| = \| l \|$. Let $DS \leftrightarrow DS'$. By Lemma 126, there exists $dl', DT'$ such that $DT \leftrightarrow DT'$, $dl' : DS' \leftrightarrow DT'$, and $\| dl' \| = \| dl \| = \| l \|$, so $DS' \xleftarrow{\rightarrow} DT'$.

Let $DT \leftrightarrow DT'$. By Lemma 126, there exists $dl', DS'$ such that $DS \leftrightarrow DS'$, $dl' : DS' \leftrightarrow DT'$, and $\| dl' \| = \| dl \| = \| l \|$, so $DS' \xleftarrow{\rightarrow} DT'$. □

**Lemma 127.** For all lenses $l : S \leftrightarrow T$, bisimilar $\leftrightarrow (\leftrightarrow^*)$, over the set of strongly unambiguous DNF regular expressions.

**Proof.** By Lemma 3 and Lemma 126. □

**Corollary 2** (Bisimilarity in Star Sequential). By Lemma 127 and Theorem 9.
Lemma 128 (\(\Leftrightarrow\) Maintained Under \(\odot\) up to \(id\) on the left). Let \(DS\) be strongly unambiguous. Let \(DT\) be strongly unambiguous. Let \(L(DS):=L(DT)\). If \(DS\Leftrightarrow DS'\), then \(DS \odot DT \Leftrightarrow DS''\) such that there exists a rewriteless DNF lens \(dl : DS' \odot DT \Leftrightarrow DS''\), and \([dl] = \{(s, s) \mid s \in L(DS \odot DT)\} \).

Proof. As \(L(DS):=L(DT)\), \(DS \odot DT\) is strongly unambiguous.
We proceed by induction on the derivation of \(\Leftrightarrow\).

Case 1 (Reflexivity).

\[
\frac{DS \Leftrightarrow DS}{DS \odot DT \Leftrightarrow DS \odot DT}
\]

By reflexivity

Furthermore, as \(DS \odot DT\) is strongly unambiguous, there exists a lens \(dl : DS \odot DT \Leftrightarrow DS \odot DT\).

Case 2 (Base).

\[
\frac{DS \Leftrightarrow DS'}{DS \Leftrightarrow DS'}
\]

So \(DS \Leftrightarrow DS'\), and by \textsc{Identity Rewrite}, \(DT \Leftrightarrow DT\).
So Lemma 122 says that there exists \(DS''\) such that

\[
\frac{DS \odot DT \Leftrightarrow DS''}{DS \odot DT \Leftrightarrow DS''}
\]

where there exists a DNF lens \(dl : DS' \odot DT \Leftrightarrow DS''\) such that \([dl] = \{(s, s) \mid s \in L(DS \odot DT)\} \).

Case 3 (Transitivity).

\[
\frac{DS \Leftrightarrow DS_1 \quad DS_1 \Leftrightarrow DS'}{DS \Leftrightarrow DS'}
\]

By IH, there exists \(DS_1''\) such that \(DS_1 \odot DT \Leftrightarrow DS_1''\), and there exists a DNF lens \(dl_1 : DS_1 \odot DT \Leftrightarrow DS_1''\), and \([dl_1] = \{(s, s) \mid s \in L(DS_1 \odot DT)\} \).

By IH, there exists \(DS''\) such that \(DS_1 \odot DT \Leftrightarrow DS''\), and there exists a DNF lens \(dl : DS_1 \odot DT \Leftrightarrow DS''\), and \([dl] = \{(s, s) \mid s \in L(DS_1 \odot DT)\} \).

By Lemma 127, as \(DS_1 \odot DT \Leftrightarrow DS''\), then there exists \(DS_1', dl_1'\) such that \(DS_1'' \Leftrightarrow DS_1'\), and \(dl_1' \Leftrightarrow DS'' \Leftrightarrow DS_1'\). With the same lenses in the composition are the identity lens, this lens is the identity lens, so \([dl_1'] = \{(s, s) \mid s \in L(DS_1 \odot DT)\} \).

Furthermore

\[
\frac{DS \odot DT \Leftrightarrow DS_1'' \quad DS_1'' \Leftrightarrow DS_1'}{DS \odot DT \Leftrightarrow DS_1'}
\]

\[\square\]

Lemma 129 (\(\Leftrightarrow\) Maintained Under \(\odot\) up to \(id\) on the right). Let \(DS\) be strongly unambiguous. Let \(DT\) be strongly unambiguous. Let \(L(DT):=L(DS)\). If \(DS \Leftrightarrow DS'\), then \(DT \odot DS \Leftrightarrow DS''\) such that there exists a rewriteless DNF lens \(dl : DT \odot DS \Leftrightarrow DS''\), and \([dl] = \{(s, s) \mid s \in L(DS \odot DT)\} \).

Proof. This is done symmetrically to Lemma 128. \[\square\]
Lemma 130 (\(\leftrightarrow\) Maintained Under \(\odot\) up to \(id\)). Let \(DS\) be strongly unambiguous. Let \(DT\) be strongly unambiguous. Let \(L(\mathbb{D}(DT)) \vdash L(\mathbb{D}(DS))\). Let \(DS \leftrightarrow^{*} DS'\). Let \(DT \leftrightarrow^{*} DT'\). Then \(DS \odot DT \leftrightarrow^{*} DS''\) such that there exists a rewriteless DNF lens \(dl : DS' \odot DT' \leftrightarrow DS''\), and \([dl] = \{(s, s) \mid s \in L(\mathbb{D}(DS \odot DT))\}\).

Proof. By Lemma 129, there exists a DNF lens \(dl_1 : DS \odot DT' \leftrightarrow DS_1\), such that \(DS \odot DT \leftrightarrow^{*} DS_1\) and \([dl_1]\) = \(\{(s, s) \mid s \in L(\mathbb{D}(DS \odot DT))\}\).

By Lemma 128, there exists a DNF lens \(dl_2 : DS' \odot DT' \leftrightarrow DS_2\), such that \(DS \odot DT' \leftrightarrow^{*} DS_2\) and \([dl_2]\) = \(\{(s, s) \mid s \in L(\mathbb{D}(DS \odot DT))\}\).

By Lemma 127, as \(\mathbb{D}(DS \odot DT') \leftrightarrow^{*}\mathbb{D}(DS_2)\), there exists a DNF lens \(dl'_1 : DS_2 \leftrightarrow DS'_1\) with \([dl'_1]\) = \(\{(s, s) \mid s \in L(\mathbb{D}(DS \odot DT))\}\) and \(DS_1 \leftrightarrow^{*} DS'_1\).

So \(dl_2 : DS' \odot DT' \leftrightarrow DS_2\), and \(dl'_1 : DS_2 \leftrightarrow DS'_1\). By Lemma 14, there exists a DNF lens, \(dl'' : DS' \odot DT'' \leftrightarrow DS''\). As both the lenses in the composition are the identity lens, this lens is the identity lens, so \([dl'']\) = \(\{(s, s) \mid s \in L(\mathbb{D}(DS \odot DT))\}\).

Furthermore

\[
\frac{DS \odot DT \leftrightarrow^{*} DS_1 \quad DS_1 \leftrightarrow^{*} DS'_1}{DS \odot DT \leftrightarrow^{*} DS''}
\]

\(\square\)

Corollary 3 (\(\rightarrow^{*}\) Maintained Under \(\odot\)). Let \(DS\) be strongly unambiguous. Let \(DT\) be strongly unambiguous. Let \(L(\mathbb{D}(DT)) \vdash L(\mathbb{D}(DS))\). Let \(DS \rightarrow^{*} DS'\). Let \(DT \rightarrow^{*} DT'\). Then \(DS \odot DT \rightarrow^{*} DS''\) such that there exists a rewriteless DNF lens \(dl : DS' \odot DT' \leftrightarrow DS''\), and \([dl] = \{(s, s) \mid s \in L(\mathbb{D}(DS \odot DT))\}\).

Proof. From Theorem 9 applied to Lemma 130.

\(\square\)

Lemma 131 (Pre-Confluence of Parallel Rewriting Without Reordering).

- If \(dl : DS \leftrightarrow DT, DS \leftrightarrow^{*} DS',\) and \(DT \leftrightarrow^{*} DT'\), then
  1. There exists a \(DS''\) such that \(DS' \leftrightarrow^{*} DS''\)
  2. There exists a \(DT''\) such that \(DT' \leftrightarrow^{*} DT''\)
  3. There exists a \(dl'' : DS'' \leftrightarrow DT''\) such that \([dl'']\) = \([dl]\).
- If \(al : A \leftrightarrow B, A \leftrightarrow^{*} A DS,\) and \(B \leftrightarrow^{*} A DT,\) then
  1. There exists a \(DS'\) such that \(DS \leftrightarrow^{*} DS'\)
  2. There exists a \(DT'\) such that \(DT \leftrightarrow^{*} DT'\)
  3. There exists a \(dl : DS' \leftrightarrow DT'\) such that \([dl]\) = \([al]\)

Proof. By mutual induction on the derivation of \(\leftrightarrow\) and \(\leftrightarrow^{*} A\). We will split into cases by the last step taken in each derivation.

Case 1 (\textsc{Atom Unrollstar}L, \textsc{Atom Unrollstar}R). Let \(al : A \leftrightarrow B.\) Let \(A = DS''\) and \(DT'' \leftrightarrow^{*} A(\{e\}) \oplus (DS' \odot (\{DS''\}))\) through an application of \textsc{Atom Unrollstar}L. Let \(B = DT^*\) and \(DT^* \leftrightarrow^{*} A(\{e\}) \oplus (DT \odot (\{DT^*\}))\) through an application of \textsc{Atom Unrollstar}L.

  1. Consider using \textsc{Identity Rewrite}

  \[
  (\{e\}) \oplus (DS \odot (\{DS''\})) \leftrightarrow^{*} (\{e\}) \oplus (DS \odot (\{DS''\}))
  \]

  2. Consider using \textsc{Identity Rewrite}

  \[
  (\{e\}) \oplus (DT \odot (\{DT''\})) \leftrightarrow^{*} (\{e\}) \oplus (DT \odot (\{DT''\}))
  \]

  3. By inversion, \(al = dl''\), and \(dl : DS \leftrightarrow DT.\)

  By Lemma 100, Lemma 102, and Lemma 104 \((\{e, e\}) \oplus (dl'' \odot D(\text{iterate}(dl''))) \leftrightarrow (\{e\}) \oplus (DS' \odot D(DS'')) \leftrightarrow (\{e\}) \oplus (DT' \odot D(DT''))\), which is the desired typing.
Case 2 (Atom UNRStarL, Atom UNRStarR). Let \( al : A \leftrightarrow B \). Let \( A = DS'' \) and \( DT'' \models A(\langle \epsilon \rangle) \oplus (DS' \odot (DT(DS''))) \) through an application of Atom UNRStarL. Let \( B = DT' \) and \( DT'' \models A(\langle \epsilon \rangle) \oplus (\mathcal{D}(DT') \odot DT) \) through an application of Atom UNRStarR.

(1)

\[
\frac{DS'' \models A(\langle \epsilon \rangle) \oplus (\mathcal{D}(DS'') \odot DS')}{\mathcal{D}(DS'') \models A(\langle \epsilon \rangle) \oplus (\mathcal{D}(DS'') \odot DS')}
\]

By Lemma 122 there exists a \( DS_1 \) such that \( DS' \odot \mathcal{D}(DS'') \models DS_1 \), and there exists \( dl_1 : DS_1 \leftrightarrow DS' \odot (\langle \epsilon \rangle \oplus (\mathcal{D}(DS'') \odot DS')) \) where \( dl_1 \) has identity semantics.

So by Lemma 51, \( \langle \epsilon \rangle \oplus (DS' \odot \mathcal{D}(DS'')) \models \langle \epsilon \rangle \oplus DS_1 \).

Furthermore, as \( \langle \langle \epsilon, \epsilon \rangle \rangle : \langle \epsilon \rangle \leftrightarrow \langle \epsilon \rangle \) has identity semantics, through Lemma 51 we get \( \langle \langle \epsilon, \epsilon \rangle \rangle \odot dl_1 : \langle \epsilon \rangle \oplus DS_1 \leftrightarrow \langle \epsilon \rangle \oplus (DS' \odot (\langle \epsilon \rangle \oplus (\mathcal{D}(DS'') \odot DS'))) \), which has the identity semantics.

(2)

\[
\frac{DT'' \models A(\langle \epsilon \rangle) \oplus (DT' \odot \mathcal{D}(DT''))}{\mathcal{D}(DT'') \models A(\langle \epsilon \rangle) \oplus (DT' \odot \mathcal{D}(DT''))}
\]

By Lemma 122 there exists a \( DT_2 \) such that \( DT' \odot \mathcal{D}(DT'') \models DT_2 \), and there exists \( dl_2 : (\langle \epsilon \rangle \oplus (DT' \odot \mathcal{D}(DT''))) \odot DT' \leftrightarrow DT_2 \) where \( dl_2 \) has identity semantics.

So by Lemma 51, \( \langle \epsilon \rangle \oplus (\mathcal{D}(DT'') \odot DT') \models \langle \epsilon \rangle \oplus DT_2 \).

Furthermore, as \( \langle \langle \epsilon, \epsilon \rangle \rangle : \langle \epsilon \rangle \leftrightarrow \langle \epsilon \rangle \) has identity semantics, through Lemma 51 we get \( \langle \langle \epsilon, \epsilon \rangle \rangle \odot dl_2 : \langle \epsilon \rangle \odot DT_2 \leftrightarrow \langle \epsilon \rangle \oplus (DT' \odot \mathcal{D}(DT'')) \), which has the identity semantics.

(3) As \( al : DS'' \leftrightarrow DT'' \), by inversion, \( al = \text{iterate}(dl) \), and \( dl : DS' \leftrightarrow DT' \).

Let \( dl' = \langle \langle \epsilon, \epsilon \rangle \rangle \odot (dl \odot \mathcal{D}(\text{iterate}(dl))) \) By Lemma 107, \( \| dl' \| = \| al \| \), and \( dl' : \langle \epsilon \rangle \odot (DS' \odot \mathcal{D}(DS'')) \leftrightarrow \langle \epsilon \rangle \odot (DT' \odot \mathcal{D}(DT'')) \).

Let \( dl'' = \langle \langle \epsilon, \epsilon \rangle \rangle \odot (\mathcal{D}(\text{iterate}(dl)) \odot dl) \) By Lemma 109, \( \| dl'' \| = \| \mathcal{D}(al) \| \), and \( dl'' : \langle \epsilon \rangle \odot (\mathcal{D}(DS') \odot DS') \leftrightarrow \langle \epsilon \rangle \odot (\mathcal{D}(DT') \odot DT') \).

Consider the DNF lens \( dl''' = \langle \langle \epsilon, \epsilon \rangle \rangle \odot (dl \odot dl' \odot dl''). \) \( dl''' : \langle \epsilon \rangle \odot (DS' \odot (\langle \epsilon \rangle \odot (\mathcal{D}(DS') \odot DS'))) \leftrightarrow \langle \epsilon \rangle \odot (DT' \odot (\langle \epsilon \rangle \odot (\mathcal{D}(DT') \odot DT'))), \) where \( dl''' \) has the same semantics as \( dl' \), as \( dl'' \) has the same semantics as \( \mathcal{D}(al) \).

By Lemma 118, there exists \( dl_3 : \langle \epsilon \rangle \odot (DT' \odot (\langle \epsilon \rangle \odot (\mathcal{D}(DT') \odot DT')) \leftrightarrow \langle \epsilon \rangle \odot (DT' \odot (\langle \epsilon \rangle \odot (\mathcal{D}(DT') \odot DT'))) \).

By Lemma 121, there exists \( dl_4 : \langle \epsilon \rangle \odot (\langle \epsilon \rangle \odot DT') \odot (DT' \odot \mathcal{D}(DT'') \odot DT') \leftrightarrow \langle \epsilon \rangle \odot (\langle \langle \epsilon \rangle \odot DT' \odot \mathcal{D}(DT'') \odot DT''). \)

Consider the composition of \( \langle \langle \epsilon, \epsilon \rangle \rangle \odot dl_1, dl''', dl_3, dl_4, \) and \( \langle \langle \epsilon, \epsilon \rangle \rangle \odot dl_2 \)
Because of Lemma 14, there exists a lens \( dl_5 \colon \langle \{ \epsilon \} \rangle \oplus DS_1 \leftrightarrow \langle \{ \epsilon \} \rangle \oplus DT_2 \). Furthermore, as all lenses except \( dl''' \) are the identity lens, \([dl_5] = [dl'''] = [dl]\).

**Case 3 (Atom UnrollStarR, Parallel Atom Structural Rewrite).** Let \( al : A \leftrightarrow B \). Let \( A = DS^* \) and \( DT^* \vdash_A \langle \{ \epsilon \} \rangle \oplus (DS' \odot D(DS'^*)) \) through an application of Atom UnrollStarL. Let \( B = DT^* \) and

\[
\frac{DT \vdash DT'}{DT^* \vdash_A D(DT')} \]

through an application of Parallel Atom Structural Rewrite.

From inversion, \( al = \text{iterate}(dl) \), and \( dl : DS \leftrightarrow DT \).

From Lemma 127, there exists \( DNFRegex' \) such that \( DS \vdash DS' \), such that there exists a lens \( dl'' : DS' \leftrightarrow DT' \), and \([dl''] = [dl]\)

\[
DST^* \vdash_A \langle \{ \epsilon \} \rangle \oplus (DS' \odot D(DS'^*))
\]

(1)

\[
DST^* \rightarrow A D(DT^*')
\]

(2)

By Lemma 122 there exists a \( DT_1 \) such that \( DT \odot D(DT^*) \vdash DT_1 \), and there exists \( dl_1 : DT' \odot D(DT^*) \leftrightarrow DT_1 \) where \( dl_1 \) has identity semantics.

\[
\langle \{ \epsilon \} \rangle \vdash \langle \{ \epsilon \} \rangle
\]

So by Lemma 51, \([\{ \epsilon \}] \oplus (DT \odot D(DT^*)) \vdash \langle \{ \epsilon \} \rangle \oplus DT_1 \).

Furthermore, as \([\{ \epsilon, \epsilon \}] \vdash \langle \{ \epsilon \} \rangle \) has identity semantics, through Lemma 51 we get \([\{ \epsilon, \epsilon \}] \odot dl_1 : \langle \{ \epsilon \} \rangle \odot (DT' \odot D(DT^*)) \leftrightarrow \langle \{ \epsilon \} \rangle \oplus DT_1 \), which has the identity semantics.

(3) Let \( dl''' = \langle \{ \epsilon, \epsilon \} \rangle \odot (dl'' \odot D(\text{iterate}(dl'))) \) By Lemma 107, \([dl'''] = [\text{iterate}(dl')] = [dl] \) and \([dl'' : \langle \{ \epsilon \} \rangle \odot (DS' \odot D(DS'^*)) \leftrightarrow \langle \{ \epsilon \} \rangle \oplus (DT' \odot D(DT^*)') \).

By Lemma 14, we can compose lenses, so consider \( dl'''' \), the composition of the lenses \( dl'' \) and \( \langle \{ \epsilon, \epsilon \} \rangle \odot dl_1 \). \( dl'''' : \langle \{ \epsilon \} \rangle \odot (DS' \odot D(DS'^*)) \leftrightarrow \langle \{ \epsilon \} \rangle \oplus DT_1 \). Furthermore, as all lenses in the composition except \( dl'''' \) are the identity, \([dl'''''] = [dl] = [dl] \).

**Case 4 (Atom UnrollStarR, Atom UnrollStarL).** This is easily transformed into the case of (Atom UnrollStarR, Atom UnrollStarR), and the solution to that case transformed to a solution of this case, through two applications of Lemma 18

**Case 5 (Atom UnrollStarR, Atom UnrollStarR).** This proceeds in the same way as (Atom UnrollStarL, Atom UnrollStarR).

**Case 6 (Atom UnrollStarR, Parallel Atom Structural Rewrite).** This proceeds in the same way as (Atom UnrollStarL, Parallel Atom Structural Rewrite)

**Case 7 (Parallel Atom Structural Rewrite, Atom UnrollStarL).** This is easily transformed into the case of (Atom UnrollStarR, Parallel Atom Structural Rewrite), and the solution to that case transformed to a solution of this case, through two applications of Lemma 18
Case 8 (Parallel Atom Structural Rewrite, Atom UnrollstarL). This is easily transformed into the case of (Atom UnrollstarR, Parallel Atom Structural Rewrite), and the solution to that case transformed to a solution of this case, through two applications of Lemma 18.

Case 9 (Parallel Atom Structural Rewrite, Parallel Atom Structural Rewrite). Let \( al : A \Leftrightarrow B \). Let \( A = DS' \) and

\[
\begin{align*}
\text{\( DS \vdash DS' \)} & \quad \text{\( DS' \vdash D(DS') \)}
\end{align*}
\]

through an application of Atom UnrollstarL. Let \( B = DT' \) and

\[
\begin{align*}
\text{\( DT \vdash DT' \)} & \quad \text{\( DT' \vdash A\overline{D}(DT') \)}
\end{align*}
\]

through an application of Parallel Atom Structural Rewrite.

From inversion, \( al = iterate(dl) \), and \( dl : DS \Leftrightarrow DT \).

By IH, there exists \( DS'' \), \( DT'' \), and \( dl'' \) such that \( DS' \vdash DS'', DT' \vdash DT'' \), and \( dl'' : DS'' \Leftrightarrow DT'' \) with \( \| dl'' \| = \| dl \| \).

\[
\begin{align*}
\text{\( DS' \vdash DS'' \)} & \quad \text{\( DS'' \vdash A\overline{D}(DS''') \)}
\end{align*}
\]

\[
\begin{align*}
\text{\( D(DS'') \vdash A\overline{D}(D(DS''')) \)} & \quad \text{\( D(TT''') \vdash A\overline{D}(DT''') \)}
\end{align*}
\]

\[
\begin{align*}
\text{\( DS' \vdash DS'' \)} & \quad \text{\( DS'' \vdash D(DS''') \)}
\end{align*}
\]

(1) As \( \| dl \| = \| dl'' \| \), \( \| al \| = \| iterate(dl) \| = \| iterate(DNFLens') \| \). Furthermore, as \( L(DS'') = L(DS) \) and \( L(DT'') = L(DS'''), DT''' \vdash DS'' \), and \( DS'' \vdash DT'' \). This means \( iterate(dl'') : DS'' \Leftrightarrow DT''' \).

From Lemma 103, \( \| iterate(dl'') \| = \| D(iterate(dl'')) \| \), and \( \overline{D}(iterate(iterate(dl''))) \vdash D(DS''') \Leftrightarrow D(DT''') \).

Case 10 (Identity Rewrite, Identity Rewrite). Let \( dl : DS \Leftrightarrow DT \). Let \( DS \vdash DS \) through an application of Atom UnrollstarL. Let \( DT \vdash DT \) through an application of Atom UnrollstarL.

\[
\begin{align*}
\text{\( DS \vdash DS \)} & \quad \text{\( DT \vdash DT \)}
\end{align*}
\]

(2) \( dl : DS \Leftrightarrow DT \), and \( \| dl \| = \| dl' \| \).

Case 11 (Identity Rewrite, Parallel DNF Structural Rewrite). Let \( dl : DS \Leftrightarrow DT \). Let \( DS \vdash DS \) through an application of Atom UnrollstarL. Let \( DT \vdash DT' \) through an application of Parallel DNF Structural Rewrite.

By Lemma 126, there exists \( dl', DS' \) such that \( DS \vdash DS', dl' : DS' \Leftrightarrow DT', \| dl \| = \| dl' \| \)

\[
\begin{align*}
\text{\( DS \vdash DS' \)} & \quad \text{\( DS' \vdash DS' \)}
\end{align*}
\]

(2) \( dl' : DS' \Leftrightarrow DT' \) and \( \| dl \| = \| dl' \| \)

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Case 12 (PARALLEL DNF STRUCTURAL REWRITE, IDENTITY REWRITE). This is easily transformed into the case of (IDENTITY REWRITE, PARALLEL DNF STRUCTURAL REWRITE), and the solution to that case transformed to a solution of this case, through two applications of Lemma 18.

Case 13 (PARALLEL DNF STRUCTURAL REWRITE, PARALLEL DNF STRUCTURAL REWRITE). Let $dl : DS \iff DT$. Let $DS \iff DS'$ through an application of PARALLEL DNF STRUCTURAL REWRITE. Let $DT \iff DT'$ through an application of PARALLEL DNF STRUCTURAL REWRITE.

By inversion, we have

$$
sql_i \vdash SQ_i \iff TQ_i \quad \ldots \quad sql_n \vdash SQ_n \iff TQ_n$$

$$\sigma \in S_n \quad i \neq j \Rightarrow L(SQ_i) \cap L(SQ_j) = \emptyset \quad i \neq j \Rightarrow L(TQ_i) \cap L(TQ_j) = \emptyset$$

$$(\langle sql_1 | \ldots | sql_n \rangle, \sigma) \vdash \langle SQ_1 | \ldots | SQ_n \rangle \iff \langle TQ_{\sigma(1)} | \ldots | TQ_{\sigma(n)} \rangle$$

By repeated application of Lemma 12, there exists $ds_{i,j}$ such that $ds_{i,j} \vdash A_{i,j} \iff B_{i,j}$, then there exists $ds'_{i,j}$, $DT'_{i,j}$, and $dl_{i,j}$ such that $ds_{i,j} \iff ds'_{i,j}$, $DT_{i,j} \iff DT'_{i,j}$, and $dl_{i,j} \iff dl_{i,j}$, where $\|dl_{i,j}\| = \|al_{i,j}\|$. For all $i,j$.

(1) $ds_{i,j} \iff ds'_{i,j}$, for all $i,j$

$$\langle [s_{i,j}] \rangle \iff \langle [s_{i,j}] \rangle$$

Define $DS'_i = \langle [s_{i,0}] \rangle \odot DS'_{i,1} \odot \ldots \odot DS'_{i,n_i} \odot \langle [s_{i,n_i}] \rangle$.

By repeated application of Lemma 12, there exists $DS''_i$ such that $DS_i = \langle [s_{i,0}] \rangle \odot DS_{i,1} \odot \ldots \odot DS_{i,n_i} \odot \langle [s_{i,n_i}] \rangle \iff DS''_i$, and there exists $dl_i$ such that $dl_i \vdash DS''_i \iff DS'_i$, and $dl_i$ has the identity semantics on $L(DS_i)$.

By repeated application of Lemma 12, $DS_1 \odot \ldots \odot DS_n \iff DS'_1 \odot \ldots \odot DS'_n$. Furthermore, through application of Lemma 102, $dl_1 \odot \ldots \odot dl_n \vdash DS''_1 \odot \ldots \odot DS''_n \iff DS'_1 \odot \ldots \odot DS'_n$.

(2) $DT_{\sigma(i),\sigma(j)} \iff DT'_{\sigma(i),\sigma(j)}$, for all $i,j$

$$\langle [s_{\sigma(i,0)}] \rangle \iff \langle [s_{\sigma(i,0)}] \rangle$$

Define $DT'_{\sigma(i)} = \langle [s_{\sigma(i,0)}] \rangle \odot DS'_{\sigma(i),\sigma(1)} \odot \ldots \odot DS'_{\sigma(i),\sigma(n_i)} \odot \langle [s_{\sigma(i,0)}] \rangle$. 

\[ \text{Synthesizing Bijective Lenses, Vol. } 1, \text{ No. } 1, \text{ Article } 1. \text{ Publication date: January 2018.} \]
By repeated application of Lemma 122, there exists $DT''_{\sigma(i)}$ such that $DT_{\sigma(i)} = \{1, 0\} \cup DT_{\sigma(i), \sigma(1)} \cup \cdots \cup DT_{\sigma(i), \sigma(n)} \cup \{(1, 0, 1) \} \cup DT'_{\sigma(i)}$, and there exists $dl'_{\sigma(i)}$ such that $dl'_{\sigma(i)} \downarrow \Rightarrow DT''_{\sigma(i)}$ and $dl'_{\sigma(i)}$ has the identity semantics on $L(DT_{\sigma(i)}).$

By repeated application of Lemma 51, $DT_{\sigma(1)} \oplus \cdots \oplus DT_{\sigma(n)} \oplus DT''_{\sigma(1)} \oplus \cdots \oplus DT''_{\sigma(n)}$. Furthermore, through application of Lemma 102, $dl'_{\sigma(1)} \oplus \cdots \oplus dl'_{\sigma(n)} \downarrow \Rightarrow DT'_{\sigma(1)} \oplus \cdots \oplus DT'_{\sigma(n)} \downarrow \Rightarrow DT''_{\sigma(1)} \oplus \cdots \oplus DT''_{\sigma(n)}$.

(3) As $L(A_{i,j}) = L(DS'_{i,j})$, and $L(B_{i,j}) = L(DT'_{i,j})$, then $\downarrow (s, i, j) \rightarrow L(DS'_{i,j}) \cap L(DS_j) = \{\}$. By Lemma 115, with the permutation $\sigma$, there exists a DNF lens $d_{l'} : DS'_{i} \downarrow \Rightarrow DT'_{i}$, with

$$
\llbracket d_{l'} \rrbracket = \{(s, i, j) \mid (s, i, j) \in \llbracket d_{l'} \rrbracket \}
$$

As $L(DS'_{i}) = L(DS_{i})$, $i \neq j \Rightarrow L(DS_{i}) \cap L(DS_{j}) = \{\}$. By Lemma 115, with the permutation $\sigma$, there exists a DNF lens $d_{l'} : DS'_{i} \downarrow \Rightarrow DT'_{i}$, with $\llbracket d_{l'} \rrbracket = \{(s, t) \mid \exists (i, s, t) \in \llbracket d_{l'} \rrbracket \} = \{(s, t) \mid \exists (i, s, t) \in \llbracket d_{l'} \rrbracket \} = \llbracket d_{l} \rrbracket$.

Consider the $dl'$, the composition of $dl_1 \oplus \cdots \oplus dl_n, d_{l}$, and $dl' \oplus \cdots \oplus dl_n, d_{l'} : DS'_{i} \oplus \cdots \oplus DS'_{n} \downarrow \Rightarrow DT''_{i} \oplus \cdots \oplus DT''_{n}$. Furthermore, all but $\llbracket d_{l} \rrbracket$ are identity, $\llbracket d_{l}'' \rrbracket = \llbracket d_{l} \rrbracket = \llbracket d_{l} \rrbracket$. 

\[ \square \]

**Theorem 11** (Confluence of Parallel Rewriting Without Reordering). For all lenses $l : S \leftrightarrow T$, confluent $\downarrow$ ($\leftrightarrow$).

**Proof.** Let $DS \Downarrow \Rightarrow DT$. This means there exists some $dl : DS \downarrow \Rightarrow DT$ such that $\llbracket dl \rrbracket = \llbracket l \rrbracket$. Let $DS \Downarrow \Rightarrow DS'$ and $DT \downarrow \Rightarrow DT'$. From Lemma 131, there exists a $DS'', DT'', dl'$ such that $DS' \Downarrow \Rightarrow DS''$, $DT \downarrow \Rightarrow DT''$, $dl' : DS'' \downarrow \Rightarrow DT''$, and $\llbracket dl' \rrbracket = \llbracket dl \rrbracket$. Because $\llbracket dl' \rrbracket = \llbracket dl \rrbracket = \llbracket l \rrbracket$, $DS'' \Downarrow \Rightarrow DT''$. 

**Lemma 132** (Identity is a left propagator). If $l : S \leftrightarrow T$ is a lens, then $\Downarrow \Rightarrow$ is a left propagator for $l$ with respect to $\Downarrow \Rightarrow$.

**Proof.** If $l : S \leftrightarrow T$ is a lens, by Lemma 16, $S$ is strongly unambiguous. By Lemma 45, $\llbracket S \rrbracket$ is strongly unambiguous. As such, $id_S : S \leftrightarrow S$. Consider $\Downarrow \Rightarrow$.

By Lemma 126, bisimilar $\Downarrow \Rightarrow$ ($\Downarrow \Rightarrow$).

By Theorem 11, confluent $\Downarrow \Rightarrow$ ($\Downarrow \Rightarrow$).

Let $DS_1 \Downarrow \Rightarrow DS_2$, and $DS_2 \Downarrow \Rightarrow DS_3$. So there exists $dl_1, dl_2$ such that $dl_1 : DS_1 \Downarrow \Rightarrow DS_2$ and $dl_2 : DS_2 \Downarrow \Rightarrow DS_3$, where $\llbracket dl_1 \rrbracket = \llbracket dl_2 \rrbracket$. By Lemma 14, there exists $dl_3 : DS_1 \Downarrow \Rightarrow DS_3$, with semantics $\llbracket dl_3 \rrbracket = \llbracket dl_1 \rrbracket$, as the semantics of each side of the composition was the identity relation on $L(S)$. This means $DS_1 \Downarrow \Rightarrow DS_3$.

Let $DS \Downarrow \Rightarrow DT$. So there exists $dl : DS \downarrow \Rightarrow DT$. By Lemma 12, $\llbracket l \rrbracket$ is a bijection between $L(S)$ and $L(T)$. As $\llbracket dl \rrbracket = \llbracket l \rrbracket$, $\llbracket dl \rrbracket$ is a bijection between $L(S)$ and $L(T)$. $\llbracket dl \rrbracket$ is a bijection between $L(DS)$ and $L(DT)$, by Lemma 13, so $L(DS) = L(S)$ and $L(DT) = L(T)$. As $DS$ is strongly unambiguous, there exists an identity lens by Lemma 15 $id_L : DS \Downarrow \Rightarrow DS$, such that $\llbracket id_L \rrbracket = \{(s, s) \mid s \in L(DS)\} = \{(s, s) \mid s \in L(S)\} = \llbracket id_S \rrbracket$. This means $DS \Downarrow \Rightarrow DS$. As $DT$ is strongly unambiguous, there exists an
identity lens $id_R : DT \Leftrightarrow DT$, such that $\llbracket id_R \rrbracket = \{(s, s) \mid s \in \mathcal{L}(DT)\} = \{(s, s) \mid s \in \mathcal{L}(T)\} = \llbracket id_T \rrbracket$. This means $DT \xRightarrow{id_s} DT$.

Let $DS \xLeftarrow{id_s} DT$. This means there exists $dl : DS \Leftrightarrow DT$. This means that $DS$ is strongly unambiguous, so there exists a DNF lens $dl' : DS \to DS$, with $\llbracket dl' \rrbracket = \{(s, s) \mid s \in \mathcal{L}(DS)\}$. By the same logic as above, $\mathcal{L}(DS) = \mathcal{L}(S)$, so $\llbracket dl' \rrbracket = \{(s, s) \mid s \in \mathcal{L}(S)\} = \llbracket id_s \rrbracket$.

\begin{lemma}[Identity is a right propagator]
If $l : S \Leftrightarrow T$ is a lens, then $\xRightarrow{id_s} l$ is a right propagator for $\Leftrightarrow$ with respect to $\leftrightarrow$.
\end{lemma}

\begin{proof}
By a symmetric argument to Lemma 132.
\end{proof}

\begin{lemma}[Confluence of Starred Parallel Rewriting Without Reordering]
For all lenses $l : S \Leftrightarrow T$, confluent $\Leftrightarrow (\leftrightarrow^*)$.
\end{lemma}

\begin{proof}
By Lemma 126 For all lenses $l : S \to T$, bisimilar $\Leftrightarrow (\leftrightarrow)$. For all lenses $l : S \Leftrightarrow T$, confluent $\Leftrightarrow (\leftrightarrow)$. By Lemma 132, $l \xRightarrow{id_s}$ is a left propagator for $\Leftrightarrow$. By Lemma 133, $l \xRightarrow{id_s}$ is a right propagator for $\Leftrightarrow$. By Theorem 6, confluent $\Leftrightarrow (\leftrightarrow^*)$.
\end{proof}

\begin{corollary}
For all lenses $l$, confluent $\Leftrightarrow (\to^*)$.
\end{corollary}

\begin{proof}
By Theorem 11, and Theorem 6, for all lenses $l$, confluent $\Leftrightarrow (\leftrightarrow^*)$. By Theorem 9, for all lenses $l$, confluent $\Leftrightarrow (\to^*)$.
\end{proof}

\subsection{Completeness}
Finally, with all the above machinery, all parts of confluence can be proven. The final statement is a quick one, with the bulk of the work done by proving a lemma involving rewrites and lens expressibility.

\begin{lemma}
Let $DS$ be strongly unambiguous, and let $DS \equiv_{\leftrightarrow^*} DT$. There exists $dl, DS', DT'$ such that $DS \to^* DS', DT' \to^* DT'$, $dl : DS' \Leftrightarrow DT'$, and $\llbracket dl \rrbracket = \{(s, s) \mid s \in \mathcal{L}(DS)\}$.
\end{lemma}

\begin{proof}
Proof by induction on the typing of $\equiv_{\leftrightarrow^*}$
\end{proof}

\begin{case}[Reflexivity]
Let $DS \equiv_{\leftrightarrow^*} DT$ through an application of Reflexivity. That means $DT = DS$.

Consider $DS \to^* DS$, and $DT \to^* DS$ through applications of Reflexivity.

Then, by Lemma 15, there exists a lens $dl : DS \Leftrightarrow DS$ such that $\llbracket dl \rrbracket = \{(s, s) \mid s \in \mathcal{L}(DS)\}$
\end{case}

\begin{case}[Base]
Let $DS \equiv_{\leftrightarrow^*} DT$ through an application of Base. That means $DS \leftrightarrow^* DT$.

$DT \to^* DT$ through an application of Reflexivity.

By Lemma 123, there exists a DNF regular expression, $DS'$, and a DNF lens $dl$, such that $DS \to DS'$, $dl : DS' \Leftrightarrow DT$, and $\llbracket dl \rrbracket = \{(s, s) \mid s \in \mathcal{L}(DS)\}$. Through an application of Base, $DS \leftrightarrow^* DS'$. From Theorem 9, $DS \to^* DS'$, as desired.
\end{case}

\begin{case}[Symmetry]
Let $DS \equiv_{\leftrightarrow^*} DT$ through an application of Symmetry. That means $DT \equiv_{\leftrightarrow^*} DS$.

By IH, there exists DNF regular expressions $DT'$, $DS'$, and a DNF lens $dl$ such that $DT \to^* DT'$, $DS \to^* DS'$, $dl : DT' \Leftrightarrow DS'$, and $\llbracket DT' \rrbracket = \{(s, s) \mid s \in \mathcal{L}(DT)\}$.

Because $\equiv_{\leftrightarrow^*}$ is equivalent to $\equiv^*$, $\mathcal{L}(DS) = \mathcal{L}(DT)$

By Lemma 18, there exists $dl' : DS' \Leftrightarrow DT'$, and $\llbracket dl' \rrbracket = \{(s, s) \mid s \in \mathcal{L}(DS)\}$, as desired.


Case 4 (Transitivity). Let $DS \equiv ^s \downarrow \uparrow DT$ through an application of Transitivity. That means there exists $DS'$ such that $DS \equiv ^s \downarrow \uparrow DS'$ and $DS' \equiv ^s \downarrow \uparrow DT$.

By IH, there exists DNF regular expressions $DS_1$, $DS_2$, and a DNF lens $dl_1$ such that $DS \rightarrow ^* DS_1$, $DS' \rightarrow ^* DS_2$, and $dl_1 \vdash DS_1 \Leftrightarrow DS_2$.

By IH, there exists DNF regular expressions $DS_3$, $DS_4$, and a DNF lens $dl_2$ such that $DS' \rightarrow ^* DS_3$, $DT \rightarrow ^* DS_4$, and $dl_2 \vdash DS_3 \Leftrightarrow DS_4$.

By Lemma 15, there exists a lens $dl_{id_3} : DS' \Leftrightarrow DS'$, where $\|dl_{id_3}\| = \{(s, s) \mid s \in L(DS')\}$.

Because $DS' \equiv ^i DS'$, and by Corollary 4, there exists $DS_5$ and $DS_6$, such that $DS_2 \rightarrow ^* DS_3$, $DS_3 \rightarrow ^* DS_6$, and $DS_5 \equiv ^i DS_6$. That means there exists $dl_{id_3} : DS_5 \Leftrightarrow DS_6$, where $\|dl_{id_3}\| = \{(s, s) \mid s \in L(DS')\}$.

By Corollary 2, as $DS_2 \rightarrow ^* DS_5$, and $dl_1 \vdash DS_1 \Leftrightarrow DS_2$. Because $DS_2 \rightarrow ^* DS_3$, there exists a DNF lens $dl_3$, and DNF regular expression $DS_7$ such that $DS_1 \rightarrow ^* DS_7$, $dl_3 \vdash DS_7 \Leftrightarrow DS_5$, and $\|dl_3\| = \{(s, s) \mid s \in L(DS)\}$.

By Corollary 2, as $DS_3 \rightarrow ^* DS_6$, and $dl_2 \vdash DS_3 \Leftrightarrow DS_4$. Because $DS_3 \rightarrow ^* DS_6$, there exists a DNF lens $dl_4$, and DNF regular expression $DS_8$ such that $DS_6 \rightarrow ^* DS_8$, $dl_4 \vdash DS_6 \Leftrightarrow DS_8$, and $\|dl_4\| = \{(s, s) \mid s \in L(DS')\}$. So there are lenses $dl_3 \vdash DS_7 \Leftrightarrow DS_5$, $dl_{id_3} : DS_5 \Leftrightarrow DS_6$, and $dl_4 \vdash DS_6 \Leftrightarrow DS_8$, by Lemma 14, there exists a lens $dl_5 : DS_7 \Leftrightarrow DS_6$. Because all of these have the semantics of the identity lens on DNF regular expressions with the same language, $\|dl_5\| = \{(s, s) \mid s \in L(DS)\}$. Furthermore, $DS \rightarrow ^* DS_1$ and $DS_1 \rightarrow ^* DS_7$, so $DS \rightarrow ^* DS_7$. $DT \rightarrow ^* DS_4$ and $DS_4 \rightarrow ^* DS_8$, so $DT \rightarrow ^* DS_8$, as desired.

$\square$

Lemma 136. Let $S \equiv ^s T$, and let $S$ be strongly unambiguous. There exists $dl$, $DS$, $DT$ such that $dl : DS \Leftrightarrow DT$, $\downarrow S \rightarrow ^* DS$, $\downarrow T \rightarrow ^* DT$, and $\|dl\| = \{(s, s) \mid s \in L(S)\}$

Proof. From Lemma 8, as $S \equiv ^s T$, $\downarrow S \equiv ^s ^i \downarrow T$. Because $S$ is strongly unambiguous, by Lemma 45, $\downarrow S$ is strongly unambiguous. Because of this, from Lemma 135, there exists $DS$, $DT$, and $dl$ such that $\downarrow S \rightarrow ^* DS$, $\downarrow T \rightarrow ^* DT$, and $\|dl\| = \{(s, s) \mid s \in L(\downarrow S)\}$. From Theorem 1, $L(\downarrow S) = L(S)$, as desired.

$\square$

Lemma 137. If $l : S \Leftrightarrow T$ then there exists $dl$, $DS$, $DT$ such that $dl : DS \Leftrightarrow DT$, $\downarrow S \rightarrow ^* DS$, $\downarrow T \rightarrow ^* DT$, and $\|dl\| = \|l\|$.

Proof. By induction of the typing derivation of $l : S \Leftrightarrow T$.

Let the last typing rule be an instance of Iterate Lens.

\[
\begin{align*}
\text{l : S} & \Leftrightarrow \text{T} \\
\text{iterate(l) : S'} & \Leftrightarrow \text{T'}
\end{align*}
\]

By IH, there exists $dl$, $DS$, $DT$ such that

\[
\begin{align*}
dl & : DS \Leftrightarrow DT \\
\downarrow S & \rightarrow ^* DS \\
\downarrow T & \rightarrow ^* DT \\
\|dl\| & = \|l\|
\end{align*}
\]

By Lemma 104, $\langle \{\text{iterate} (dl)\} \rangle \Leftrightarrow \langle DS' \rangle \Leftrightarrow \langle DT' \rangle$. By Corollary 1, $\langle \downarrow S' \rangle \rightarrow ^* \langle DS' \rangle$ and $\langle \downarrow T' \rangle \rightarrow ^* \langle DT' \rangle$. From this, we get
\[
\langle \text{iterate}(dl) \rangle \notimplies \langle [DS^*] \rangle \iff \langle [DT^*] \rangle \implies \langle [DS^*] \rangle \implies \langle [DT^*] \rangle
\]

By Lemma 104, \(\llbracket \langle \text{iterate}(dl) \rangle \rrbracket = \{(s_1 \cdot \ldots \cdot s_n, t_1 \cdot \ldots \cdot t_n) \mid (s_i, t_i) \in \llbracket dl \rrbracket \} = \{(s_1 \cdot \ldots \cdot s_n, t_1 \cdot \ldots \cdot t_n) \mid (s_i, t_i) \in \llbracket l \rrbracket \} = \llbracket \text{iterate}(l) \rrbracket\)

Let the last typing rule be an instance of \textsc{Constant Lens}.

\begin{align*}
\text{const}(s_1, s_2) \\
\langle \langle (s_1, s_2) \rangle \rangle \notimplies \langle [s_1] \rangle \iff \langle [s_2] \rangle & \iff \langle [s_1] \rangle \implies \langle [s_2] \rangle \\
\langle [s_2] \rangle \notimplies \langle [s_1] \rangle & \iff \langle [s_1] \rangle \implies \langle [s_2] \rangle
\end{align*}

\(\llbracket \langle (s_1, s_2) \rangle \rrbracket = \{s_1, s_2\} = \llbracket \text{const}(s_1, s_2) \rrbracket\)

Let the last typing rule be an instance of \textsc{Concat Lens}.

\[
\begin{array}{c}
l_1 : S_1 \iff T_1 \\
l_2 : S_2 \iff T_2
\end{array}
\]

\[
\text{concat}(l_1, l_2) : S_1 \cdot S_2 \iff T_1 \cdot T_2
\]

By IH, there exists \(dl_1, DS_1, DT_1, dl_2, DS_2\), and \(DT_2\) such that

\[
\begin{align*}
dl_1 & : DS_1 \iff DT_1 \\
\llbracket S_1 \implies *DS_1 \\
\llbracket T_1 \implies *DT_1 \\
\llbracket dl_1 \rrbracket & = \llbracket l_1 \rrbracket \\
dl_2 & : DS_2 \iff DT_2 \\
\llbracket S_2 \implies *DS_2 \\
\llbracket T_2 \implies *DT_2 \\
\llbracket dl_2 \rrbracket & = \llbracket l_2 \rrbracket
\end{align*}
\]

From Lemma 100, \(dl_1 \odot dl_2 \notimplies DS_1 \odot DS_2 \iff DT_1 \odot DT_2\).

By Corollary 3, and Lemma 11, there exists a DNF regular expression, \(DS_L\), and a DNF lens, \(dl_L\), such that \(dl_L : DS_1 \iff DS_1 \odot DS_2\), where \(\llbracket dl_L \rrbracket = \{(s, s) \mid s \in L(\llbracket DS_1 \odot DS_2 \rrbracket)\}\). Furthermore, \(\llbracket (DS_1 \odot DS_2) \implies *DS_L\).

By Corollary 3, there exists a DNF regular expression, \(DS_R\), and a DNF lens, \(dl_R\), such that \(dl_R : DT_1 \odot DT_2 \iff DS_R\), where \(\llbracket dl_R \rrbracket = \{(s, s) \mid s \in L(\llbracket (DT_1 \odot DT_2) \rrbracket)\}\). Furthermore, \(\llbracket (DT_1 \odot DT_2) \implies *DS_R\).

By Lemma 14, as \(dl_L : DS_L \iff DS_1 \odot DS_2, dl_1 \odot dl_2 : DS_1 \odot DS_2 \iff DT_1 \odot DT_2,\) and \(dl_R : DT_1 \odot DT_2 \iff DS_R\) there exists a DNF Lens \(dl : DS_L \iff DS_R\), with semantics of the composition of the three lenses. Because the left and right lenses are the identity lenses, \(\llbracket dl \rrbracket = \llbracket dl_1 \odot dl_2 \rrbracket\).

By Lemma 100, \(\llbracket dl \rrbracket = \llbracket dl_1 \odot dl_2 \rrbracket = \{(s_1 \cdot s_2, t_1 \cdot t_2) \mid (s_1, t_1) \in \llbracket dl_1 \rrbracket \land (s_2, t_2) \in \llbracket dl_2 \rrbracket\} = \{(s_1 \cdot s_2, t_1 \cdot t_2) \mid (s_1, t_1) \in \llbracket l_1 \rrbracket \land (s_2, t_2) \in \llbracket l_2 \rrbracket\} = \llbracket \text{concat}(l_1, l_2) \rrbracket\).
\[ dl : DS_L \leftrightarrow DS_R \]
\[ \Downarrow (S_1 \cdot S_2) \rightarrow^* DS_L \]
\[ \Downarrow (T_1 \cdot T_2) \leftrightarrow DS_L \]
\[ \llbracket dl \rrbracket = \llbracket \text{concat}(l_1, l_2) \rrbracket \]

Let the last typing rule be an instance of Or Lens.

\[
\begin{array}{c}
l_1 : S_1 \leftrightarrow T_1 \\
l_2 : S_2 \leftrightarrow T_2 \\
o r(l_1, l_2) : S_1 | S_2 \leftrightarrow T_1 | T_2
\end{array}
\]

By IH, there exists \( dl_1, DS_1, DT_1, dl_2, DS_2, \) and \( DT_2 \) such that

\[
\begin{array}{c}
dl_1 : DS_1 \leftrightarrow DT_1 \\
\Downarrow S_1 \rightarrow^* DS_1 \\
\Downarrow T_1 \rightarrow^* DT_1 \\
\llbracket dl_1 \rrbracket = \llbracket l_1 \rrbracket \\
dl_2 : DS_2 \leftrightarrow DT_2 \\
\Downarrow S_2 \rightarrow^* DS_2 \\
\Downarrow T_2 \rightarrow^* DT_2 \\
\llbracket dl_2 \rrbracket = \llbracket l_2 \rrbracket
\end{array}
\]

From Lemma 102, \( dl_1 \oplus dl_2 : DS_1 \oplus DS_2 \leftrightarrow DT_1 \oplus DT_2 \). By Lemma 76, \( \Downarrow S_1 | S_2 \rightarrow^* DS_1 \oplus DS_2 \) and \( \Downarrow T_1 | T_2 \rightarrow^* DT_1 \oplus DT_2 \).

By Lemma 102, \( \llbracket dl_1 \oplus dl_2 \rrbracket = \{ (s, t) \mid (s, t) \in \llbracket dl_1 \rrbracket \lor (s, t) \in \llbracket dl_2 \rrbracket \} = \{ (s, t) \mid (s, t) \in \llbracket l_1 \rrbracket \lor (s, t) \in \llbracket l_2 \rrbracket \} = \llbracket \text{or}(l_1, l_2) \rrbracket \).

\[
\begin{array}{c}
l_1 \oplus l_2 : DS_1 \oplus DS_2 \leftrightarrow DT_1 \oplus DT_2 \\
\Downarrow (S_1 | S_2) \rightarrow^* DS_1 \oplus DS_2 \\
\Downarrow (T_1 | T_2) \rightarrow^* DT_1 \oplus DT_2 \\
\llbracket dl_1 \oplus dl_2 \rrbracket = \llbracket \text{or}(l_1, l_2) \rrbracket
\end{array}
\]

Let the last typing rule be an instance of Swap Lens.

\[
\begin{array}{c}
l_1 : S_1 \leftrightarrow T_1 \\
l_2 : S_2 \leftrightarrow T_2 \\
\text{swap}(l_1, l_2) : S_1 \cdot S_2 \leftrightarrow T_2 \cdot T_1
\end{array}
\]
By IH, there exists $dl_1$, $DS_1$, $DT_1$, $dl_2$, $DS_2$, and $DT_2$ such that

\[
dl_1 : DS_1 \leftrightarrow DT_1
\]
\[
\downarrow S_1 \ldots DT_1
\]
\[
[dl_1] = [l_1]
\]
\[
dl_2 : DS_2 \leftrightarrow DT_2
\]
\[
\downarrow S_2 \ldots DT_2
\]
\[
[dl_2] = [l_2]
\]
\[
dl_1 \circ dl_2 : DS_1 \circ DS_2 \leftrightarrow DT_2 \circ DT_1.
\]

From Lemma 101, $dl_1 \circ dl_2 : DS_1 \circ DS_2 \leftrightarrow DT_2 \circ DT_1$.

By Corollary 3, and Lemma 11, there exists a DNF regular expression, $DS_L$, and a DNF lens, $dl_L$, such that $dl_L : DS_L \leftrightarrow DS_1 \circ DS_2$, where $[dl_L] = \{(s, s) \mid s \in L(\downarrow (DS_1 \circ DS_2))\}$. Furthermore, $\downarrow (DS_1 \circ DS_2) \rightarrow DS_L$.

By Corollary 3, there exists a DNF regular expression, $DS_R$, and a DNF lens, $dl_R$, such that $dl_R : DT_2 \circ DT_1 \leftrightarrow DS_R$, where $[dl_R] = \{(s, s) \mid s \in L(\downarrow (DT_2 \circ DT_1))\}$ Furthermore, $\downarrow (DT_2 \circ DT_1) \rightarrow DS_R$.

By Lemma 14, as $dl_L : DS_L \leftrightarrow DS_1 \circ DS_2$, $dl_1 \circ dl_2 : DS_1 \circ DS_2 \leftrightarrow DT_2 \circ DT_1$, and $dl_R : DT_2 \circ DT_1 \leftrightarrow DS_R$ there exists a DNF Lens $dl : DS_L \leftrightarrow DS_R$, with semantics of the composition of the three lenses. Because the left and right lenses are the identity lenses, $[dl] = [dl_1 \circ dl_2]$.

Let the last rule be an instance of COMPOSE LENS.

\[
l_1 : S_1 \leftrightarrow S_2 \quad l_2 : S_2 \leftrightarrow S_3
\]
\[
l_2 \circ l_1 : S_1 \leftrightarrow S_3
\]

By induction assumption, there exists $dl_1$, $DS_1$, $DS_2$, $dl_2$, $DT_2$, and $DS_3$ such that
\[ dl_1 : DS_1 \leftrightarrow DS_2 \\
\| S_1 \rightarrow^* DS_1 \\
\| S_2 \rightarrow^* DS_2 \\
\llbracket dl_1 \rrbracket = \llbracket l_1 \rrbracket \\
dl_2 : DS_2 \leftrightarrow DS_3 \\
\| S_2 \rightarrow^* DS'_2 \\
\| S_3 \rightarrow^* DS_3 \\
\llbracket dl_2 \rrbracket = \llbracket l_2 \rrbracket \]

From Lemma 15, there exists a rewriteless DNF lens \( dl_{id} : \| S_2 \leftrightarrow \| S_2 \) where \( \llbracket dl_{id} \rrbracket = \{(s, s) \mid s \in \mathcal{L}(S_2)\} \). From Corollary 4, we know that, as \( \| S_2 \rightarrow^* DS_2 \) and as \( \| S_2 \rightarrow^* DT_2 \), there must exist some \( DS'_2, DT'_2 \) such that \( DS_2 \rightarrow^* DS'_2 \) and \( DT_2 \rightarrow^* DT'_2 \) and there exists a rewriteless DNF lens \( dl'_{id} : DS'_2 \leftrightarrow DT'_2 \) where \( \llbracket dl'_{id} \rrbracket = \{(s, s) \mid s \in \mathcal{L}(S_2)\} \). From Corollary 2 and Corollary 2, there exists \( DS'_1, DS'_3, dl'_1 \), and \( dl'_2 \) such that \( DS_1 \rightarrow^* DS'_1 \), \( DS_3 \rightarrow^* DS'_3 \), \( dl'_1 \equiv DS'_1 \), \( dl'_2 \equiv DS'_3 \), \( \llbracket dl'_1 \rrbracket = \llbracket dl'_2 \rrbracket \) and \( \llbracket dl'_1 \rrbracket = \llbracket dl'_2 \rrbracket \). From Lemma 14 rewriteless DNF lenses are closed under composition, so there exists a rewriteless DNF lens \( dl' : DS'_1 \leftrightarrow DS'_3 \) where \( \llbracket dl' \rrbracket = \llbracket dl'_1 \rrbracket \circ \llbracket dl'_2 \rrbracket \rrbracket = \llbracket dl'_1 \rrbracket \circ \llbracket dl'_2 \rrbracket = \llbracket l_2 \rrbracket \circ \llbracket l_1 \rrbracket = \llbracket l_2 \circ l_1 \rrbracket \). Furthermore, \( \| S_1 \rightarrow^* DS \rightarrow^* DS'_1 \) so \( \| S_1 \rightarrow^* DS_1 \). \( \| S_3 \rightarrow^* DS_3 \rightarrow^* DS'_3 \) so \( \| S_3 \rightarrow^* DS_3 \).

\[ dl' : DS'_1 \leftrightarrow DS'_3 \\
\| S_1 \rightarrow^* DS'_1 \\
\| S_2 \rightarrow^* DS'_3 \\
\llbracket dl' \rrbracket = \llbracket l_1 \circ l_2 \rrbracket \]

Let the last rule be an instance of Rewrite Regex Lens.

\[
\begin{array}{ccc}
l : S \leftrightarrow T & S \equiv S' & T \equiv T' \\
l : S' \leftrightarrow T' & & \\
\end{array}
\]

By IH, there exists \( dl, DS, DT \) such that

\[
dl : DS \leftrightarrow DT \\
\| S \rightarrow^* DS \\
\| T \rightarrow^* DT \\
\llbracket dl \rrbracket = \llbracket l \rrbracket \\
\]

As \( S' \equiv S \), and \( S \) is strongly unambiguous, from Lemma 136 there exists a rewriteless DNF lens \( dl_{S,S} : DS \leftrightarrow DS \) such that \( \| S \rightarrow^* DS \), \( \| S' \rightarrow^* DS \), and \( \llbracket dl_{S,S} \rrbracket = \{(s, s) \mid s \in \mathcal{L}(\| S)\} \).

As \( T \equiv T' \), and \( T \) is strongly unambiguous, from Lemma 136 there exists a rewriteless DNF lens \( dl_{T,T} : DT \leftrightarrow DT \) such that \( \| T \rightarrow^* DT \), \( \| T' \rightarrow^* DT' \), and \( \llbracket dl_{T,T} \rrbracket = \{(s, s) \mid s \in \mathcal{L}(\| T)\} \).
From Lemma 15, there exists a lens \( \overline{dl} \vdash S \leftrightarrow S \). As \( \overline{S} \rightarrow S \) and \( \overline{S} \rightarrow DS \), by Corollary 4, there exists some \( \overline{dl} \vdash \overline{DS} \leftrightarrow DS \), such that \( \overline{DS} \rightarrow \overline{DS} \), \( DS \rightarrow \overline{DS} \), and \( \| \overline{dl}\| = \| \overline{dl} \| \). From Corollary 2, there exists a \( \overline{dl}_{S,S} \vdash \overline{DS}' \leftrightarrow \overline{DS} \) such that \( \overline{DS}' \rightarrow \overline{DS}' \) and \( \| \overline{dl}_{S,S} \| = \| \overline{dl}_{S,S} \| \).

From Lemma 15, there exists a lens \( \overline{dl}_{\overline{T},\overline{T}} \vdash \overline{T} \leftrightarrow T \). As \( \overline{T} \rightarrow \overline{T}DT \) and \( \overline{T} \rightarrow \overline{TDT} \), by Corollary 4, there exists some \( \overline{dl}_{\overline{T},\overline{T}} \vdash \overline{TDT} \leftrightarrow \overline{TDT} \), such that \( \overline{TDT} \rightarrow \overline{TDT} \), \( \overline{TDT} \rightarrow \overline{TDT} \), and \( \| \overline{dl}_{\overline{T},\overline{T}} \| = \| \overline{dl}_{\overline{T},\overline{T}} \| \). From Corollary 2, there exists a \( \overline{dl}_{\overline{T},\overline{T}} \vdash \overline{TDT} \leftrightarrow \overline{TDT} \) such that \( \overline{TDT} \rightarrow \overline{TDT} \) and \( \| \overline{dl}_{\overline{T},\overline{T}} \| = \| \overline{dl}_{\overline{T},\overline{T}} \| \).

As \( DS \rightarrow \overline{DS} \) and \( DT \rightarrow \overline{DT} \), by Corollary 4 there exists a \( \overline{dl} \vdash DS \leftrightarrow DT \) such that \( \overline{DS} \rightarrow \overline{DS} \), \( \overline{DT} \rightarrow \overline{DT} \), and \( \| \overline{dl} \| = \| \overline{dl} \| \).

From Corollary 2, there exists \( \overline{dl}_{S} \vdash \overline{DS} \leftrightarrow DS \) such that \( \overline{DS} \rightarrow \overline{DS} \) and \( \| \overline{dl}_{S} \| = \| \overline{dl}_{S} \| \). From Corollary 2, there exists \( \overline{dl}_{\overline{T},\overline{T}} \vdash \overline{TDT} \leftrightarrow \overline{TDT} \) such that \( \overline{TDT} \rightarrow \overline{TDT} \) and \( \| \overline{dl}_{\overline{T},\overline{T}} \| = \| \overline{dl}_{\overline{T},\overline{T}} \| \). From Corollary 2, there exists \( \overline{dl}_{\overline{T},\overline{T}} \vdash \overline{TDT} \leftrightarrow \overline{TDT} \) such that \( \overline{TDT} \rightarrow \overline{TDT} \) and \( \| \overline{dl}_{\overline{T},\overline{T}} \| = \| \overline{dl}_{\overline{T},\overline{T}} \| \).

From Lemma 14, there exists a lens \( \overline{dl} \vdash \overline{DS} \leftrightarrow \overline{DS} \). Because the semantics of all lenses in the composition for \( dl \) were all the identity relation, \( \| dl \| = \| dl \| \). Furthermore, \( \overline{S} \rightarrow _* \overline{DS} \) and \( \overline{T} \rightarrow _* \overline{DT} \), so we have

\[
\overline{dl} : \overline{DS} \leftrightarrow \overline{DS} \\
\overline{S} \rightarrow _* \overline{DS} \\
\overline{T} \rightarrow _* \overline{DT} \\
\| \overline{dl} \| = \| \overline{dl} \| 
\]

**Theorem 12.** If there exists a derivation for \( l : S \leftrightarrow T \), then there exists a DNF lens \( dl \) such that \( dl : (\overline{S}) \leftrightarrow (\overline{T}) \) and \( \| dl \| = \| dl \| \).

**Proof.** By Lemma 137, there exists \( dl, DS, DT \) such that \( \overline{dl} : \overline{DS} \leftrightarrow DT \), \( \overline{S} \rightarrow _* \overline{DS} \), \( \overline{T} \rightarrow _* DT \), and \( \| dl \| = \| dl \| \). Because of that, we have the derivation

\[
dl : DS \leftrightarrow DT \\
\overline{S} \rightarrow _* \overline{DS} \\
\overline{T} \rightarrow _* \overline{DT} \\
dl : \overline{S} \leftrightarrow \overline{T}
\]

**B.12 Algorithm Correctness**

We use an auxiliary data structure of a set-of-examples-parse-tree to define the orderings.

**Definition 19.** We use \( il \) to denote list of ints.

**Definition 20.** We use \( ils \) to denote a set of int lists.

**Definition 21.** We use \( sils \) to denote a set of string and int list pairs. We also require that the int lists are distinct.

**Definition 22.** To get the strings out of \( sils \), we use *projectstrings*. In particular, 

\[
\text{projectstrings}(\{(s_1, il_1), \ldots, (s_n, il_n)\}) = \{s_1, \ldots, s_n\}
\]
Definition 23. To get the int lists out of sils, we use projectils. In particular, projectils(\{ (s_1, il_1), \ldots, (s_n, il_n) \}) = \{ il_1, \ldots, il_n \}.

Definition 24. Define an exampled atom, exampled sequence, and exampled atoms. We need to define some general orderings for a number of strings which match it embedded.

We build the typing derivation of the form sils \in DS \leadsto EDS to express that the strings projectstrings(sils), labelled by the identifiers projectils(sils) when they have their parse trees embedded in DS, generate EDS. Similarly for SQ and ESQ, and A and EA.

Definition 25.

\[
\begin{align*}
\{ (s_{1,1}, 1 :: il_1), \ldots, (s_{1,n_1}, n_1 :: il_1), \ldots, (s_{m,1}, 1 :: il_m), \ldots, (s_{m,n_m}, n_m :: il_m) \} & \in DS \leadsto EDS \\
\{ (s_{1,1} \cdot \ldots \cdot s_{1,n_1} :: il_1), \ldots, (s_{m,1} \cdot \ldots \cdot s_{m,n_m} :: il_m) \} & \in DS' \leadsto (EDS^*, \{ il_1, \ldots, il_m \}) \\
\{ (s_{1,1}, il_1), \ldots, (s_{m,1}, il_m) \} & \in A_1 \leadsto EA_1 \quad \ldots \quad \{ (s_{1,n_1}, il_1), \ldots, (s_{m,n_m}, il_m) \} \in A_n \leadsto EA_n \\
\{ (s_{0} \cdot s_{1,1} \cdot \ldots \cdot s_{1,n_1} :: s_{0}', il_1), \ldots, (s_{0} \cdot s_{m,1} \cdot \ldots \cdot s_{m,n_m} :: s_{0}', il_m) \} & \in \{ s_{0} \cdot A_1 \cdot \ldots \cdot A_n \cdot s_{0}' \} \leadsto (s_{0} :: A_1 \cdot \ldots \cdot A_n :: s_{0}') \\
\{ sil_{1}, \ldots, sil_{n} \} & \in \langle SQ_1 \mid \ldots \mid SQ_n \rangle \leadsto (\langle ESQ_1 \mid \ldots \mid ESQ_n \rangle, \text{projectils}(\bigcup_{i \in [1,n]} sil_i))
\end{align*}
\]

This is a big typing derivation, and we feel it is clear that, when a DNF regular expression is strongly unambiguous, this typing derivation is unique for a given set of strings and DNF regular expression, so it is functional from the first two arguments of the derivation. Furthermore, we can perform this function by doing case analysis on all the possible ways the string is split up (though it is slow). We perform this function by performing this embedding the function into a NFA matching algorithm. We elide these details.

Definition 26. Define EMBEDEXAMPLES as the function from DNF Regex DS and intlist labelled examples sils to exampled DNF regex, such that sils \in DS \leadsto EMBEDEXAMPLES(sils, DS)

Now, we are going to build up the machinery to define an ordering on exampled DNF regular expressions, exampled sequences, and exampled atoms. We need to define some general orderings first.

Definition 27. Let \leq be an ordering on A. Let [x_1, \ldots, x_n] be a list of As. Define sorting([x_1; \ldots; x_n], \leq) as a permutation \sigma \in S_n such that \sigma(i) \leq \sigma(j) \Rightarrow x_{\sigma(i)} \leq x_{\sigma(j)}.

Definition 28. Let \leq be an ordering on A. Let [x_1; \ldots; x_n] be a list of As. Define sort([x_1; \ldots; x_n], \leq) = [x_{\sigma(1)}; \ldots; x_{\sigma(n)}] where \sigma = \text{sorting}([x_1; \ldots; x_n], \leq).

Definition 29. Let \leq_1 be an ordering on A_1. Let \leq_2 be an ordering on A_2. Define the product ordering (\leq_1, \leq_2) on A_1 \times A_2 as the lexicographic ordering on the two elements.

Definition 30. Let \leq be an ordering on A. We write [\leq] for the lexicographic ordering on A List.
Property 2. \([x_1; \ldots; x_n] \leq [y_1; \ldots; y_m] \) and \([y_1; \ldots; y_m] \leq [x_1; \ldots; x_n]\) if, and only if \(n = m\) and for all \(i \in [1, n]\) \(x_i \leq y_i\) and \(y_i \leq x_i\)

Definition 31. Let \(\leq\) be an ordering on \(A\). Define \([\leq]\) as the ordering on \(A\ List\) as: \([x_1; \ldots; x_n] \leq [y_1; \ldots; y_m]\) if \(\text{sort}(x_1; \ldots; x_n) \leq \text{sort}(y_1; \ldots; y_m)\). We also use \([\leq]\) to operate on sets, by first converting the set to a list, then using that ordering.

Definition 32. Define an ordering on int list sets, \(\leq_{\text{inlistset}}\), as \([\{\leq\}]\), where \(\leq\) is the usual on integers.

Lemma 138. If \(\sigma_1 = \text{sort}([x_1; \ldots; x_n])\) and \(\sigma_2 = \text{sort}([y_1; \ldots; y_m])\), then \(n = m\), \(x_i \leq y_{(\sigma_1^{-1} \circ \sigma_2)(i)}\), and \(y_{(\sigma_1^{-1} \circ \sigma_2)(i)} \leq x_i\)

Proof. Let \(\sigma_1 = \text{sort}([x_1; \ldots; x_n])\) and \(\sigma_2 = \text{sort}([y_1; \ldots; y_m])\). This means that \([x_{\sigma_1(1)}; \ldots; x_{\sigma_1(n)}] \leq [y_{\sigma_2(1)}; \ldots; y_{\sigma_2(m)}]\) and \([y_{\sigma_2(1)}; \ldots; y_{\sigma_2(m)}] \leq [x_{\sigma_1(1)}; \ldots; x_{\sigma_1(n)}]\).

By the above property about dictionary orderings, this means that \(n = m\) and \(x_{\sigma_1(i)} \leq y_{\sigma_2(i)}\) and \(y_{\sigma_2(i)} \leq x_{\sigma_1(i)}\).

Consider the permutation \(\sigma = \sigma_1^{-1} \circ \sigma_2\). We know \(x_{\sigma_1(i)} \leq y_{\sigma_2(i)}\) and \(y_{\sigma_2(i)} \leq x_{\sigma_1(i)}\). By reordering through the permutation \(\sigma_1^{-1}\), we get \(x_{\sigma_1^{-1} \circ \sigma_2(i)} \leq y_{\sigma_1^{-1} \circ \sigma_2(i)}\) and \(y_{\sigma_1^{-1} \circ \sigma_2(i)} \leq x_{\sigma_1^{-1} \circ \sigma_2(i)}\). By simplifying we get \(x_i \leq y_{\sigma(i)}\) and \(y_{\sigma(i)} \leq x_i\).

Lemma 139. \([x_1; \ldots; x_n] \leq [y_1; \ldots; y_m]\) and \([y_1; \ldots; y_m] \leq [x_1; \ldots; x_n]\) if, and only if \(n = m\) and there exists a permutation \(\sigma\) such that \(x_i \leq y_{\sigma(i)}\) and \(y_{\sigma(i)} \leq x_i\).

Proof.

Case 1 (\(\Rightarrow\)). By Lemma 138.

Case 2 (\(\Leftarrow\)). Let \(n = m\) and \(\sigma\) be a permutation such that \(x_i \leq y_{\sigma(i)}\) and \(y_{\sigma(i)} \leq x_i\).

We know the number of equivalence classes in the two lists is equal, as otherwise there would be some equivalence class in one that is not in the other, a contradiction with the assumption.

We proceed by induction on the number of equivalence classes:

Base Case: no equivalence classes, no elements, trivially true.

Induction Step: Let there be \(n + 1\) equivalence classes. Let \(\sigma_1 = \text{sort}([x_1; \ldots; x_n])\) and \(\sigma_2 = \text{sort}([y_1; \ldots; y_n])\).

Consider the largest equivalence class. We know that all except that equivalence class must map to each other, so when we remove that equivalence class, we get that all except the largest elements are ordered with \(\leq\), by IH. Adding those elements back in, we know they must go at the end.

Furthermore, they must have the same number of elements on each side \(k\), else we contradict the assumption. This means that in \(\text{sort}([x_1; \ldots; x_n], a)\) and \(\text{sort}([y_1; \ldots; y_n])\) are ordered such that the \(j\)th element in the \(x\) list is equivalent to the \(j\)th element in the \(y\) list, until the end, but the last \(k\) elements are all equivalent as they are all the largest equivalence class, so we are done.

\[\square\]

Now we can define what \(\leq_{\text{Atom}}\), \(\leq_{\text{Seq}}\) and \(\leq_{\text{DNF}}\) are, mutually.
Definition 33.

- We say \((EDS^\ast, ils_1) \leq_{\text{Atom}}^{\text{ex}} (EDT^\ast, ils_2)\) if 
  \((EDS, ils_1) \leq_{\text{DNF}_{\text{intlistset}}}^{\text{ex}} (EDT, ils_2)\).
- We say \(\{(s_0 \cdot EA_1 \cdot \ldots \cdot EA_n \cdot s_n\}, ils_1\) \leq_{\text{Seq}}^{\text{ex}} \[[t_0 \cdot EB_1 \cdot \ldots \cdot EB_m \cdot t_n\], ils_2\) if 
  \([(EA_1; \ldots; EA_n], ils_1) \leq_{\text{ex}}^{\text{Atom}} ([EB_1; \ldots; EB_m], ils_2)\).
- We say \(\langle ESQ_1 \mid \ldots \mid ESQ_n \rangle, ils_1\) \leq_{\text{DNF}}^{\text{ex}} \[[ETQ_1 \mid \ldots \mid ETQ_n], ils_2\) if 
  \([\langle ESQ_1; \ldots; ESQ_n\rangle, ils_1) \leq_{\text{ex}}^{\text{intlistset}} ([ETQ_1; \ldots; ETQ_m], ils_2)\).

Now, using this we provide the more formal definition of algorithm, with the formal use of the examples in Algorithm 4. We do not include information about user defined data types.

Lemma 140.

- Let A and B be strongly unambiguous atoms. Let \(sils_1\) be a string int list set. Let \(sils_2\) be a string int list set. Let \(EA\) and \(EB\) be exampled atoms. Let \(sils_1 \in A \rightsquigarrow EA\). Let \(sils_2 \in B \rightsquigarrow EB\). If \(\text{RIGIDSYNTHATOM}(EA, EB)\) returns an atom lens, Some \(al\), then \(\text{projectils}(sils_1) = \text{projectils}(sils_2)\), \(al \mapsto A \leftrightarrow B\), and for each \((s, t)\) pair with the same int list in \(sils_1\) and \(sils_2\), \(s, t \in \llbracket al\rrbracket\).
- Let \(SQ\) and \(TQ\) be strongly unambiguous sequences. Let \(sils_1\) be a string int list set. Let \(sils_2\) be a string int list set. Let \(SQ\) and \(TQ\) be exampled sequences. Let \(sils_1 \in SQ \rightsquigarrow ESQ\). Let \(sils_2 \in TQ \rightsquigarrow ETQ\). If \(\text{RIGIDSYNTHSEQ}(ESQ, ETQ)\) returns a sequence lens, Some \(sql\), then \(\text{projectils}(sils_1) = \text{projectils}(sils_2)\), \(sql \mapsto SQ \leftrightarrow TQ\), and for each \((s, t)\) pair with the same int list in \(sils_1\) and \(sils_2\), \((s, t) \in \llbracket sql\rrbracket\).
- Let \(DS\) and \(DT\) be strongly unambiguous DNF regular expressions. Let \(sils_1\) be a string int list set. Let \(sils_2\) be a string int list set. Let \(EDS\) and \(EDT\) be exampled DNF regular expressions. Let \(sils_1 \in EDS \rightsquigarrow EDS\). Let \(sils_2 \in EDT \rightsquigarrow EDT\). If \(\text{RIGIDSYNTHINTERNAL}(EDS, EDT)\) returns a DNF lens, Some \(dl\), then \(\text{projectils}(sils_1) = \text{projectils}(sils_2)\), \(dl \mapsto DS \leftrightarrow DT\), and for each \((s, t)\) pair with the same int list in \(sils_1\) and \(sils_2\), \((s, t) \in \llbracket dl\rrbracket\).

Proof.

Case 1 (atom). Unfolding definitions.

Let \(sils_1 = \{(s'_1, il'_1), \ldots, (s'_m, il'_m)\}\).
Let \(sils_2 = \{(t'_1, il'_1), \ldots, (t'_{m'}, il'_{m'})\}\).

By inversion on \(sils_1 \in A \rightsquigarrow EA\) and \(sils_2 \in B \rightsquigarrow EB\), we know that
\(\{(s'_{1,1}, 1 : il'_1); \ldots; (s'_{1,n'_1}, n'_1 :: il'_1); \ldots; (s'_{m',1}, 1 :: il'_{m'}); \ldots; (s'_{m',n'_{m'}}, n'_{m'} :: il'_{m'})\} \in DS \rightsquigarrow EDS\),
\(\{(t'_{1,1}, 1 :: il'_1); \ldots; (t'_{1,n'_1}, n'_1 :: il'_1); \ldots; (t'_{m',1}, 1 :: il'_{m'}); \ldots; (s'_{m',n'_{m'}}, n'_{m'} :: il'_{m'})\} \in DT \rightsquigarrow EDT\),
\(s'_{1,1} \cdot \ldots \cdot s'_{1,n'_1} = s_1\),
\(t'_{1,1} \cdot \ldots \cdot t'_{1,n'_1} = t_1\),
\(EA = (EDS^\ast, \{il'_1, \ldots, il'_m\})\), and
\(EB = (EDT^\ast, \{il'_1, \ldots, il'_m\})\).

As \(\text{RIGIDSYNTHATOM}\) returns, then it must return \(\text{iterate}(\text{RIGIDSYNTHINTERNAL}(EDS, EDT))\).

By IH that means that
\(\text{projectils}\{(s'_{1,1}, 1 :: il'_1); \ldots; (s'_{1,n'_1}, n'_1 :: il'_1); \ldots; (s'_{m',1}, 1 :: il'_{m'}); \ldots; (s'_{m',n'_{m'}}, n'_{m'} :: il'_{m'})\}\)
\(= \text{projectils}\{(t'_{1,1}, 1 :: il'_1); \ldots; (t'_{1,n'_1}, n'_1 :: il'_1); \ldots; (t'_{m',1}, 1 :: il'_{m'}); \ldots; (s'_{m',n'_{m'}}, n'_{m'} :: il'_{m'})\}\).

So, by reindexing,
\(\text{projectils}\{(s_{1,1}, 1 :: il_1); \ldots; (s_{1,n_1}, n_1 :: il_1); \ldots; (s_{m,1}, 1 :: il_m); \ldots; (s_{m,n_m}, n_m :: il_m)\}\)
\(= \text{projectils}\{(t_{1,1}, 1 :: il_1); \ldots; (t_{1,n_1}, n_1 :: il_1); \ldots; (t_{m,1}, 1 :: il_m); \ldots; (s_{m,n_m}, n_m :: il_m)\}\).
Algorithm 4 RigidSynth

1: function RigidSynthATOM((EDS*, ils1), (EDT*, ils2))
   2: if ils1 ⋈ intlistset ils2 ∨ ils2 ⋈ intlistset ils1 then
      return None
   3: match RigidSynthINTERNAL(EDS, EDT) with
      4: Some dl → return iterate(dl)
      5: None → return None

7: function RigidSynthSEQ(ESQ, ETQ)
   8: ([s0 · EA1 · · · EA n · s n], ils1) ← ESQ
   9: ([t0 · EB1 · · · EBm · tm], ils2) ← ETQ
10: if ils1 ⋈ intlistset ils2 ∨ ils2 ⋈ intlistset ils1 then
      return None
11: if n ≠ m then
      return None
12: σ1 ← sorting(≤Atom, [EA1 · · · EA n])
13: σ2 ← sorting(≤Atom, [EB1 · · · EB n])
14: σ ← σ1−1 ◦ σ2
15: EABs ← Zip([EA1 · · · EA n], [EBσ(1) · · · EBσ(n)])
16: alos ← Map(RigidSynthATOM, EABs)
17: match ALLSOME(alos) with
18: Some [al1 · · · al n] → return Some ([(s0, t0) · al1 · · · al n · (sn, tn)], σ−1)
19: None → return None

22: function RigidSynthINTERNAL(EDS, EDT)
23: ([ESQ1 | · · · | ESQn], ils1) ← EDS
24: ([ETQ1 | · · · | ETQm], ils2) ← EDT
25: if ils1 ⋈ intlistset ils2 ∨ ils2 ⋈ intlistset ils1 then
      return None
26: if n ≠ m then
      return None
27: σ1 ← sorting(≤Seq, [ESQ1 | · · · | ESQn])
28: σ2 ← sorting(≤Seq, [ETQ1 | · · · | ETQn])
29: σ ← σ1−1 ◦ σ2
30: ETQs ← Zip([ESQ1 | · · · | ESQn], [ETQσ(1) | · · · | ETQσ(n)])
31: sqlos ← Map(RigidSynthSEQ, ETQs)
32: match ALLSOME(sqlos) with
33: Some [sql1 | · · · | sql n] → return Some ((sql1 | · · · | sql n), σ−1)
34: None → return None

37: function RigidSynth(DS, DT, exs)
38: [(s1, t1); · · · ; (sn, tn)] ← exs
39: EDS ← EmbedExamples([[1], s1]; · · · ; [[n], sn]), DS)
40: EDT ← EmbedExamples([[1], t1]; · · · ; [[n], tn]), DT)
41: return RigidSynthINTERNAL(EDS, EDT)
sils₁ = {(s₁, il₁), . . . , (sₘ, ilₘ)}, and
sils₂ = {(t₁, il₁), . . . , (tₘ, ilₘ)}, so we have
\(\text{projectils}((s₁, il₁), . . . , (sₘ, ilₘ))) = \text{projectils}((t₁, il₁), . . . , (tₘ, ilₘ)))\), as desired. (also we know by the condition which returns None)

Furthermore, by IH, for all \(i, j\), we have \((sᵢₗ, tᵢₗ) \in \{dl\}\). This means that \((sᵢ₁, . . . , sᵢₙ, tᵢ₁, . . . , tᵢₙ) \in \{dl\}\) for all \(i\), so \((sᵢ, tᵢ) \in \{dl\}\) for all \(i\).

Furthermore, by IH, \(dl \vdash DS \Leftrightarrow DT\). Thusly, as \(A\) and \(B\) are strongly unambiguous, \(DS\) and \(DT\) are unambiguously iterable, so we have

\[
\begin{array}{c|c|c|c}
\text{dl} \vdash DS & \Leftrightarrow DT & DS^* & DT^* \\
\hline
\text{iterate}(dl) \vdash DS & \Leftrightarrow DT & & \\
\end{array}
\]

Case 2 (sequence). Unfolding definitions.

Let sils₁ = \{(sᵢ', ilᵢ'), . . . , (sᵣ', ilᵣ')\}.
Let sils₂ = \{(tᵢ', ilᵢ'), . . . , (tᵣ', ilᵣ')\}

By inverison on sils₁ ∈ SQ ⊑ ESQ and sils₂ ∈ TQ ⊑ ETQ, we know that
\(\{(sᵢ', ilᵢ'); . . . ; (sᵣ', ilᵣ')\} ∈ A_i ⊑ E_A_i,\)
\(\{(tᵢ', ilᵢ'); . . . ; (tᵣ', ilᵣ')\} ∈ B_i ⊑ E_B_i,\)
\(sᵦ · sᵢ' . . . · sᵣ' = sᵦ,\)
\(tᵦ · tᵢ' . . . · tᵣ' = tᵦ',\)

\(ESQ = ([sᵦ', Aᵦ₁ . . . ; E_Aᵦ₁ · sᵦ], \{ilᵢ', . . . , ilᵣ'\}),\)

\(ETQ = ([sᵦ', E_B₁ . . . ; E_Bᵦ₁ · sᵦ'], \{ilᵢ', . . . , ilᵣ'\})\)

Let \(σ₁ = \text{sorting}([Aᵦ₁ . . . ; E_Aᵦ₁]), \)
Let \(σ₂ = \text{sorting}([E_B₁ . . . ; E_Aᵦ₁]), \)
Let \(σ = σ₁⁻¹ ∘ σ₂.\)

As \text{RigidsynthAtom}(Eₐᵦ₁, EBₐᵦ₁) must be true, which means that for each \(i \in [1, n],\)
\text{RigidsynthAtom}(Eₐᵦ₁, EBₐᵦ₁) returns a lens, \(alᵦ₁.\)

By IH, that means that
\(\text{projectils}((sᵦ', ilᵦ'), . . . ; (sᵣ', ilᵣ'))\)
= \(\text{projectils}((tᵦ', ilᵦ'), . . . ; (tᵣ', ilᵣ'))\), which immediately implies that \(m' = m''.\)

So, by reindexing, and aligning the int lists, \(\text{projectils}((s₁, il₁), . . . , (sₘ, ilₘ))\)
= \(\text{projectils}((t₁, il₁), . . . , (tₘ, ilₘ))\),
\(sils₁ = \{(s₁, il₁), . . . , (sₘ, ilₘ)\},\)
\(sils₂ = \{(t₁, il₁), . . . , (tₘ, ilₘ)\},\) so we have
\(\text{projectils}((s₁, il₁), . . . , (sₘ, ilₘ))\)
= \(\text{projectils}((t₁, il₁), . . . , (tₘ, ilₘ))\), as desired. (also we know by the condition which returns None)

Furthermore, by IH, for all \((i, j)\), we have \((sᵢᵦᵣ, tᵢᵦᵣ) \in \{alᵦᵣ\}\). By the definition of sequence lens semantics, \((sᵦ', sᵢ₁ . . . sᵣ₁, tᵦ', tᵢ₁ . . . tᵣ₁, σ⁻¹(σ(ᵦ₁)) . . . σ⁻¹(σ(ᵦₙ)) · tᵦ') \in \{([sᵦ', tᵦ') · alᵦ₁ . . . alᵦₙ · (sᵦ', tᵦ')] · σ⁻¹] \}

By simplifying, we get \((sᵦ', sᵦ₁ . . . sᵦₙ, tᵦ', tᵦ₁ . . . tᵦₙ) \in \{([sᵦ', tᵦ') · alᵦ₁ . . . alᵦₙ · (sᵦ', tᵦ')] · σ⁻¹] \}

By simplifying even more, we get \((sᵦ₁, tᵦ₁) \in \{([sᵦ', tᵦ') · alᵦ₁ . . . alᵦₙ · (sᵦ', tᵦ')] · σ⁻¹] \), as desired.

Furthermore, by IH, \(alᵦ₁ ⊨ Aᵦ₁ \Leftrightarrow Bᵦᵦ₁.\) Thusly, as \(SQ\) and \(TQ\) are strongly unambiguous, they are sequence unambiguously concatenable, so we have:

\[
\begin{align*}
alᵦ₁ ⊨ Aᵦ₁ & \Leftrightarrow Bᵦᵦ₁ \\
\overset{1}{sᵦ'} · Aᵦ₁ . . . Aᵦₙ · sᵦ' \overset{1}{tᵦ'} · Bᵦᵦ₁ . . . Bᵦᵦ₁ · tᵦ' \\
& \{([sᵦ', tᵦ') · alᵦ₁ . . . alᵦₙ · (sᵦ', tᵦ')] · σ⁻¹\} \Leftrightarrow \{sᵦ' · Bᵦᵦ₁ . . . Bᵦᵦ₁ · sᵦ'\}
\end{align*}
\]

Which, by simplifying is:
\[
al_i \models A_i \iff B_{\sigma(i)} \quad \vdash (s'_0; A_1; \ldots; A_n; s'_n) \quad \vdash (t'_0; B_1; \ldots; B_n; t'_n)
\]

\[
((s'_0, t'_0) \cdot a_1 \cdot \ldots \cdot a_n \cdot (s'_n, t'_n)), \sigma^{-1}) \models [s'_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot s'_n] \iff [s'_0 \cdot B_1 \cdot \ldots \cdot B_n \cdot s'_n]
\]

As desired.

Case 3 (dnfregex). Unfolding definitions.

Let \(sils_1 = \{(s'_0, t'_0), \ldots, (s'_m, t'_m)\}\).

Let \(sils_2 = \{(t'_0, i'_0), \ldots, (t'_m, i'_m)\}\)

By invarion on \(sils_1 \in DS \iff EDS\) and \(sils_2 \in DT \iff EDT\), we know that
\[
S_i = \{(s'_0, t'_0), \ldots, (s'_m, t'_m)\}, \quad S'_i \in ESQ_i \iff ETQ_i,
\]

\[
\bigcup_{i \in [1,n]} S_i = sils_1,
\]
\[
\bigcup_{i \in [1,n]} S'_i = sils_2,
\]
\[
EDS = \langle ESQ_1 \mid \ldots \mid ESQ_n \rangle, \quad \text{and}
\]
\[
EDT = \langle ETQ_1 \mid \ldots \mid ETQ_n \rangle
\]

Let \(\sigma_1 = \text{solving}(\{EA_1; \ldots; EA_n\})\). Let \(\sigma_2 = \text{solving}(\{EA_1; \ldots; EA_n\})\). Let \(\sigma = \sigma_1^{-1} \circ \sigma_2\).

As \(\text{RigidsynthInternal}\) returns, \(\text{Allsome(silos)}\) must be true, which means that for each \(i \in [1,n]\), \(\text{Rigidsyntheseq}(ESQ_i, ETQ_{\sigma(i)})\) returns a lens, \(sql_i\).

By IH, that means that \(\text{projectils}(S_i) = \text{projectils}(S'_i)\). This means that, \(\text{projectils}(sils_1) = \text{projectils}(sils_2)\), as they are each the union of all these sets. We also know this through the fact we return a value, as desired. This immediately implies that \(m' = m''\). Furthermore, we can use this to reindex \(S_i = S_i = \{(s'_i, t'_i), \ldots, (s'_m, t'_m)\}\), and \(S'_i = S'_i = \{(t'_i, i'_i), \ldots, (t'_m, i'_m)\}\).

Furthermore, by IH, for all \((i,j)\), we have \((s_j, t_j) \notin \langle sql_i \mid \ldots \mid sql_n \rangle\).

\[
\langle \langle sql_1 \mid \ldots \mid sql_n \rangle, \sigma^{-1} \rangle = \bigcup_{i \in [1,n]} \langle sql_i \rangle
\]

Let \((s_j, t_j)\) arbitrary from \(sils_1\) and \(sils_2\) sharing the same int list \(il_i\). By them being the union of all \(S_i\) and \(S'_i\), there exists some \(i', j\) such that \(s_i = s_{i,j}\) and \(t_i = t_{i,j}\). As such, \((s_i, t_i) \in \langle \langle sql_1 \mid \ldots \mid sql_n \rangle, \sigma^{-1} \rangle\), as desired.

Furthermore, by IH, \(sql_i \models SQ_i \iff TQ_{\sigma(i)}\). Thusly, as \(DS\) and \(DT\) are strongly unambiguous, they are pairwise disjoint in sequences, so we have:

\[
sql_i \models SQ_i \iff TQ_{\sigma(i)} \quad i \neq j \implies L(SQ_i) \cap L(SQ_j) = \emptyset \quad i \neq j \implies L(TQ_i) \cap L(TQ_j) = \emptyset
\]

\[
\langle \langle sql_1 \mid \ldots \mid sql_n \rangle, \sigma^{-1} \rangle \models \langle (SQ_1 \mid \ldots \mid SQ_n) \rangle \iff \langle (TQ_1 \mid \ldots \mid TQ_n) \rangle
\]

Which, by simplifying is:

\[
sql_i \models SQ_i \iff TQ_{\sigma(i)} \quad i \neq j \implies L(SQ_i) \cap L(SQ_j) = \emptyset \quad i \neq j \implies L(TQ_i) \cap L(TQ_j) = \emptyset
\]

\[
\langle \langle sql_1 \mid \ldots \mid sql_n \rangle, \sigma^{-1} \rangle \models \langle (SQ_1 \mid \ldots \mid SQ_n) \rangle \iff \langle (TQ_1 \mid \ldots \mid TQ_n) \rangle
\]

As desired.

\[\square\]

**Lemma 141.** Let \(DS\) and \(DT\) be strongly unambiguous DNF regular expressions. Let \([s_1, t_1]; \ldots; [s_n, t_n]\) be a list of input-output examples. If \(\text{Rigidsynth}(EDS, EDT)\) returns a DNF lens, *Some dl*, then \(dl \models DS \iff DT\), and \((s_i, t_i) \in \langle dl \rangle\).

**Proof.** Let \([s_1, t_1]; \ldots; [s_n, t_n]\) = \(\text{exs}\).

Let \(EDS = \text{EmbedExamples}(\{(i, [s_1]); \ldots; ([n], s_n)\}, DS)\)
Let $EDT = \text{EmbedExamples}([[1], t_1]; \ldots; ([n], t_n), DT)$

This means that $[[([1], s_1); \ldots; ([n], s_n)] \in DS \rightsquigarrow EDS$ and $[[([1], t_1); \ldots; ([n], t_n)] \in DT \rightsquigarrow EDT$.

As RigidSynth returns a lens, then RigidSynthInternal(EDS, EDT) must return a lens.

By the above statements, Lemma 140 applies, so the DNF lens returned by it satisfies $dl \downarrow DS \leftrightarrow DT$, and $(s_i, t_i) \in \|dl\|$. $DS$ and $DT$ are strongly unambiguous.

As such the conditions for Lemma 143 apply so $EDS \leq_{DNF}^{\times} EDS$ and $EDT \leq_{DNF}^{\times} EDS$.

By Lemma 144, that means that RigidSynthInternal(EDS, EDT) returns a DNF lens, so then too does RigidSynth.

Let $DS$ and $DT$ be strongly unambiguous DNF regular expressions. Let $sils_1$ be a string int list set. Let $sils_2$ be a string int list set. Let $EDS$ and $EDT$ be exampled DNF regular expressions. Let $sils_1 \in EDS \rightsquigarrow EDS$. Let $sils_2 \in EDT \rightsquigarrow EDT$. If RigidSynthInternal(EDS, EDT) returns a sequence lens, Some $dl$, then $\text{projectils}(sils_1) = \text{projectils}(sils_2)$, $dl \downarrow DS \leftrightarrow DT$, and for each $(s, t)$ pair with the same int list in $sils_1$ and $sils_2$, $(s, t) \in \|dl\|$.

\[ \square \]

**Lemma 142.** Let $DS$ and $DT$ be strongly unambiguous DNF regular expressions. Let $[(s_1, t_1); \ldots; (s_m, t_m)]$ be a list of input-output examples. If SynthDNFLens(EDS, EDT) returns a DNF lens, Some $dl$, then $dl : DS \leftrightarrow DT$, and $(s_i, t_i) \in \|dl\|$.

**Proof.** If SynthDNFLens returns, then there must be some $(S', T')$ popped from the queue which returned a DNF lens $dl = \text{RigidSynthInternal}(S', T')$. These regular expressions are such that $DS \rightarrow_{DS} S'$ and $DT \rightarrow_{DS} T'$, as the regular expressions are always either the originals, or have been added to the queue from an expansion on a previously popped element. Inductively, everything in the queue is an expansion on $DS$ or $DT$.

As $DS$ and $DT$ are strongly unambiguous DNF regular expressions, $DS \rightarrow_{DS} S'$ means that $DS'$ is strongly unambiguous, and similarly for $DT$ and $T'$.

So, by Lemma 141, $dl \downarrow DS' \iff DT'$. Furthermore, for each $(s_i, t_i)$ pair in the examples $(s_i, t_i) \in \|dl\|$.

Lastly, we can then build the typing derivation:

\[
\begin{array}{ccc}
dl \downarrow DS & \iff DT' & DS \rightarrow_{DS} S' \\
& & DT \rightarrow_{DS} DT'
\end{array}
\]

\[ \square \]

**Theorem 13.** For all lenses $l$, regular expressions $S$ and $T$, and examples $exs$, if $l = \text{SynthLens}(S, T, exs)$, then $l : S \equiv T$ and for all $(s, t)$ in exs, $(s, t) \in \|l\|$.

**Proof.** As SynthLenses returns, then SynthDNFLens($DS, DT$) returns, where $DS = \Downarrow S$ and $DT = \Downarrow T$. As Validate($S, T, exs$) does not error out, then we know $S$ and $T$ are strongly unambiguous, and $exs$ matches them. This then means that $DS$ and $DT$ are also strongly unambiguous.

As such, by Lemma 142, we know that $dl = \text{SynthDNFLens}(DS, DT)$ returns, that $dl : DS \leftrightarrow DS$, and that for each $(s, t)$ in the examples, $(s, t) \in \|dl\|$. By Theorem 2, there exists $S'$ and $T'$ such that $\Downarrow dl$ and $\Downarrow S' = DS$ and $\Downarrow T' = DT$ and $\|\Downarrow dl\| = \|dl\|$. As $\Downarrow S = \Downarrow S'$ and $\Downarrow T = \Downarrow T'$, we have $S \equiv S'$ and $T \equiv T'$.

\[
\Downarrow dl : S' \iff T' \quad S \equiv S' \quad T \equiv T'
\]

\[
\Downarrow dl : S \iff T
\]

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
Furthermore, as \(|dl| = |\uparrow dl|\), we have for each \((s, t)\) in the examples, \((s, t) \in |\uparrow dl|\), as desired.

**Lemma 143.**

- Let \(A\) and \(B\) be strongly unambiguous atoms. Let \(sils_1\) be a string int list set. Let \(sils_2\) be a string int list set. Let \(projectils(sils_1) = projectils(sils_2)\). Let \(EA\) and \(EB\) be exampled atoms. Let \(sils_1 \in A \leadsto EA\). Let \(sils_2 \in B \leadsto EB\). If \(al \equiv A \Leftrightarrow B\), and for each \((s, t)\) pair with the same int list in \(sils_1\) and \(sils_2\), \((s, t) \in |dl|\), then \(EA \leq^{\text{ex} Atom} EB\) and \(EB \leq^{\text{ex} Atom} EA\).
- Let \(SQ\) and \(TQ\) be strongly unambiguous sequences. Let \(sils_1\) be a string int list set. Let \(sils_2\) be a string int list set. Let \(projectils(sils_1) = projectils(sils_2)\). Let \(ESQ\) and \(ETQ\) be exampled sequences. Let \(sils_1 \in SQ \leadsto ESQ\). Let \(sils_2 \in TQ \leadsto ETQ\). If \(sql \equiv SQ \Leftrightarrow TQ\), and for each \((s, t)\) pair with the same int list in \(sils_1\) and \(sils_2\), \((s, t) \in |sql|\), then \(ESQ \leq^{\text{ex} Seq} ETQ\) and \(ETQ \leq^{\text{ex} Seq} ESQ\).
- Let \(DS\) and \(DT\) be strongly unambiguous DNF regular expressions. Let \(sils_1\) be a string int list set. Let \(sils_2\) be a string int list set. Let \(projectils(sils_1) = projectils(sils_2)\). Let \(EDS\) and \(EDT\) be exampled DNF regular expressions. Let \(sils_1 \in EDS \leadsto EDS\). Let \(sils_2 \in EDT \leadsto EDT\). If \(dl \equiv DS \Leftrightarrow DT\), and for each \((s, t)\) pair with the same int list in \(sils_1\) and \(sils_2\), \((s, t) \in |dl|\), then \(EDS \leq^{\text{ex} DNF} EDT\) and \(EDT \leq^{\text{ex} DNF} EDS\).

**Proof.** We proceed by mutual induction

**Case 1 (atom case).** Unfolding definitions.

Let \(sils_1 = \{(s_1, il_1), \ldots, (s_m, il_m)\}\).

Let \(sils_2 = \{(t_1, il_1), \ldots, (t_m, il_m)\}\).

By inversion on \(al \equiv A \Leftrightarrow B\), we know \(DS^* = A\), \(DT^* = B\), \(iterate(dl) = al\), and \(dl \equiv DS \Leftrightarrow DT\).

By inversion on \(\uparrow iterate(dl)\), there exist \(s_1, \ldots, s_m\) such that \(s_1, \ldots, s_m, il_m \equiv s_m\) and \(t_1, \ldots, t_m \equiv t_m\) through \(s_1, \ldots, s_m, il_m \equiv s_m\) and \(t_1, \ldots, t_m \equiv t_m\) such that \((s_{ij}, t_{ij}) \in |dl|\).

Furthermore, this means that \(s_{ij} \in L(DS)\) and \(t_{ij} \in L(DT)\).

By inversion on \(sils_1 \equiv A \leadsto EA\) and \(sils_2 \equiv B \leadsto EB\), we know that

\[\{(s'_1, 1 : il_1), \ldots, (s'_{n_1}, il'_1); \ldots; (s'_1, 1 : il_m); \ldots; (s'_{m_1}, n_1 : il_m); \ldots; (s'_{n_1}, n_1 : il_m)\} \in DS \leadsto EDS,\]

\[\{(t'_1, 1 : il_1); \ldots; (t'_n, n : il_1); \ldots; (t'_1, 1 : il_m); \ldots; (t'_n, n : il_m)\} \in DT \leadsto EDT,\]

\[s'_{1,1} \ldots s'_{1,n_1} = s_1,\]

\[t'_{1,1} \ldots t'_{1,n_1} = s_1,\]

\[EA = (EDS^*, \{il_1, \ldots, il_m\}),\]

\[EB = (EDT^*, \{il_1, \ldots, il_m\}).\]

As \(A\) and \(B\) are strongly unambiguous, \(n_j = n'_j = n''_j\), \(s_{ij} = s'_1, t_{ij} = t'_1\), and \(DS\) and \(DT\) are strongly unambiguous.

By inspection

\[\text{projectils}(\{(s'_1, 1 : il_1); \ldots; (s'_{n_1}, n : il_1); \ldots; (s_m, 1 : il_m); \ldots; (s_{m_n}, n : il_m)\})\]

\[= \text{projectils}(\{(t'_1, 1 : il_1); \ldots; (t'_{n}, n : il_1); \ldots; (t_m, 1 : il_m); \ldots; (t_{m_n}, n : il_m)\})\]

Using the above facts, we have satisfied the preconditions to use the IH, so \(EDS \leq^{\text{ex} DNF} EDT\) and \(EDT \leq^{\text{ex} DNF} EDS\).

Furthermore, \(\{il_1, \ldots, il_m\} \leq \text{intlistset} \{il_1, \ldots, il_m\}\), so \(EA \leq^{\text{ex} Atom} EB\) and \(EB \leq^{\text{ex} Atom} EA\).

**Case 2 (sequence case).** Unfolding definitions.

Let \(sils_1 = \{(s_1, il_1), \ldots, (s_m, il_m)\}\).

Let \(sils_2 = \{(t_1, il_1), \ldots, (t_m, il_m)\}\).

By inversion on \(sql \equiv SQ \Leftrightarrow TQ\), we know \(\left[s'_0 \cdot A_1 \cdot A_n \cdot s'_n\right] = SQ, \left[t'_0 \cdot B_{\sigma(1)} \cdot \ldots \cdot B_{\sigma(n)} \cdot t'_n\right] = TQ,\)

\[\left([s'_{0}, t'_{0}] \cdot al_1 \cdot \ldots \cdot al_n \cdot (s'_{n}, t'_{n}), \sigma\right) = sql, \text{ and } al_i : A_i \Leftrightarrow B_i.\]

, Vol. 1, No. 1, Article 1. Publication date: January 2018.
By inversion on \(\| [(s'_0, t'_0) \cdot al_1 \ldots al_n \cdot (s'_n, t'_n)], \sigma]\), there exist \(s'_0 \cdot s_{1,1} \ldots \cdot s_{1,n} \cdot s'_n = s_1\) through
\(s'_0 \cdot s_{m,1} \ldots \cdot s_{m,n} \cdot s'_n = s_m\) and \(t'_0 \cdot t_{1,\sigma(1)} \ldots \cdot t_{1,\sigma(n)} \cdot t'_n = t_1\) through \(t'_0 \cdot t_{m,\sigma(1)} \ldots \cdot s_{m,\sigma(n)} \cdot t'_n = t_m\)
such that \((s_{i,j}, t_{i,j}) \in \| al_i\|\).

Furthermore, this means that \(s_{i,j} \in L(A_i)\) and \(t_{i,j} \in L(B_i)\).

By inversion on \(sils_1 \in SQ \leadsto ESQ\) and \(sils_2 \in TQ \leadsto ETQ\), we know that
\({(s'_{i,j}, il_1)}; \ldots ; {(s'_{m,j}, il_m)} \in A_i \leadsto EA_i\),
\({(t'_{i,j}, il_1)}; \ldots ; {(t'_{m,j}, il_m)} \in B_i \leadsto EB_i\),
\(s'_0 \cdot s'_{i,j} \ldots \cdot s'_{n} = s_i\),
\(t'_0 \cdot t'_{1,j} \ldots \cdot t'_{n} = t_j\),
\(ESQ = \{(s'_0 \cdot EA_1 \ldots \cdot EA_n \cdot s'_n), \{il_1, \ldots , il_m\}\}\), and
\(ETQ = \{(s'_0 \cdot EB_{\sigma(1)} \ldots \cdot EB_{\sigma(n)} \cdot s'_n), \{il_1, \ldots , il_m\}\}\).

As \(SQ\) and \(TQ\) are strongly unambiguous, \(s_{i,j} = s'_{i,j}, t_{i,j} = t'_{i,j}\), and \(A_i\) and \(B_i\) are strongly unambiguous.

By inspection \(projectils((s_{i,j}, il_1); \ldots ; (s_{m,j}, il_m)) = projectils((t_{i,j}, il_1); \ldots ; (t_{m,j}, il_m))\).

Using the above facts, we have satisfied the preconditions to use the IH, so \(EA_i \leq_{\text{Atom}}^{\text{ex}} EB_i\) and
\(EB_i \leq_{\text{Atom}}^{\text{ex}} EA_i\).

As such, from Lemma 139, \([EA_1; \ldots ; EA_n]\{\leq_{\text{Atom}}^{\text{ex}}\}[EB_{\sigma(1)}; \ldots ; EB_{\sigma(n)}]\).

This means that, as \(\{il_1, \ldots , il_m\} \models_{\text{initial state}} \{il_1, \ldots , il_m\}\), we have
\([\{EA_1; \ldots ; EA_n\}, \{il_1, \ldots , il_m\}] \models_{\text{Atom}} \{[EB_{\sigma(1)}; \ldots ; EB_{\sigma(n)}], \{il_1, \ldots , il_m\}\}\).

Which means that \(ESQ \leq_{\text{Seq}}^{\text{ex}} ETQ\).

Case 3 (DNF regex case). Unfolding definitions.

Let \(sils_1 = \{(s_1, il_1), \ldots , (s_m, il_m)\}\).

Let \(sils_2 = \{(t_1, il_1), \ldots , (t_m, il_m)\}\).

By inversion on \(dl : DS \Rightarrow DT\), we know \(\{SQ_1 | \ldots | SQ_n\} = DS\), \(\{TQ_{\sigma(1)} | \ldots | TQ_{\sigma(n)}\} = DT\),
\(\{(sql_1 | \ldots | sql_m), \sigma\} = sql\), and \(sql_i : SQ_i \Rightarrow TQ_i\).

By inversion on \(\| [(sql_1 | \ldots | sql_m), \sigma]\), there exist \({(s_{1,1}, t_{1,1})}, \ldots , {(s_{1,m}, t_{1,m})}\} = S_1\) through
\({(s_{m,1}, t_{m,1})}, \ldots , {(s_{m,m}, t_{m,m})}\} = S_m\) such that \(\bigcup_{i \in [1,n]} S_i = \{(s_1, t_1), \ldots , (s_m, t_m)\}\), such that \(S_i \subset\)
\(\|sql_i\|\), also with this reindexing Furthermore, this means that \(s_{i,j} \in L(SQ_i)\) and \(t_{i,j} \in L(TQ_i)\).

By inversion on \(sils_1 \in DS \leadsto EDS\) and \(sils_2 \in DT \leadsto EDT\), we know that
\({(s'_{1,1}, il'_{1,1})}, \ldots , {(s'_{1,m}, il'_{1,m})}\} \in EDS \leadsto ESQ_1,
\({(t'_{1,1}, il'_{1,1})}, \ldots , {(t'_{1,m}, il'_{1,m})}\} \in EDT \leadsto ETQ_1,
\bigcup_{i \in [1,n]} \{(s'_i, il'_i)|, \ldots , (s'_m, il'_m)\} = \{s_1, \ldots , s_m\}\)
\(\bigcup_{i \in [1,n]} \{(t'_i, il'_i)|, \ldots , (t'_m, il'_m)\} = \{t_1, \ldots , t_m\}\).

\(EDS = \{(ESQ_1 | \ldots | ESQ_n), \{il_1, \ldots , il_m\}\}\), and
\(EDT = \{(ETQ_{\sigma(1)} | \ldots | ETQ_{\sigma(n)}), \{il_1, \ldots , il_m\}\}\).

As \(DS\) and \(DT\) are strongly unambiguous, \{s_{i,1}, \ldots , s_{i,m}\} = \{s'_{i,1}, \ldots , s'_{i,m}\}\).

By aligning with their int lists (which are unique), we get \(ils_i = \{(s'_{i,1}, il'_{i,1})|, \ldots , (s'_{i,m}, il'_{i,m})|\} = \{(s_{i,1}, il_{1,1})|, \ldots , (s_{i,m}, il_{i,m})|\}\)
and
\(ils'_i = \{(t'_{i,1}, il'_{i,1})|, \ldots , (t'_{i,m}, il'_{i,m})|\} = \{(t_{i,1}, il_{1,1})|, \ldots , (t_{i,m}, il_{i,m})|\}\).

By inspection \(projectils((s_{i,1}, il_{1,1})|, \ldots , (s_{i,m}, il_{i,m})|) = projectils((t_{i,1}, il_{1,1})|, \ldots , (t_{i,m}, il_{i,m})|)\).

Using the above facts, we have satisfied the preconditions to use the IH, so \(ESQ_i \leq_{\text{Seq}}^{\text{ex}} ETQ_i\) and
\(ETQ_i \leq_{\text{Seq}}^{\text{ex}} ESQ_i\).

As such, from Lemma 139, \([ESQ_1; \ldots ; ESQ_n]\{\leq_{\text{Seq}}^{\text{ex}}\}[ETQ_{\sigma(1)}; \ldots ; ETQ_{\sigma(n)}]\).
This means that, as \( \{i_1, \ldots, i_m\} \leq \text{inlistset} \{i_1, \ldots, i_m\} \), we have
\[
(ESQ_1; \ldots; ESQ_n), \{i_1, \ldots, i_m\} \left( \{i_1, \ldots, i_m\} \right) \leq \text{inlistset} \left( \{ETQ_{\sigma(1)}; \ldots; ETQ_{\sigma(n)}\}, \{i_1, \ldots, i_m\} \right).
\]
Which means that \( EDS \leq_{\text{DNF}}^\text{exs} EDT \). □

**Lemma 144.**

- Let \( EA \) and \( EB \) be exampled atoms. If \( EA \leq_{\text{Atom}}^\text{exs} EB \) and \( EB \leq_{\text{Atom}}^\text{exs} EA \), then \( \text{RigidSynthAtom} \) returns an atom lens.
- Let \( ESQ \) and \( ETQ \) be exampled atoms. If \( ESQ \leq_{\text{Seq}}^\text{exs} ETQ \) and \( ETQ \leq_{\text{Seq}}^\text{exs} ESQ \), then \( \text{RigidSynthSeq} \) returns a sequence lens.
- Let \( EDS \) and \( EDT \) be exampled atoms. If \( EDS \leq_{\text{DNF}}^\text{exs} EDT \) and \( EDT \leq_{\text{DNF}}^\text{exs} EDS \), then \( \text{RigidSynthInternal} \) returns a DNF lens.

**Proof.**

*Case 1* (atoms). Let \((EDS^*, ils) = EA \). Let \((EDT^*, ils') = EB \). As \( EA \leq_{\text{Atom}}^\text{exs} EB \), we have \( EDS \leq_{\text{DNF}}^\text{exs} EDT \). By IH, this means that \( \text{RigidSynthInternal}(EDS, EDT) \) returns a DNF lens, \( dl \), this means that the match goes to the first, returning \( \text{iterate}(dl) \), an atom lens.

*Case 2* (seqs). Let \( ([s_0 \cdot EA_1 \cdot \ldots \cdot EA_n \cdot s_n], ils) = ESQ \) and \( ([t_0 \cdot EB_1 \cdot \ldots \cdot EB_m \cdot t_m], ils') = ETQ \).

Let \( \sigma_1 = \text{sorting}([EA_1; \ldots; EA_n,]) \) and \( \sigma_2 = \text{sorting}([EB_1; \ldots; EB_m,]) \). As \( ESQ \leq_{\text{Seq}}^\text{exs} ETQ \) and \( ETQ \leq_{\text{Seq}}^\text{exs} ESQ \), we know \( n = m \). By Lemma 138 \( \sigma = \sigma_1^{-1} \circ \sigma_2 \) is such that \( EA_1 \leq EB_{\sigma(i)} \) and \( EB_{\sigma(i)} \leq EA_i \).

By IH, this means that calling \( \text{RigidSynthAtom}(EA_i, EB_{\sigma(i)}) \) returns an atom lens, \( al_i \). As such, \( \text{AllSome} \) on all these atom lens options returns a list of atom lenses. This then returns the sequence lens \( ([s_0, t_0] \cdot al_1 \cdot \ldots \cdot al_n \cdot (s_n, t_n)], \sigma^{-1}) \).

*Case 3* (dnf regexps). Let \((ESQ_1 | \ldots | ESQ_n), ils) = EDS \) and \((ETQ_1 | \ldots | EB_m), ils') = ETQ \).

Let \( \sigma_1 = \text{sorting}([ESQ_1; \ldots; ESQ_n,]) \) and \( \sigma_2 = \text{sorting}([ETQ_1; \ldots; ETQ_m,]) \). As \( EDS \leq_{\text{DNF}}^\text{exs} EDT \) and \( ETQ \leq_{\text{DNF}}^\text{exs} EDS \), we know \( n = m \). By Lemma 138 \( \sigma = \sigma_1^{-1} \circ \sigma_2 \) is such that \( EA_i \leq EB_{\sigma(i)} \) and \( EB_{\sigma(i)} \leq EA_i \).

By IH, this means that calling \( \text{RigidSynthSeq}(ESQ_i, ETQ_{\sigma(i)}) \) returns a sequence lens, \( sql_i \). As such, \( \text{AllSome} \) on all these sequence lens options returns a list of sequence lenses. This then returns the DNF lens \( ([sql_1 | \ldots | sql_n], \sigma^{-1}) \).

□

**Lemma 145.** Let \( DS \) and \( DT \) be DNF regexes. Let \( exs \) be a set of string pairs. If there exists a DNF lens \( dl \vdash DS \leftrightarrow DT \) such that \( exs \in [dl] \), then \( \text{RigidSynth}(DS, DT, exs) \) returns a DNF lens.

**Proof.** Let \( ([s_1, t_1]; \ldots; [s_n, t_n]) = exs \).

Let \( EDS = \text{EmbedExamples}([([1], s_1); \ldots; ([n], s_n)], DS) \)
Let \( EDT = \text{EmbedExamples}([([1], t_1); \ldots; ([n], t_n)], DT) \)

We know that \( \text{EmbedExamples} \) doesn’t fail, as each \( s_i \in L(DS) \) and each \( t_i \in L(DT) \).

This means that \( ([1], s_1); \ldots; ([n], s_n) \in DS \sim EDS \) and \( ([1], t_1); \ldots; ([n], t_n) \in DT \sim EDT \).

As \( dl \vdash DS \leftrightarrow DT \), we know that \( DS \) and \( DT \) are strongly unambiguous.

As such the conditions for Lemma 143 apply so \( EDS \leq_{\text{DNF}}^\text{exs} EDT \) and \( EDT \leq_{\text{DNF}}^\text{exs} EDS \).

By Lemma 144, that means that \( \text{RigidSynthInternal}(EDS, EDT) \) returns a DNF lens, so then so too does \( \text{RigidSynth} \).

□
Lemma 146. Given DNF regular expressions $DS$ and $DT$, and a set of examples $exs$, if there exists a DNF lens $dl$ such that $dl : DS \iff DT$ and for all $(s, t)$ in $exs$, $(s, t) \in \llbracket dl \rrbracket$, then $\text{SynthDNFLens}(DS, DT, exs)$ will return a DNF lens.

Proof. If $dl : DS \iff DT$, then there exist DNF regular expressions $DS'$ and $DT'$ such that $dl : DS' \iff DT'$, $DS_\rightarrow DS' \rightarrow DS'$, and $DT_\rightarrow DT' \rightarrow DT'$.

However, we may not perform the same single rewrites as $DS_\rightarrow DS' \rightarrow DT$, because we infer certain expansions, which are then taken earlier. However, confluence allows us to reorganize the order of the expansions (though this may possibly increase the total number of expansions, and change the ultimate DNF lens), as was done in the rewrite case of DNF completeness.

As such, we know that, there exist two DNF regular expressions $DS''$ and $DT''$, and a DNF lens $dl''$ such that $DS_\rightarrow DS'' \rightarrow DT_\rightarrow DT''$, $dl'' : DS'' \iff DT''$, $\llbracket dl'' \rrbracket = \llbracket dl \rrbracket$, and $(DS'', DT'')$ are defined by the queue.

If there are any $(DS'', DT''')$ pairs enumerated before $(DS'', DT'')$ such that $\text{RigidSynth}(DS'', DT''')$ returns a lens, then that lens is returned, and we are done.

If not, eventually $(DS'', DT'')$ are enumerated.

By Lemma 145, as $dl'' : DS'' \iff DT''$, $\text{RigidSynth}(DS'', DT'')$ returns. This is then immediately returned by $\text{SynthDNFLens}$. \hfill \qed

Theorem 14. Given regular expressions $S$ and $T$, and a set of examples $exs$, if there exists a lens $l$ such that $l : S \iff T$ and for all $(s, t)$ in $exs$, $(s, t) \in \llbracket l \rrbracket$, then $\text{SynthLens}(S, T, exs)$ will return a lens.

Proof. Let $l : S \iff T$. By Theorem 3, there exists a DNF lens $dl : \llbracket S \iff T \rrbracket$, such that $\llbracket dl \rrbracket = \llbracket l \rrbracket$. In particular, this means that $exs \subset \llbracket dl \rrbracket$. This means, from Lemma 146, $\text{SynthDNFLens}(\llbracket S \rrbracket, \llbracket T \rrbracket)$ returns a lens, $dl''$. From Theorem 2, we can then convert using $\llbracket \cdot \rrbracket$, to get a lens which we return. \hfill \qed

B.13 Additional Proofs

In the paper, some claims were made that aren’t necessarily the main theorems. In this area we prove those claims.

Definition 34. Let $\rightarrow$ be a rewrite rule on regular expressions. We define $\rightarrow_\downarrow$ as the rewrite relation on DNF regular expressions defined by $\llbracket S \rightarrow_\downarrow T \rrbracket$ if $S \rightarrow T$.

Definition 35. We overload $\llbracket \cdot \rrbracket$ to extend to regular expressions with rewriteless DNF lenses after conversion to DNF form. In particular, $S \llbracket \rightarrow_\downarrow \llbracket T \rrbracket$ if $S \rightarrow T$.

Lemma 147. $\text{confluent}_{\downarrow}$ ($\rightarrow$) implies $\text{confluent}_{\downarrow}$ ($\rightarrow_\downarrow$).

Proof. Let $dl : DS \iff DT$, with $\llbracket dl \rrbracket = \llbracket l \rrbracket$. Let $DS_\rightarrow DS_1$ and $DT_\rightarrow DT_2$. This means that there exists $S, T, S_1$, and $T_2$ such that $S \rightarrow S_2$, $T \rightarrow T_2$, $\llbracket S \rrbracket = DS$, $\llbracket T \rrbracket = DT$, $\llbracket S_1 \rrbracket = DS_2$, and $\llbracket T_2 \rrbracket = DT_2$. This means, as $\text{confluent}_{\downarrow}$ ($\rightarrow$), there exists $S_3, T_3$ such that there exists $dl_3 : \llbracket S_3 \iff T_3 \rrbracket$ where $\llbracket dl_3 \rrbracket = \llbracket dl \rrbracket$. This means that $DS_2 \rightarrow_\downarrow DS_3$, and $\llbracket T_2 \rrbracket \rightarrow_\downarrow DS_3$, and as before there exists $dl_3 : \llbracket S_3 \iff T_3 \rrbracket$ where $\llbracket dl_3 \rrbracket = \llbracket dl \rrbracket$. As such, $\text{confluent}_{\downarrow}$ ($\rightarrow_\downarrow$). \hfill \qed

Lemma 148. $\text{confluent}_{\downarrow}$ ($\rightarrow_\downarrow$) implies $\text{confluent}_{\downarrow}$ ($\rightarrow$).

Proof. Let $dl : \llbracket S \iff T \rrbracket$. Let $S \rightarrow S_1$ and $T \rightarrow T_2$. This means that $\llbracket S \rightarrow_\downarrow S_2 \rrbracket$ and $\llbracket T \rightarrow_\downarrow T_2 \rrbracket$. As $\text{confluent}_{\downarrow}$ ($\rightarrow_\downarrow$), then there exists $dl_3$, $DS_3$ and $DT_3$ such that $\llbracket S_2 \rightarrow_\downarrow DS_3$, $\llbracket T_2 \rightarrow_\downarrow DS_3$, $\llbracket T_2 \rightarrow_\downarrow DS_3$.
Lemma 149. If bisimilar \( \vdash (\rightarrow) \), then bisimilar \( \vdash (\rightarrow_{\|}) \).

Proof. Let \( dl \vdash DS \leftrightarrow DT \), with \( \llbracket dl \rrbracket = \llbracket l \rrbracket \). Let \( DS \rightarrow_{\|} DS_2 \). This means that there exists \( S \) and \( S_2 \) such that \( S \rightarrow S_2 \), \( \llbracket S \rrbracket = DS \), and \( \llbracket S_2 \rrbracket = DS_2 \). This means, as confluent \( \vdash (\rightarrow) \), there exists \( T_2 \) such that there exists \( dl_2 \vdash \llbracket S_2 \rrbracket \leftrightarrow \llbracket T_2 \rrbracket \) where \( \llbracket dl_2 \rrbracket = \llbracket dl \rrbracket \), and \( \llbracket DT \rrbracket \rightarrow \llbracket T_2 \rrbracket \). Symmetrically for if \( DT \rightarrow_{\|} DT_2 \), so bisimilar \( \vdash (\rightarrow_{\|}) \).

Lemma 150. If bisimilar \( \vdash (\rightarrow_{\|}) \), then bisimilar \( \vdash (\rightarrow) \).

Proof. Let \( dl \vdash \llbracket S \rrbracket \leftrightarrow \llbracket T \rrbracket \), with \( \llbracket dl \rrbracket = \llbracket l \rrbracket \). Let \( S \rightarrow S_2 \). This means that \( \llbracket S \rrbracket \rightarrow_{\|} \llbracket S_2 \rrbracket \). This means that, as bisimilar \( \vdash (\rightarrow_{\|}) \) there exists \( DT_2 \), \( dl_2 \) such that \( \llbracket T \rrbracket \rightarrow_{\|} \llbracket DT_2 \rrbracket \), and \( \llbracket dl_2 \rrbracket = \llbracket dl \rrbracket \). As \( \llbracket T \rrbracket \rightarrow_{\|} \llbracket DT_2 \rrbracket \), we have \( DT_2 = \llbracket T_2 \rrbracket \), where \( T \rightarrow T_2 \). Symmetrically for if \( T \rightarrow T_2 \), so bisimilar \( \vdash (\rightarrow) \).

Lemma 151. If \( \rightarrow \) doesn’t introduce ambiguity, then neither does \( \rightarrow_{\|} \).

Proof. Let \( DS \rightarrow_{\|} DT \), where \( DS \) is strongly unambiguous. This means \( DS = \llbracket S \rrbracket \) and \( DT = \llbracket T \rrbracket \), and \( S \rightarrow T \). As \( \llbracket S \rrbracket \) is strongly unambiguous iff \( S \) is, \( S \) is strongly unambiguous. By assumption, we now have \( T \) is strongly unambiguous, so then \( \llbracket T \rrbracket \) also is.

Lemma 152. If \( \rightarrow_{\|} \) doesn’t introduce ambiguity, then neither does \( \rightarrow \).

Proof. Let \( S \rightarrow T \), where \( S \) is strongly unambiguous. This means \( \llbracket \text{Regex} \rrbracket \rightarrow_{\|} \llbracket T \rrbracket \). By assumption, that means \( \llbracket T \rrbracket \) is strongly unambiguous. As \( \llbracket T \rrbracket \) strongly unambiguous iff \( T \) strongly unambiguous, by assumption we have \( T \) is strongly unambiguous.

Lemma 153. \( \rightarrow_{\|} \) maintains the language if, and only if \( \rightarrow \) does.

Proof. \( S \rightarrow T \). \( \mathcal{L}(S) = \mathcal{L}(\llbracket S \rrbracket) \). \( \mathcal{L}(T) = \mathcal{L}(\llbracket T \rrbracket) \). \( \llbracket S \rrbracket = \mathcal{L}(\llbracket T \rrbracket) \). \( \llbracket S \rrbracket \rightarrow_{\|} \llbracket T \rrbracket \). \( DS \rightarrow_{\|} DT \).

Thus, we have confluence, bisimilarity, unambiguity retention, language retention on rewrites for regular expressions, iff we also do for the DNF rewrites they induce. As such, because confluence and bisimilarity, alongside not introducing ambiguity or changing the language, is sufficient on DNF rewrites, then it is sufficient on rewrite rules for regular expressions as a sufficient condition is fulfilled.