COS 423, April 2, 1996
Handout Number 8

Solutions to Midterm

Problem 1 For (a) the answer is YES; we need only note that $2^n < 3+e^n < 8^n$ (for $n > 1$) and hence $n < \log_2(3 + e^n) < 3n$. For (b) either YES or NO is accepted, depending on which definition one uses for the O notation. If $f(n) = O(g(n))$ means $|f(n)| < g(n)$ for all sufficiently large n, then the answer is YES since $|\cos((n+\sqrt{2})\pi)| \leq 1$; if it means $f(n)$ must be positive in addition, then the answer is NO. The textbooks use the O notation only for positive $f(n)$, but many people use the former convention too. For (c), the answer is YES, since $\log_2(\log^* n) \leq \log_2 n$ for all $n > 4$, which implies $(\log_2 n)^2 \log_2(\log^* n) \leq (\log_2 n)^3$, and hence by exponentiating both sides we get $(\log^* n)^{(\log_2 n)^2} \leq 2^{(\log_2 n)^3}$.

Problem 2. People give two solutions.

Method 1 In Dijkstra's algorithm, we initially set $V \leftarrow \{n\}$. Let $\delta(j)$ store the value of the shortest path from n to j via only vertices in V; initially $\delta(j) = w(n, j)$ for $1 \leq j < n$. Clearly, the first iteration selects vertex 1 because $\delta(1) = \sqrt{n^2 + 1^2} < \sqrt{n^2 + j^2} = \delta(j)$ for all $1 < j < n$. Then update $V \leftarrow V \cup \{1\} = \{n, 1\}$ and for each $j \notin V$,

$$\begin{aligned} \delta(j) &\leftarrow \min\{w(n,j), \delta(1) + w(1,j)\} \\ &= \min\{\sqrt{n^2+j^2}, \sqrt{n^2+1^2} + \sqrt{1^2+j^2}\} \\ &= \sqrt{n^2+j^2}. \end{aligned}$$

Thus, $\delta(j)$ remains $w(n, j)$. Similarly, in the ith iteration for each $i > 1$, vertex i will be selected; the updating gives $V = \{n, 1, 2, \cdots, i\}$, and $\delta(j)$ remains $w(n, j)$ for all $j \notin V$. This proves that the shortest path from n to vertex j is equal to the final value $\delta(j) = w(n, j)$ for all $1 \leq j < n$.

Method 2 The shortest path from n to j is the one-edge path $n \rightarrow j$, since any path of length more than 1 has length

$$\sqrt{n^2+i_1^2} + \sqrt{i_1^2+i_2^2} + \cdots + \sqrt{i_k^2+j^2} \geq \sqrt{n^2+i_1^2} + \sqrt{i_k^2+j^2} > \sqrt{n^2+j^2}.$$

Remark One can also obtain the answer by apply the following general theorem: if the positive weight function w satisfies the *triangular ineqalities* $w(i,k) \leq w(i,j)+w(j,k)$ for all distinct i, j, k, then the length of the shortest path from i to j is equal to $w(i, j)$.

Problem 3 We refer edges by their *weights*. Applying Kruskal's algorithm, we get in turn edges $1, 2, 3, 5, 6$, which form the minimum spanning tree T. To obtain the second smallest spanning tree D, we use the result from Problem 24-1 that D can be obtained from T by adding some edge $j \notin T$ and delete some edge $k_j \in T$ (clearly k_j must be the largest-weight edge in the cycle created in T by adding j). The weigth of the new spanning tree is equal to weight of $T + j - k_j$. Clearly, we want to pick j with the smallest $j - k_j$. If we choose $j = 7, 8, 10, 11, 12, 13$, then k_j are $3, 5, 6, 6, 6, 6$, and the differences are $4, 3, 4, 5, 6, 7$, respectively. Thus, the second smallest spanning tree is obtained from T by adding edge 8 and deleting edge 5.