PRINCETON UNIVERSITY COS 522: COMPUTATIONAL COMPLEXITY

Homework 4

Out: *April 16* Due: *April 25*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. Also, limit your answers to one handwritten page, preferably less —you just need to give enough detail to convince me. If some question doesn't seem correct or solvable, indicate why.

For this problem set, assume that circuits consist of $\vee$ *and* $\wedge$ *gates with fanin* 2 *and the inputs are* $x_1, x_2, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}$*. In other words, the* $\neg$ *gates have been pushed down to the inputs. The depth of the circuit is the number of edges (wires) in the longest path from output to an input.*

§1 Let $\mathcal{A}$ be a class of deterministic algorithms for a problem and I be the set of possible inputs. Both sets are finite. Let $\text{cost}(A, x)$ denote the cost of running algorithm A on input x. Prove Yao's lemma:

$$\max_{\mathcal{D}} \min_{A \in \mathcal{A}} \mathbf{E}_{x \in \mathcal{D}}[\text{cost}(A, x)] \quad = \quad \min_{\mathcal{P}} \max_{x \in I} \mathbf{E}_{A \in \mathcal{P}}[\text{cost}(A, x)],$$

where $\mathcal{D}$ is a probability distribution on I and $\mathcal{P}$ is a probability distribution on $\mathcal{A}$.

Does this result hold if the class of algorithms and inputs is infinite?

§2 For any graph G with n vertices, consider the following communication problem: Player 1 receives a clique C in G, and Player 2 receives an independent set I. They have to communicate in order to determine $|C \cap I|$. (Note that this number is either 0 or 1.) Prove an $O(\log^2 n)$ upperbound on the communication complexity.

Can you improve your upperbound or prove a lower bound better than $\Omega(\log n)$?

§3 Associate the following communication problem with any function $f : \{0,1\}^n \to \{0,1\}$. Player 1 gets any input x such that $f(x) = 0$ and player 2 gets any input y such that $f(y) = 1$. They have to communicate in order to determine a bit position i such that $x_i \neq y_i$.

Show that the communication complexity of this problem is *exactly* the minimum depth of any circuit that computes f.

§4 Use the previous question to show that computing the parity of n bits requires depth at least $2 \log n$.

§5 Show that in any circuit with m edges and depth d, there are $km/logd$ edges whose removal yields a circuit of depth less than $d/2^{k-1}$.