4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights
Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from $s$ to $t$.

**edge-weighted digraph**

4→5 0.35  
5→4 0.35  
4→7 0.37  
5→7 0.28  
7→5 0.28  
5→1 0.32  
0→4 0.38  
0→2 0.26  
7→3 0.39  
1→3 0.29  
2→7 0.34  
6→2 0.40  
3→6 0.52  
6→0 0.58  
6→4 0.93

**shortest path from 0 to 6**

0→2 0.26  
2→7 0.34  
7→3 0.39  
3→6 0.52
Google maps
Car navigation
Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

Shortest path variants

Which vertices?
- **Single source**: from one vertex \( s \) to every other vertex.
- **Source-sink**: from one vertex \( s \) to another \( t \).
- **All pairs**: between all pairs of vertices.

Restrictions on edge weights?
- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

Cycles?
- No directed cycles.
- No "negative cycles."

Simplifying assumption. Shortest paths from \( s \) to each vertex \( v \) exist.
4.4 **Shortest Paths**

- APIs
  - shortest-paths properties
  - Dijkstra's algorithm
  - edge-weighted DAGs
  - negative weights
Weighted directed edge API

```
public class DirectedEdge

    DirectedEdge(int v, int w, double weight)  // weighted edge v→w
        int from()                // vertex v
        int to()                  // vertex w
        double weight()           // weight of this edge
        String toString()         // string representation

Idiom for processing an edge e: int v = e.from(), w = e.to();
```
Weighted directed edge: implementation in Java

Similar to Edge for undirected graphs, but a bit simpler.

```java
public class DirectedEdge {
    private final int v, w;
    private final double weight;

    public DirectedEdge(int v, int w, double weight) {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int from() {
        return v;
    }

    public int to() {
        return w;
    }

    public int weight() {
        return weight;
    }
}
```
Edge-weighted digraph API

public class EdgeWeightedDigraph

    EdgeWeightedDigraph(int V)  // edge-weighted digraph with V vertices
    EdgeWeightedDigraph(In in)  // edge-weighted digraph from input stream
    void addEdge(DirectedEdge e)  // add weighted directed edge e
    Iterable<DirectedEdge> adj(int v)  // edges pointing from v
    int V()  // number of vertices
    int E()  // number of edges
    Iterable<DirectedEdge> edges()  // all edges
    String toString()  // string representation

Conventions. Allow self-loops and parallel edges.
Edge-weighted digraph: adjacency-lists representation

tinyEWD.txt

V
E

adj

Bag objects

reference to a DirectedEdge object
**Edge-weighted digraph: adjacency-lists implementation in Java**

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```java
public class EdgeWeightedDigraph
{
    private final int V;
    private final Bag<DirectedEdge>[] adj;

    public EdgeWeightedDigraph(int V)
    {
        this.V = V;
        adj = (Bag<DirectedEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<DirectedEdge>();
    }

    public void addEdge(DirectedEdge e)
    {
        int v = e.from();
        adj[v].add(e);
    }

    public Iterable<DirectedEdge> adj(int v)
    {   return adj[v];   }
}
```

add edge $e = v \rightarrow w$ to only $v$'s adjacency list
Single-source shortest paths API

**Goal.** Find the shortest path from \( s \) to every other vertex.

```java
public class SP {
    SP(EdgeWeightedDigraph G, int s) // shortest paths from s in graph G
    double distTo(int v) // length of shortest path from s to v
    Iterable <DirectedEdge> pathTo(int v) // shortest path from s to v
    boolean hasPathTo(int v) // is there a path from s to v?
}
```

```java
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
    StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
    for (DirectedEdge e : sp.pathTo(v))
        StdOut.print(e + " ");
    StdOut.println();
}
```
Single-source shortest paths API

**Goal.** Find the shortest path from $s$ to every other vertex.

```java
public class SP

SP(EdgeWeightedDigraph G, int s) // shortest paths from s in graph G
double distTo(int v) // length of shortest path from s to v
Iterable <DirectedEdge> pathTo(int v) // shortest path from s to v
boolean hasPathTo(int v) // is there a path from s to v?
```

```bash
% java SP tinyEWD.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38 4->5 0.35 5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26 2->7 0.34 7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38 4->5 0.35
0 to 6 (1.51): 0->2 0.26 2->7 0.34 7->3 0.39 3->6 0.52
0 to 7 (0.60): 0->2 0.26 2->7 0.34
```
4.4 Shortest Paths

- APIs
  - shortest-paths properties
  - Dijkstra's algorithm
  - edge-weighted DAGs
  - negative weights
4.4 **Shortest Paths**

- APIs
- *shortest-paths* properties
- *Dijkstra's algorithm*
- edge-weighted DAGs
- negative weights
Data structures for single-source shortest paths

**Goal.** Find the shortest path from \( s \) to every other vertex.

**Observation.** A shortest-paths tree (SPT) solution exists. Why?

**Consequence.** Can represent the SPT with two vertex-indexed arrays:
- \( \text{distTo}[v] \) is length of shortest path from \( s \) to \( v \).
- \( \text{edgeTo}[v] \) is last edge on shortest path from \( s \) to \( v \).

\[
\begin{array}{|c|c|c|}
| \text{edgeTo[]} | \text{distTo[]} | \\
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>null</td>
</tr>
<tr>
<td>1</td>
<td>5-&gt;1 0.32</td>
</tr>
<tr>
<td>2</td>
<td>0-&gt;2 0.26</td>
</tr>
<tr>
<td>3</td>
<td>7-&gt;3 0.37</td>
</tr>
<tr>
<td>4</td>
<td>0-&gt;4 0.38</td>
</tr>
<tr>
<td>5</td>
<td>4-&gt;5 0.35</td>
</tr>
<tr>
<td>6</td>
<td>3-&gt;6 0.52</td>
</tr>
<tr>
<td>7</td>
<td>2-&gt;7 0.34</td>
</tr>
</tbody>
</table>
\end{array}
\]

shortest-paths tree from 0

parent-link representation
Data structures for single-source shortest paths

**Goal.** Find the shortest path from $s$ to every other vertex.

**Observation.** A shortest-paths tree (SPT) solution exists. Why?

**Consequence.** Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$ is length of shortest path from $s$ to $v$.
- $\text{edgeTo}[v]$ is last edge on shortest path from $s$ to $v$.

```java
public double distTo(int v)
{
    return distTo[v];
}

public Iterable<DirectedEdge> pathTo(int v)
{
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
        path.push(e);
    return path;
}
```
Edge relaxation

Relax edge \( e = v \rightarrow w \).

- \( \text{distTo}[v] \) is length of shortest known path from \( s \) to \( v \).
- \( \text{distTo}[w] \) is length of shortest known path from \( s \) to \( w \).
- \( \text{edgeTo}[w] \) is last edge on shortest known path from \( s \) to \( w \).
- If \( e = v \rightarrow w \) gives shorter path to \( w \) through \( v \), update both \( \text{distTo}[w] \) and \( \text{edgeTo}[w] \).

\( v \rightarrow w \) successfully relaxes
**Edge relaxation**

Relax edge \( e = v \rightarrow w \).

- \( \text{distTo}[v] \) is length of shortest known path from \( s \) to \( v \).
- \( \text{distTo}[w] \) is length of shortest known path from \( s \) to \( w \).
- \( \text{edgeTo}[w] \) is last edge on shortest known path from \( s \) to \( w \).
- If \( e = v \rightarrow w \) gives shorter path to \( w \) through \( v \), update both \( \text{distTo}[w] \) and \( \text{edgeTo}[w] \).

```java
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```
Shortest-paths optimality conditions

**Proposition.** Let $G$ be an edge-weighted digraph. Then $\text{distTo}[]$ are the shortest path distances from $s$ iff:

- For each vertex $v$, $\text{distTo}[v]$ is the length of some path from $s$ to $v$.
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.

**Pf.** $\iff$ [necessary]

- Suppose that $\text{distTo}[w] > \text{distTo}[v] + e.\text{weight}()$ for some edge $e = v \rightarrow w$.
- Then, $e$ gives a path from $s$ to $w$ (through $v$) of length less than $\text{distTo}[w]$. 

\[ \text{distTo}[v] = 3.1 \]
\[ \text{distTo}[w] = 7.2 \]
Shortest-paths optimality conditions

**Proposition.** Let $G$ be an edge-weighted digraph. Then $\text{distTo}[]$ are the shortest path distances from $s$ iff:

- For each vertex $v$, $\text{distTo}[v]$ is the length of some path from $s$ to $v$.
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.

**Pf.** $\Rightarrow$ [ sufficient ]

- Suppose that $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = w$ is a shortest path from $s$ to $w$.
- Then,
  \[
  \begin{align*}
  \text{distTo}[v_1] & \leq \text{distTo}[v_0] + e_1.\text{weight}() \quad \text{for } e_1 = \text{ith edge on shortest path from } s \text{ to } w \\
  \text{distTo}[v_2] & \leq \text{distTo}[v_1] + e_2.\text{weight}() \\
  \vdots \\
  \text{distTo}[v_k] & \leq \text{distTo}[v_{k-1}] + e_k.\text{weight}()
  \end{align*}
  \]

- Add inequalities; simplify; and substitute $\text{distTo}[v_0] = \text{distTo}[s] = 0$:
  \[
  \text{distTo}[w] = \text{distTo}[v_k] \leq e_1.\text{weight}() + e_2.\text{weight}() + \ldots + e_k.\text{weight}()
  \]

- Thus, $\text{distTo}[w]$ is the weight of shortest path to $w$. $\blacksquare$
Generic shortest-paths algorithm

**Generic algorithm (to compute SPT from s)**

- Initialize \( \text{distTo}[s] = 0 \) and \( \text{distTo}[v] = \infty \) for all other vertices.

- Repeat until optimality conditions are satisfied:
  - Relax any edge.

**Proposition.** Generic algorithm computes SPT (if it exists) from \( s \).

**Pf sketch.**

- Throughout algorithm, \( \text{distTo}[v] \) is the length of a simple path from \( s \) to \( v \) (and \( \text{edgeTo}[v] \) is last edge on path).
- Each successful relaxation decreases \( \text{distTo}[v] \) for some \( v \).
- The entry \( \text{distTo}[v] \) can decrease at most a finite number of times. \( \blacksquare \)
Generic shortest-paths algorithm

**Generic algorithm (to compute SPT from s)**

- Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.
- Repeat until optimality conditions are satisfied:
  - Relax any edge.

**Efficient implementations.** How to choose which edge to relax?

**Ex 1.** Dijkstra's algorithm (nonnegative weights).

**Ex 2.** Topological sort algorithm (no directed cycles).

**Ex 3.** Bellman-Ford algorithm (no negative cycles).
4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra’s algorithm
- edge-weighted DAGs
- negative weights
4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights
“Do only what only you can do.”

“In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.”

“The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.”

“It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration.”

“APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.”
"Object-oriented programming is an exceptionally bad idea which could only have originated in California."

-- Edsger Dijkstra
Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges pointing from that vertex.

![Diagram of an edge-weighted digraph with edges labeled with their weights]
Dijkstra's algorithm demo

• Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \texttt{distTo[]} value).

• Add vertex to tree and relax all edges pointing from that vertex.

\[
\begin{array}{c|cc}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
\hline
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow1 \\
2 & 14.0 & 5\rightarrow2 \\
3 & 17.0 & 2\rightarrow3 \\
4 & 9.0 & 0\rightarrow4 \\
5 & 13.0 & 4\rightarrow5 \\
6 & 25.0 & 2\rightarrow6 \\
7 & 8.0 & 0\rightarrow7 \\
\end{array}
\]

\textit{shortest-paths tree from vertex \( s \)}
Dijkstra's algorithm visualization
Dijkstra's algorithm visualization
Dijkstra's algorithm: correctness proof

**Proposition.** Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

**Pf.**
- Each edge \( e = v \rightarrow w \) is relaxed exactly once (when \( v \) is relaxed), leaving \( \text{distTo}[w] \leq \text{distTo}[v] + e\text{.weight}() \).
- Inequality holds until algorithm terminates because:
  - \( \text{distTo}[w] \) cannot increase
  - \( \text{distTo}[v] \) will not change

\[ \text{distTo[] values are monotone decreasing} \]
\[ \text{we choose lowest distTo[] value at each step (and edge weights are nonnegative)} \]

- Thus, upon termination, shortest-paths optimality conditions hold. ■
Dijkstra's algorithm: Java implementation

```java
public class DijkstraSP {
    private DirectedEdge[] edgeTo;
    private double[] distTo;
    private IndexMinPQ<Double> pq;

    public DijkstraSP(EdgeWeightedDigraph G, int s) {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];
        pq = new IndexMinPQ<Double>(G.V());

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        pq.insert(s, 0.0);
        while (!pq.isEmpty()) {
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}
```

relax vertices in order of distance from s
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
        if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
        else pq.insert(w, distTo[w]);
    }
}
Computing spanning trees in graphs

Dijkstra’s algorithm seem familiar?
- Prim’s algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a graph’s spanning tree.

Main distinction: Rule used to choose next vertex for the tree.
- Prim’s: Closest vertex to the tree (via an undirected edge).
- Dijkstra’s: Closest vertex to the source (via a directed path).

Note: DFS and BFS are also in this family of algorithms.
Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: \( V \) insert, \( V \) delete-min, \( E \) decrease-key.

<table>
<thead>
<tr>
<th>PQ implementation</th>
<th>insert</th>
<th>delete-min</th>
<th>decrease-key</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>unordered array</td>
<td>1</td>
<td>( V )</td>
<td>1</td>
<td>( V^2 )</td>
</tr>
<tr>
<td>binary heap</td>
<td>( \log V )</td>
<td>( \log V )</td>
<td>( \log V )</td>
<td>( E \log V )</td>
</tr>
<tr>
<td>d-way heap (Johnson 1975)</td>
<td>( d \log_d V )</td>
<td>( d \log_d V )</td>
<td>( \log_d V )</td>
<td>( E \log_{E/V} V )</td>
</tr>
<tr>
<td>Fibonacci heap (Fredman–Tarjan 1984)</td>
<td>( 1^\dagger )</td>
<td>( \log V^\dagger )</td>
<td>( 1^\dagger )</td>
<td>( E + V \log V )</td>
</tr>
</tbody>
</table>

\( ^\dagger \) amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.
4.4 **Shortest Paths**

- APIs
- shortest-paths properties
- *Dijkstra’s algorithm*
- edge-weighted DAGs
- negative weights
4.4 **Shortest Paths**

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights
Acyclic edge-weighted digraphs

Q. Suppose that an edge-weighted digraph has no directed cycles. Is it easier to find shortest paths than in a general digraph?

A. Yes!
Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.

An edge-weighted DAG
Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.

shortest-paths tree from vertex s

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>5→2</td>
</tr>
<tr>
<td>3</td>
<td>17.0</td>
<td>2→3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td>13.0</td>
<td>4→5</td>
</tr>
<tr>
<td>6</td>
<td>25.0</td>
<td>2→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
Shortest paths in edge-weighted DAGs

**Proposition.** Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

**Pf.**
- Each edge $e = v \rightarrow w$ is relaxed exactly once (when $v$ is relaxed), leaving $\text{distTo}[w] \leq \text{distTo}[v] + e\text{.weight()}$.
- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$ cannot increase \hspace{1cm} $\text{distTo[]}$ values are monotone decreasing
  - $\text{distTo}[v]$ will not change \hspace{1cm} because of topological order, no edge pointing to $v$ will be relaxed after $v$ is relaxed

- Thus, upon termination, shortest-paths optimality conditions hold. ■
public class AcyclicSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;

    public AcyclicSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V());

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        Topological topological = new Topological(G);
        for (int v : topological.order())
            for (DirectedEdge e : G.adj(v))
                relax(e);
    }
}

Shortest paths in edge-weighted DAGs
Content-aware resizing

Seam carving.  [Avidan and Shamir]  Resize an image without distortion for display on cell phones and web browsers.

http://www.youtube.com/watch?v=vIFCV2spKtg
Content-aware resizing

**Seam carving.** [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

*In the wild.* Photoshop CS 5, Imagemagick, GIMP, ...
Content-aware resizing

To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.
Content-aware resizing

To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.
Content-aware resizing

To remove vertical seam:
- Delete pixels on seam (one in each row).
Content-aware resizing

To remove vertical seam:
  • Delete pixels on seam (one in each row).
Formulate as a shortest paths problem in edge-weighted DAGs.

- Negate all weights.
- Find shortest paths.
- Negate weights in result.

**Key point.** Topological sort algorithm works even with negative weights.
Longest paths in edge-weighted DAGs: application

Parallel job scheduling. Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

<table>
<thead>
<tr>
<th>job</th>
<th>duration</th>
<th>must complete before</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>41.0</td>
<td>1  7  9</td>
</tr>
<tr>
<td>1</td>
<td>51.0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>50.0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36.0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>38.0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>45.0</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>21.0</td>
<td>3  8</td>
</tr>
<tr>
<td>7</td>
<td>32.0</td>
<td>3  8</td>
</tr>
<tr>
<td>8</td>
<td>32.0</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>29.0</td>
<td>4  6</td>
</tr>
</tbody>
</table>
Critical path method

**CPM.** To solve a parallel job-scheduling problem, create edge-weighted DAG:

- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

<table>
<thead>
<tr>
<th>job</th>
<th>duration</th>
<th>must complete before</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>41.0</td>
<td>1 7 9</td>
</tr>
<tr>
<td>1</td>
<td>51.0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>50.0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36.0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>38.0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>45.0</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>21.0</td>
<td>3 8</td>
</tr>
<tr>
<td>7</td>
<td>32.0</td>
<td>3 8</td>
</tr>
<tr>
<td>8</td>
<td>32.0</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>29.0</td>
<td>4 6</td>
</tr>
</tbody>
</table>
**Critical path method**

**CPM.** Use longest path from the source to schedule each job.
4.4 **Shortest Paths**

- APIs
- shortest-paths properties
- Dijkstra’s algorithm
- edge-weighted DAGs
- negative weights
4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights
Shortest paths with negative weights: failed attempts

**Dijkstra.** Doesn’t work with negative edge weights.

Dijkstra selects vertex 3 immediately after 0. But shortest path from 0 to 3 is 0→1→2→3.

**Re-weighting.** Add a constant to every edge weight doesn’t work.

Adding 9 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

**Conclusion.** Need a different algorithm.
Negative cycles

**Def.** A **negative cycle** is a directed cycle whose sum of edge weights is negative.

\[
\begin{align*}
\text{digraph} & \\
4 \rightarrow 5 & 0.35 \\
5 \rightarrow 4 & -0.66 \\
4 \rightarrow 7 & 0.37 \\
5 \rightarrow 7 & 0.28 \\
7 \rightarrow 5 & 0.28 \\
5 \rightarrow 1 & 0.32 \\
0 \rightarrow 4 & 0.38 \\
0 \rightarrow 2 & 0.26 \\
7 \rightarrow 3 & 0.39 \\
1 \rightarrow 3 & 0.29 \\
2 \rightarrow 7 & 0.34 \\
6 \rightarrow 2 & 0.40 \\
3 \rightarrow 6 & 0.52 \\
6 \rightarrow 0 & 0.58 \\
6 \rightarrow 4 & 0.93
\end{align*}
\]

**Proposition.** A SPT exists iff no negative cycles.

assuming all vertices reachable from \(s\)
Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.
Repeat V times:
  - Relax each edge.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

an edge-weighted digraph
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

shortest-paths tree from vertex $s$
Bellman-Ford algorithm visualization

passes
4

7

10

13

SPT
Bellman-Ford algorithm: analysis

Bellman–Ford algorithm

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.

Repeat $V$ times:
  - Relax each edge.

Proposition. Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

Pf idea. After pass $i$, found shortest path containing at most $i$ edges.
Bellman-Ford algorithm: practical improvement

**Observation.** If \( \text{distTo}[v] \) does not change during pass \( i \), no need to relax any edge pointing from \( v \) in pass \( i+1 \).

**FIFO implementation.** Maintain queue of vertices whose \( \text{distTo}[] \) changed.

\[ \text{be careful to keep at most one copy} \]
\[ \text{of each vertex on queue (why?)} \]

**Overall effect.**
- The running time is still proportional to \( E \times V \) in worst case.
- But much faster than that in practice.
Single source shortest-paths implementation: cost summary

<table>
<thead>
<tr>
<th>algorithm</th>
<th>restriction</th>
<th>typical case</th>
<th>worst case</th>
<th>extra space</th>
</tr>
</thead>
<tbody>
<tr>
<td>topological sort</td>
<td>no directed cycles</td>
<td>(E + V)</td>
<td>(E + V)</td>
<td>(V)</td>
</tr>
<tr>
<td>Dijkstra (binary heap)</td>
<td>no negative weights</td>
<td>(E \log V)</td>
<td>(E \log V)</td>
<td>(V)</td>
</tr>
<tr>
<td>Bellman–Ford</td>
<td>no negative cycles</td>
<td>(E \ V)</td>
<td>(E \ V)</td>
<td>(V)</td>
</tr>
<tr>
<td>Bellman–Ford (queue-based)</td>
<td>no negative cycles</td>
<td>(E + V)</td>
<td>(E \ V)</td>
<td>(V)</td>
</tr>
</tbody>
</table>

**Remark 1.** Directed cycles make the problem harder.

**Remark 2.** Negative weights make the problem harder.

**Remark 3.** Negative cycles makes the problem intractable.
Finding a negative cycle

**Negative cycle.** Add two method to the API for SP.

```java
boolean hasNegativeCycle() // is there a negative cycle?
Iterable <DirectedEdge> negativeCycle() // negative cycle reachable from s
```

**digraph**

```
digraph
4->5 0.35
5->4 -0.66
4->7 0.37
5->7 0.28
7->5 0.28
5->1 0.32
0->4 0.38
0->2 0.26
7->3 0.39
1->3 0.29
2->7 0.34
6->2 0.40
3->6 0.52
6->0 0.58
6->4 0.93
```

Negative cycle (-0.66 + 0.37 + 0.28)

5->4->7->5

Shortest path from 0 to 6

```
s
5
1
3
7
0
2
4
6
```

Negative cycle (-0.66 + 0.37 + 0.28)
Finding a negative cycle

**Observation.** If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.

**Proposition.** If any vertex $v$ is updated in phase $v$, there exists a negative cycle (and can trace back `edgeTo[v]` entries to find it).

**In practice.** Check for negative cycles more frequently.
Negative cycle application: arbitrage detection

**Problem.** Given table of exchange rates, is there an arbitrage opportunity?

<table>
<thead>
<tr>
<th></th>
<th>USD</th>
<th>EUR</th>
<th>GBP</th>
<th>CHF</th>
<th>CAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>USD</td>
<td>1</td>
<td>0.741</td>
<td>0.657</td>
<td>1.061</td>
<td>1.011</td>
</tr>
<tr>
<td>EUR</td>
<td>1.350</td>
<td>1</td>
<td>0.888</td>
<td>1.433</td>
<td>1.366</td>
</tr>
<tr>
<td>GBP</td>
<td>1.521</td>
<td>1.126</td>
<td>1</td>
<td>1.614</td>
<td>1.538</td>
</tr>
<tr>
<td>CHF</td>
<td>0.943</td>
<td>0.698</td>
<td>0.620</td>
<td>1</td>
<td>0.953</td>
</tr>
<tr>
<td>CAD</td>
<td>0.995</td>
<td>0.732</td>
<td>0.650</td>
<td>1.049</td>
<td>1</td>
</tr>
</tbody>
</table>

**Ex.** $1,000 \Rightarrow 741$ Euros $\Rightarrow 1,012.206$ Canadian dollars $\Rightarrow \$1,007.14497.$

\[1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497\]
Negative cycle application: arbitrage detection

Currency exchange graph.
- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is > 1.

0.741 * 1.366 * .995 = 1.00714497

Challenge. Express as a negative cycle detection problem.
Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logs.

- Let weight of edge $v \rightarrow w$ be $-\ln$ (exchange rate from currency $v$ to $w$).
- Multiplication turns to addition; $>1$ turns to $<0$.
- Find a directed cycle whose sum of edge weights is $<0$ (negative cycle).

Remark. Fastest algorithm is extraordinarily valuable!
Shortest paths summary

Dijkstra’s algorithm.
- Nearly linear-time when weights are nonnegative.
- Generalization encompasses DFS, BFS, and Prim.

Acyclic edge-weighted digraphs.
- Arise in applications.
- Faster than Dijkstra’s algorithm.
- Negative weights are no problem.

Negative weights and negative cycles.
- Arise in applications.
- If no negative cycles, can find shortest paths via Bellman-Ford.
- If negative cycles, can find one via Bellman-Ford.

Shortest-paths is a broadly useful problem-solving model.
4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights
4.4 **Shortest Paths**

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights