4.3 Minimum Spanning Trees

- introduction
- greedy algorithm
- edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- context
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Minimum spanning tree

Given. Undirected graph $G$ with positive edge weights (connected).

Def. A spanning tree of $G$ is a subgraph $T$ that is both a tree (connected and acyclic) and spanning (includes all of the vertices).

Goal. Find a min weight spanning tree.
**Minimum spanning tree**

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A spanning tree of $G$ is a subgraph $T$ that is both a tree (connected and acyclic) and spanning (includes all of the vertices).

**Goal.** Find a min weight spanning tree.

![Diagram of a graph with labeled edges and a note that it is not connected.](image)
Minimum spanning tree

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A spanning tree of $G$ is a subgraph $T$ that is both a tree (connected and acyclic) and spanning (includes all of the vertices).

**Goal.** Find a min weight spanning tree.

![Diagram of a graph with edges labeled with weights.](image)

*not acyclic*
**Minimum spanning tree**

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A *spanning tree* of $G$ is a subgraph $T$ that is both a *tree* (connected and acyclic) and *spanning* (includes all of the vertices).

**Goal.** Find a min weight spanning tree.

![Graph Image]

spanning tree T: $\text{cost} = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7$

**Brute force.** Try all spanning trees?
Network design

MST of bicycle routes in North Seattle

http://www.flickr.com/photos/ewedistrict/21980840
Models of nature

MST of random graph

http://algo.inria.fr/broutin/gallery.html
MST describes arrangement of nuclei in the epithelium for cancer research

http://www.bccrc.ca/ci/ta01_archlevel.html
Medical image processing

MST dithering

http://www.flickr.com/photos/quasimondo/2695389651
Applications

MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, computer, road).

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Simplifying assumptions

Simplifying assumptions.

- Edge weights are distinct.
- Graph is connected.

Consequence. MST exists and is unique.
Cut property

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets.
Def. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.


### Cut property: correctness proof

**Def.** A **cut** in a graph is a partition of its vertices into two (nonempty) sets.

**Def.** A **crossing edge** connects a vertex in one set with a vertex in the other.

**Cut property.** Given any cut, the crossing edge of min weight is in the MST.

**Pf.** Suppose min-weight crossing edge \( e \) is not in the MST.

- Adding \( e \) to the MST creates a cycle.
- Some other edge \( f \) in cycle must be a crossing edge.
- Removing \( f \) and adding \( e \) is also a spanning tree.
- Since weight of \( e \) is less than the weight of \( f \), that spanning tree is lower weight.
- Contradiction. □

---

The MST does not contain \( e \)  
Adding \( e \) to MST creates a cycle
Greedy MST algorithm demo

- Start with all edges colored gray.
- Find cut with no black crossing edges; color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

![Greedy MST algorithm demo](image)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7</td>
<td>0.16</td>
</tr>
<tr>
<td>2-3</td>
<td>0.17</td>
</tr>
<tr>
<td>1-7</td>
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<td>0.32</td>
</tr>
<tr>
<td>2-7</td>
<td>0.34</td>
</tr>
<tr>
<td>4-5</td>
<td>0.35</td>
</tr>
<tr>
<td>1-2</td>
<td>0.36</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
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<tr>
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<td>6-0</td>
<td>0.58</td>
</tr>
<tr>
<td>6-4</td>
<td>0.93</td>
</tr>
</tbody>
</table>

an edge-weighted graph
Greedy MST algorithm demo

- Start with all edges colored gray.
- Find cut with no black crossing edges; color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

**MST edges**

0–2  5–7  6–2  0–7  2–3  1–7  4–5
Proposition. The greedy algorithm computes the MST.

Pf.
- Any edge colored black is in the MST (via cut property).
- Fewer than \( V - 1 \) black edges \( \Rightarrow \) cut with no black crossing edges.
  (consider cut whose vertices are one connected component)
Proposition. The greedy algorithm computes the MST.

Efficient implementations. Choose cut? Find min-weight edge?

Ex 1. Kruskal's algorithm. [stay tuned]
Ex 2. Prim's algorithm. [stay tuned]
Ex 3. Borůvka's algorithm.
Removing two simplifying assumptions

Q. What if edge weights are not all distinct?
A. Greedy MST algorithm still correct if equal weights are present! (our correctness proof fails, but that can be fixed)

Q. What if graph is not connected?
A. Compute minimum spanning forest = MST of each component.

can independently compute MSTs of components
Greed is good

Gordon Gecko (Michael Douglas) address to Teldar Paper Stockholders in Wall Street (1986)
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Weighted edge API

Edge abstraction needed for weighted edges.

```java
public class Edge implements Comparable<Edge>

    Edge(int v, int w, double weight)  // create a weighted edge v-w
        int either()                    // either endpoint
        int other(int v)                // the endpoint that's not v
        int compareTo(Edge that)       // compare this edge to that edge
        double weight()                // the weight
        String toString()              // string representation

    v          weight          w
```

Idiom for processing an edge e: `int v = e.either(), w = e.other(v);`
public class Edge implements Comparable<Edge> {
    private final int v, w;
    private final double weight;

    public Edge(int v, int w, double weight) {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int either() {
        return v;
    }

    public int other(int vertex) {
        if (vertex == v) return w;
        else return v;
    }

    public int compareTo(Edge that) {
        if (this.weight < that.weight) return -1;
        else if (this.weight > that.weight) return +1;
        else return 0;
    }
}

Weighted edge: Java implementation

constructor

either endpoint

other endpoint

compare edges by weight
## Edge-weighted graph API

```java
public class EdgeWeightedGraph {
    EdgeWeightedGraph(int V) // create an empty graph with V vertices
    EdgeWeightedGraph(In in) // create a graph from input stream
    void addEdge(Edge e) // add weighted edge e to this graph
    Iterable<Edge> adj(int v) // edges incident to v
    Iterable<Edge> edges() // all edges in this graph
    int V() // number of vertices
    int E() // number of edges
    String toString() // string representation
}
```

**Conventions.** Allow self-loops and parallel edges.
Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of Edge lists.

tinyEWG.txt

```
8
16

4 5 0.35
4 7 0.37
5 7 0.28
0 7 0.16
1 5 0.32
0 4 0.38
2 3 0.17
1 7 0.19
0 2 0.26
1 2 0.36
1 3 0.29
2 7 0.34
6 2 0.40
3 6 0.52
6 0 0.58
6 4 0.93
```
Edge-weighted graph: adjacency-lists implementation

```java
public class EdgeWeightedGraph {
    private final int V;
    private final Bag<Edge>[] adj;

    public EdgeWeightedGraph(int V) {
        this.V = V;
        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Edge>();
    }

    public void addEdge(Edge e) {
        int v = e.either(), w = e.other(v);
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<Edge> adj(int v) {
        return adj[v];
    }
}
```

same as Graph, but adjacency lists of Edges instead of integers

collector

add edge to both adjacency lists
Q. How to represent the MST?

```
public class MST

MST(EdgeWeightedGraph G) // constructor

Iterable<Edge> edges() // edges in MST

double weight() // weight of MST
```

Minimum spanning tree API

tinyEWG.txt

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```
Minimum spanning tree API

Q. How to represent the MST?

```java
public class MST
{
    MST(EdgeWeightedGraph G) Constructor
    Iterable<Edge> edges edges in MST
    double weight() weight of MST
}
```

```java
public static void main(String[] args)
{
    In in = new In(args[0]);
    EdgeWeightedGraph G = new EdgeWeightedGraph(in);
    MST mst = new MST(G);
    for (Edge e : mst.edges())
        StdOut.println(e);
    StdOut.printf("%2f\n", mst.weight());
}
```

% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
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Kruskal's algorithm demo

Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.

graph edges sorted by weight

<table>
<thead>
<tr>
<th>Edge</th>
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<tbody>
<tr>
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Kruskal's algorithm demo

Consider edges in ascending order of weight.
  - Add next edge to tree $T$ unless doing so would create a cycle.

```
0-7  0.16
2-3  0.17
1-7  0.19
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1-5  0.32
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1-2  0.36
4-7  0.37
0-4  0.38
6-2  0.40
3-6  0.52
6-0  0.58
6-4  0.93
```
Kruskal's algorithm: visualization
**Kruskal's algorithm: correctness proof**

**Proposition.** [Kruskal 1956] Kruskal's algorithm computes the MST.

**Pf.** Kruskal's algorithm is a special case of the greedy MST algorithm.
- Suppose Kruskal's algorithm colors the edge $e = v - w$ black.
- Cut = set of vertices connected to $v$ in tree $T$.
- No crossing edge is black.
- No crossing edge has lower weight. Why?

![Diagram showing edge addition to tree](diagram.png)
Kruskal's algorithm: implementation challenge

Challenge. Would adding edge $v\rightarrow w$ to tree $T$ create a cycle? If not, add it.

How difficult?

- $E + V$
- $V$
- $\log V$
- $\log^* V$
- $1$

run DFS from $v$, check if $w$ is reachable
(T has at most $V - 1$ edges)

use the union-find data structure!
Kruskal's algorithm: implementation challenge

Challenge. Would adding edge $v$–$w$ to tree $T$ create a cycle? If not, add it.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in $T$.
- If $v$ and $w$ are in same set, then adding $v$–$w$ would create a cycle.
- To add $v$–$w$ to $T$, merge sets containing $v$ and $w$.

Case 1: adding $v$–$w$ creates a cycle

Case 2: add $v$–$w$ to $T$ and merge sets containing $v$ and $w$
Kruskal's algorithm: Java implementation

```java
public class KruskalMST {
    private Queue<Edge> mst = new Queue<Edge>();

    public KruskalMST(EdgeWeightedGraph G) {
        MinPQ<Edge> pq = new MinPQ<Edge>();
        for (Edge e : G.edges())
            pq.insert(e);
        UF uf = new UF(G.V());
        while (!pq.isEmpty() && mst.size() < G.V()-1) {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (!uf.connected(v, w)) {
                uf.union(v, w);
                mst.enqueue(e);
            }
        }
    }

    public Iterable<Edge> edges() {
        return mst;
    }
}
```

- **build priority queue**
- **greedily add edges to MST**
- **edge v–w does not create cycle**
- **merge sets**
- **add edge to MST**
Proposition. Kruskal's algorithm computes MST in time proportional to $E \log E$ (in the worst case).

Pf. 

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>time per op</th>
</tr>
</thead>
<tbody>
<tr>
<td>build pq</td>
<td>1</td>
<td>$E \log E$</td>
</tr>
<tr>
<td>delete-min</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
<tr>
<td>union</td>
<td>$V$</td>
<td>$\log^* V$</td>
</tr>
<tr>
<td>connected</td>
<td>$E$</td>
<td>$\log^* V$</td>
</tr>
</tbody>
</table>

† amortized bound using weighted quick union with path compression

recall: $\log^* V \leq 5$ in this universe

Remark. If edges are already sorted, order of growth is $E \log^* V$. 
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Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

![Edge-weighted graph](image)

an edge-weighted graph

<table>
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</table>
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

- 0-7
- 1-7
- 0-2
- 2-3
- 5-7
- 4-5
- 6-2
Prim’s algorithm: visualization
Prim's algorithm: proof of correctness

**Proposition.** [Jarník 1930, Dijkstra 1957, Prim 1959] Prim's algorithm computes the MST.

**Pf.** Prim's algorithm is a special case of the greedy MST algorithm.
  - Suppose edge $e = \min\ weight$ edge connecting a vertex on the tree to a vertex not on the tree.
  - Cut = set of vertices connected on tree.
  - No crossing edge is black.
  - No crossing edge has lower weight.

*edge $e = 7-5$ added to tree*
Prim's algorithm: implementation challenge

Challenge. Find the min weight edge with exactly one endpoint in $T$.

How difficult?

- $E$  
  try all edges
- $V$
- $\log E$  
  use a priority queue!
- $\log^* E$
- $1$

1-7 is min weight edge with exactly one endpoint in $T$
Prim's algorithm: lazy implementation

Challenge. Find the min weight edge with exactly one endpoint in \( T \).

Lazy solution. Maintain a PQ of edges with (at least) one endpoint in \( T \).
- Key = edge; priority = weight of edge.
- Delete-min to determine next edge \( e = v–w \) to add to \( T \).
- Disregard if both endpoints \( v \) and \( w \) are marked (both in \( T \)).
- Otherwise, let \( w \) be the unmarked vertex (not in \( T \)):
  - add to PQ any edge incident to \( w \) (assuming other endpoint not in \( T \))
  - add \( e \) to \( T \) and mark \( w \)

1-7 is min weight edge with exactly one endpoint in \( T \)

priority queue of crossing edges

1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58
Prim's algorithm (lazy) demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

an edge-weighted graph

<table>
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</tbody>
</table>
Prim's algorithm (lazy) demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2
Prim's algorithm: lazy implementation

```java
public class LazyPrimMST {
    private boolean[] marked; // MST vertices
    private Queue<Edge> mst; // MST edges
    private MinPQ<Edge> pq; // PQ of edges

    public LazyPrimMST(WeightedGraph G) {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);

        while (!pq.isEmpty() && mst.size() < G.V() - 1) {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (marked[v] && marked[w]) continue;
            mst.enqueue(e);
            if (!marked[v]) visit(G, v);
            if (!marked[w]) visit(G, w);
        }
    }
}
```

- Assume G is connected
- Repeatedly delete the min weight edge e = v–w from PQ
- Ignore if both endpoints in T
- Add edge e to tree
- Add v or w to tree
Prim's algorithm: lazy implementation

```java
private void visit(WeightedGraph G, int v) {
    marked[v] = true;
    for (Edge e : G.adj(v))
        if (!marked[e.other(v)])
            pq.insert(e);
}

public Iterable<Edge> mst() {
    return mst;
}
```

- add v to T
- for each edge e = v–w, add to PQ if w not already in T
Lazy Prim's algorithm: running time

**Proposition.** Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to $E$ (in the worst case).

**Pf.**

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>binary heap</th>
</tr>
</thead>
<tbody>
<tr>
<td>delete min</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
<tr>
<td>insert</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
</tbody>
</table>
**Prim's algorithm: eager implementation**

**Challenge.** Find min weight edge with exactly one endpoint in $T$.

**Eager solution.** Maintain a PQ of *vertices* connected by an edge to $T$, where priority of vertex $v$ = weight of shortest edge connecting $v$ to $T$.

- Delete min vertex $v$ and add its associated edge $e = v-w$ to $T$.
- Update PQ by considering all edges $e = v-x$ incident to $v$
  - ignore if $x$ is already in $T$
  - add $x$ to PQ if not already on it
  - **decrease priority** of $x$ if $v-x$ becomes shortest edge connecting $x$ to $T$
Prim's algorithm (eager) demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.
Prim's algorithm (eager) demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2
Indexed priority queue

Associate an index between 0 and \(N-1\) with each key in a priority queue.
- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

```
public class IndexMinPQ<Key extends Comparable<Key>> {

    IndexMinPQ(int N) {
        // create indexed priority queue with indices 0, 1, ..., N-1
    }

    void insert(int i, Key key) {
        // associate key with index i
    }

    void decreaseKey(int i, Key key) {
        // decrease the key associated with index i
    }

    boolean contains(int i) {
        // is i an index on the priority queue?
    }

    int delMin() {
        // remove a minimal key and return its associated index
    }

    boolean isEmpty() {
        // is the priority queue empty?
    }

    int size() {
        // number of entries in the priority queue
    }

```
Indexed priority queue implementation

**Implementation.**

- Start with same code as MinPQ.
- Maintain parallel arrays `keys[]`, `pq[]`, and `qp[]` so that:
  - `keys[i]` is the priority of `i`
  - `pq[i]` is the index of the key in heap position `i`
  - `qp[i]` is the heap position of the key with index `i`
- Use `swim(qp[i])` implement `decreaseKey(i, key)`. 

```
<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>keys[i]</td>
<td>A</td>
<td>S</td>
<td>0</td>
<td>R</td>
<td>T</td>
<td>I</td>
<td>N</td>
<td>G</td>
<td>-</td>
</tr>
<tr>
<td>pq[i]</td>
<td>-</td>
<td>0</td>
<td>6</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>qp[i]</td>
<td>1</td>
<td>5</td>
<td>4</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>-</td>
</tr>
</tbody>
</table>
```

```
  A
 / \  
 N   G
   / \ 
  S   I
   /  \ 
 R   T
   / 
 O   
```

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Prim's algorithm: running time

Depends on PQ implementation: $V$ insert, $V$ delete-min, $E$ decrease-key.

<table>
<thead>
<tr>
<th>PQ implementation</th>
<th>insert</th>
<th>delete-min</th>
<th>decrease-key</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>$1$</td>
<td>$V$</td>
<td>$1$</td>
<td>$V^2$</td>
</tr>
<tr>
<td>binary heap</td>
<td>$\log V$</td>
<td>$\log V$</td>
<td>$\log V$</td>
<td>$E \log V$</td>
</tr>
<tr>
<td>d–way heap (Johnson 1975)</td>
<td>$\log_d V$</td>
<td>$d \log_d V$</td>
<td>$\log_d V$</td>
<td>$E \log_{E/V} V$</td>
</tr>
<tr>
<td>Fibonacci heap (Fredman–Tarjan 1984)</td>
<td>$1^\dagger$</td>
<td>$\log V^\dagger$</td>
<td>$1^\dagger$</td>
<td>$E + V \log V$</td>
</tr>
</tbody>
</table>

$^\dagger$ amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.
4.3 Minimum Spanning Trees

- introduction
- greedy algorithm
- edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- context
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### Does a linear-time MST algorithm exist?

**Deterministic compare-based MST algorithms**

<table>
<thead>
<tr>
<th>Year</th>
<th>Worst Case</th>
<th>Discovered by</th>
</tr>
</thead>
<tbody>
<tr>
<td>1975</td>
<td>$E \log \log V$</td>
<td>Yao</td>
</tr>
<tr>
<td>1976</td>
<td>$E \log \log V$</td>
<td>Cheriton-Tarjan</td>
</tr>
<tr>
<td>1984</td>
<td>$E \log^* V, E + V \log V$</td>
<td>Fredman-Tarjan</td>
</tr>
<tr>
<td>1986</td>
<td>$E \log (\log^* V)$</td>
<td>Gabow-Galil-Spencer-Tarjan</td>
</tr>
<tr>
<td>1997</td>
<td>$E \alpha(V) \log \alpha(V)$</td>
<td>Chazelle</td>
</tr>
<tr>
<td>2000</td>
<td>$E \alpha(V)$</td>
<td>Chazelle</td>
</tr>
<tr>
<td>2002</td>
<td>Optimal</td>
<td>Pettie-Ramachandran</td>
</tr>
<tr>
<td>20xx</td>
<td>$E$</td>
<td>???</td>
</tr>
</tbody>
</table>

**Remark.** Linear-time randomized MST algorithm (Karger-Klein-Tarjan 1995).
Euclidean MST

Given $N$ points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.

**Brute force.** Compute $\sim N^2/2$ distances and run Prim's algorithm.

**Ingenuity.** Exploit geometry and do it in $\sim c N \log N$. 
Scientific application: clustering

**k-clustering.** Divide a set of objects classify into k coherent groups.

**Distance function.** Numeric value specifying "closeness" of two objects.

**Goal.** Divide into clusters so that objects in different clusters are far apart.

Applications.

- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster $10^9$ sky objects into stars, quasars, galaxies.

outbreak of cholera deaths in London in 1850s (Nina Mishra)
Single-link clustering

**k-clustering.** Divide a set of objects classify into $k$ coherent groups.  
**Distance function.** Numeric value specifying "closeness" of two objects.

**Single link.** Distance between two clusters equals the distance between the two closest objects (one in each cluster).

**Single-link clustering.** Given an integer $k$, find a $k$-clustering that maximizes the distance between two closest clusters.
Single-link clustering algorithm

“Well-known” algorithm in science literature for single-link clustering:

- Form $V$ clusters of one object each.
- Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
- Repeat until there are exactly $k$ clusters.

Observation. This is Kruskal's algorithm (stop when $k$ connected components).

Alternate solution. Run Prim's algorithm and delete $k-1$ max weight edges.
Dendrogram of cancers in human

Tumors in similar tissues cluster together.

Reference: Botstein & Brown group
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