Machine Learning Basics
Lecture 3: Perceptron

Princeton University COS 495

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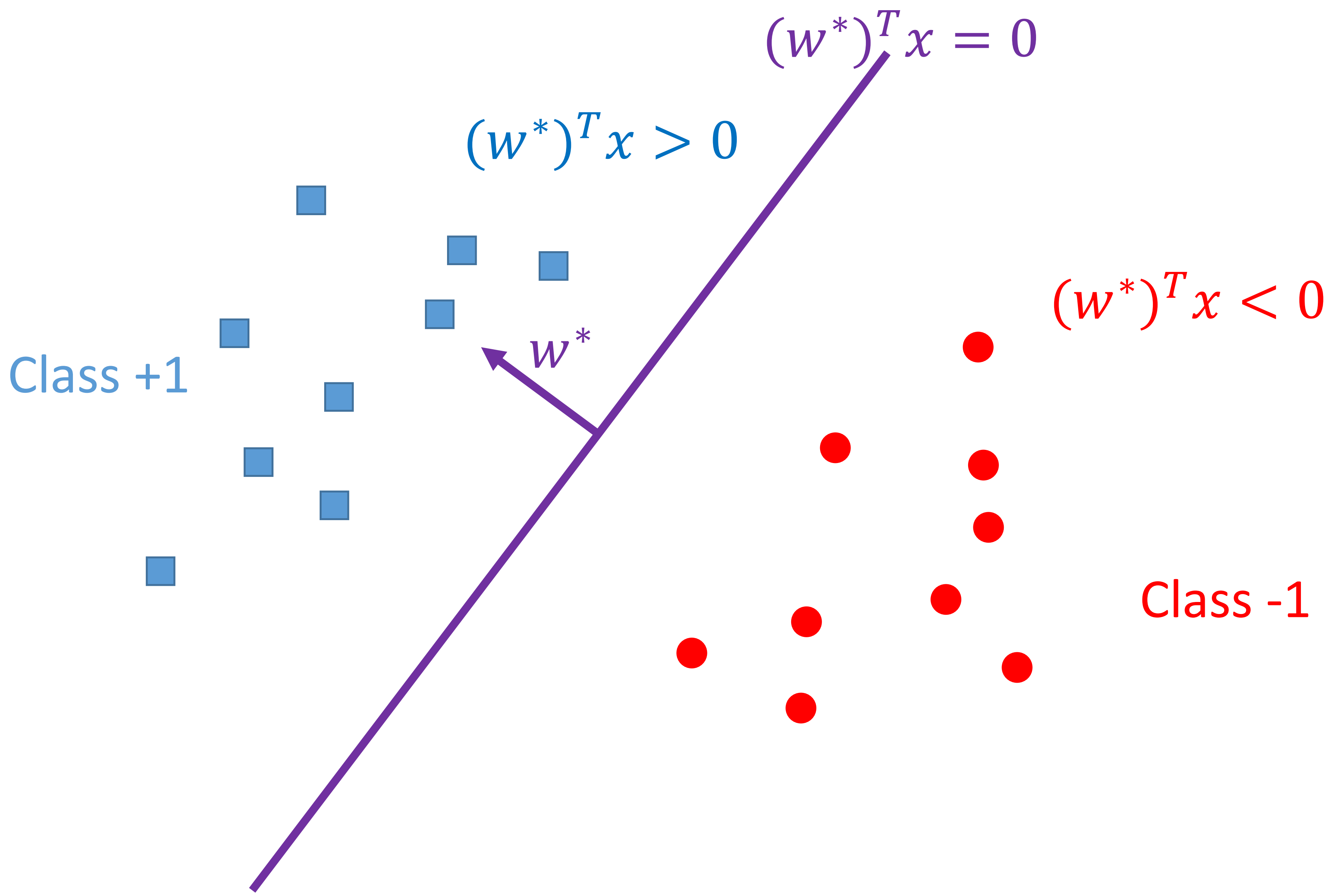
Perceptron

Overview

- Previous lectures: (Principle for loss function) MLE to derive loss
  - Example: linear regression; some linear classification models
- This lecture: (Principle for optimization) local improvement
  - Example: Perceptron; SGD

Task



Attempt

- Given training data $\{(x_i, y_i): 1 \leq i \leq n\}$ i.i.d. from distribution D
- Hypothesis $f_w(x) = w^T x$
    - $y = +1$ if $w^T x > 0$
    - $y = -1$ if $w^T x < 0$
- Prediction: $y = \text{sign}(f_w(x)) = \text{sign}(w^T x)$

- Goal: minimize classification error

Perceptron Algorithm

- Assume for simplicity: all x_i has length 1

1. Start with the all-zeroes weight vector $\mathbf{w}_1 = \mathbf{0}$, and initialize t to 1.
2. Given example $\mathbf{x}$, predict positive iff $\mathbf{w}_t \cdot \mathbf{x} > 0$.
3. On a mistake, update as follows:
   - Mistake on positive: $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \mathbf{x}$.
   - Mistake on negative: $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t - \mathbf{x}$.

   $t \leftarrow t + 1$.

Perceptron: figure from the lecture note of Nina Balcan

Intuition: correct the current mistake

- If mistake on a positive example

$$w_{t+1}^T x = (w_t + x)^T x = w_t^T x + x^T x = w_t^T x + 1$$

- If mistake on a negative example

$$w_{t+1}^T x = (w_t - x)^T x = w_t^T x - x^T x = w_t^T x - 1$$

The Perceptron Theorem

- Suppose there exists w^* that correctly classifies $\{(x_i, y_i)\}$
- W.L.O.G., all x_i and w^* have length 1, so the minimum distance of any example to the decision boundary is

$$\gamma = \min_i |(w^*)^T x_i|$$

- Then Perceptron makes at most $\left(\frac{1}{\gamma}\right)^2$ mistakes

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Need not be i.i.d. !

Do not depend on n, the length of the data sequence!

Analysis

- First look at the quantity $w_t^T w^*$

- Claim 1: $w_{t+1}^T w^* \geq w_t^T w^* + \gamma$
- Proof: If mistake on a positive example x

$$w_{t+1}^T w^* = (w_t + x)^T w^* = w_t^T w^* + x^T w^* \geq w_t^T w^* + \gamma$$

- If mistake on a negative example

$$w_{t+1}^T w^* = (w_t - x)^T w^* = w_t^T w^* - x^T w^* \geq w_t^T w^* + \gamma$$

Analysis

- Next look at the quantity $||w_t||$

Negative since we made a mistake on x

- Claim 2: $||w_{t+1}||^2 \leq ||w_t||^2 + 1$
- Proof: If mistake on a positive example x

$$||w_{t+1}||^2 = ||w_t + x||^2 = ||w_t||^2 + ||x||^2 + 2w_t^T x$$

Analysis: putting things together

- Claim 1: $w_{t+1}^T w^* \geq w_t^T w^* + \gamma$
- Claim 2: $||w_{t+1}||^2 \leq ||w_t||^2 + 1$

After M mistakes:

- $w_{M+1}^T w^* \geq \gamma M$
- $||w_{M+1}|| \leq \sqrt{M}$
- $w_{M+1}^T w^* \leq ||w_{M+1}||$

So $\gamma M \leq \sqrt{M}$, and thus $M \leq \left(\frac{1}{\gamma}\right)^2$

Intuition

- Claim 1: $w_{t+1}^T w^* \geq w_t^T w^* + \gamma$
- Claim 2: $||w_{t+1}||^2 \leq ||w_t||^2 + 1$

The correlation gets larger. Could be:

1. w_{t+1} gets closer to w^*
2. w_{t+1} gets much longer

Rules out the bad case "2. w_{t+1} gets much longer"

Some side notes on Perceptron

History

Frank Rosenblatt

1928–1969

Rosenblatt's perceptron played an important role in the history of machine learning. Initially, Rosenblatt simulated the perceptron on an IBM 704 computer at Cornell in 1957, but by the early 1960s he had built special-purpose hardware that provided a direct, parallel implementation of perceptron learning. Many of his ideas were encapsulated in "Principles of Neurodynamics: Perceptrons and the Theory of Brain Mechanisms" published in 1962. Rosenblatt's work was criticized by Marvin Minksy, whose objections were published in the book "Perceptrons", co-authored with Seymour Papert. This book was widely misinterpreted at the time as showing that neural networks were fatally flawed and could only learn solutions for linearly separable problems. In fact, it only proved such limitations in the case of single-layer networks such as the perceptron and merely conjectured (incorrectly) that they applied to more general network models. Unfortunately, however, this book contributed to the substantial decline in research funding for neural computing, a situation that was not reversed until the mid-1980s. Today, there are many hundreds, if not thousands, of applications of neural networks in widespread use, with examples in areas such as handwriting recognition and information retrieval being used routinely by millions of people.

Figure from *Pattern Recognition and Machine Learning*, Bishop

Note: connectionism vs symbolism

- Symbolism: AI can be achieved by representing concepts as symbols
  - Example: rule-based expert system, formal grammar

- Connectionism: explain intellectual abilities using connections between neurons (i.e., artificial neural networks)
  - Example: perceptron, larger scale neural networks

Symbolism example: Credit Risk Analysis

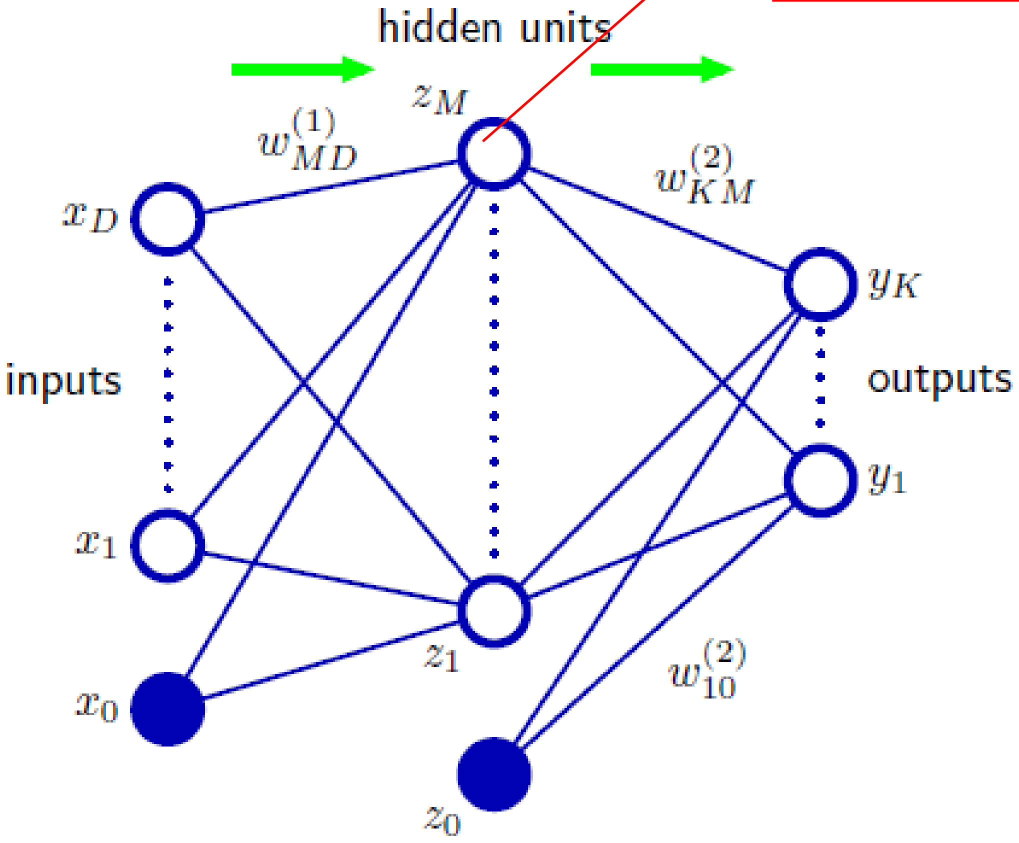
```
If   Other-Delinquent-Accounts > 2, and
     Number-Delinquent-Billing-Cycles > 1
Then Profitable-Customer? = No
     [Deny Credit Card application]

If   Other-Delinquent-Accounts = 0, and
     (Income > $30k)  OR  (Years-of-Credit > 3)
Then Profitable-Customer? = Yes
     [Accept Credit Card application]
```

Example from Machine learning lecture notes by Tom Mitchell

Connectionism example

Network diagram for the two-layer neural network corresponding to (5.7). The input, hidden, and output variables are represented by nodes, and the weight parameters are represented by links between the nodes, in which the bias parameters are denoted by links coming from additional input and hidden variables x_0 and z_0. Arrows denote the direction of information flow through the network during forward propagation.



Neuron/perceptron

Figure from *Pattern Recognition and machine learning*, Bishop

Note: connectionism v.s. symbolism

- Formal theories of logical reasoning, grammar, and other higher mental faculties compel us to think of the mind as a machine for rule-based manipulation of highly structured arrays of symbols. What we know of the brain compels us to think of human information processing in terms of manipulation of a large unstructured set of numbers, the activity levels of interconnected neurons.

---- The Central Paradox of Cognition (Smolensky et al., 1992)

Note: online vs batch

- Batch: Given training data $\{(x_i, y_i): 1 \leq i \leq n\}$, typically i.i.d.

- Online: data points arrive one by one
  - 1. The algorithm receives an unlabeled example x_i
  - 2. The algorithm predicts a classification of this example.
  - 3. The algorithm is then told the correct answer y_i, and update its model

Stochastic gradient descent (SGD)

Gradient descent

- Minimize loss $\hat{L}(\theta)$, where the hypothesis is parametrized by θ

- Gradient descent
  - Initialize θ_0
  - $\theta_{t+1} = \theta_t - \eta_t \nabla \hat{L}(\theta_t)$

Stochastic gradient descent (SGD)

- Suppose data points arrive one by one

- $\hat{L}(\theta) = \frac{1}{n}\sum_{t=1}^{n} l(\theta, x_t, y_t)$, but we only know $l(\theta, x_t, y_t)$ at time t

- Idea: simply do what you can based on local information
  - Initialize θ_0
  - $\theta_{t+1} = \theta_t - \eta_t \nabla l(\theta_t, x_t, y_t)$

Example 1: linear regression

- Find $f_w(x) = w^T x$ that minimizes $\hat{L}(f_w) = \frac{1}{n}\sum_{t=1}^{n}(w^T x_t - \ y_t)^2$
- $l(w, x_t, y_t) = \frac{1}{n}(w^T x_t - \ y_t)^2$

- $w_{t+1} = w_t - \eta_t \nabla l(w_t, x_t, y_t) = w_t - \frac{2\eta_t}{n}(w_t^T x_t - \ y_t)x_t$

Example 2: logistic regression

- Find w that minimizes

$$\hat{L}(w) = -\frac{1}{n}\sum_{y_t=1}\log\sigma(w^T x_t) - \frac{1}{n}\sum_{y_t=-1}\log[1-\sigma(w^T x_t)]$$

$$\hat{L}(w) = -\frac{1}{n}\sum_{t}\log\sigma(y_t w^T x_t)$$

$$l(w, x_t, y_t) = \frac{-1}{n}\log\sigma(y_t w^T x_t)$$

Example 2: logistic regression

- Find w that minimizes

$$l(w, x_t, y_t) = \frac{-1}{n} \log\sigma(y_t w^T x_t)$$

$$w_{t+1} = w_t - \eta_t \nabla l(w_t, x_t, y_t) = w_t + \frac{\eta_t}{n} \frac{\sigma(a)\big(1-\sigma(a)\big)}{\sigma(a)} y_t x_t$$

Where $a = y_t w_t^T x_t$

Example 3: Perceptron

- Hypothesis: $y = \text{sign}(w^T x)$
- Define hinge loss

$$l(w, x_t, y_t) = -y_t w^T x_t \ \mathbb{I}[\text{mistake on } x_t]$$

$$\hat{L}(w) = -\sum_t y_t w^T x_t \ \mathbb{I}[\text{mistake on } x_t]$$

$$w_{t+1} = w_t - \eta_t \nabla l(w_t, x_t, y_t) = w_t + \eta_t y_t x_t \ \mathbb{I}[\text{mistake on } x_t]$$

Example 3: Perceptron

- Hypothesis: $y = \mathrm{sign}(w^T x)$

$$w_{t+1} = w_t - \eta_t \nabla l(w_t, x_t, y_t) = w_t + \eta_t y_t x_t \, \mathbb{I}[\text{mistake on } x_t]$$

- Set $\eta_t = 1$. If mistake on a positive example

$$w_{t+1} = w_t + y_t x_t = w_t + x$$

- If mistake on a negative example

$$w_{t+1} = w_t + y_t x_t = w_t - x$$

Pros & Cons

Pros:

- Widely applicable
- Easy to implement in most cases
- Guarantees for many losses
- Good performance: error/running time/memory etc.

Cons:

- No guarantees for non-convex opt (e.g., those in deep learning)
- Hyper-parameters: initialization, learning rate

Mini-batch

- Instead of one data point, work with a small batch of b points

$$(x_{tb+1}, y_{tb+1}), \ldots, (x_{tb+b}, y_{tb+b})$$

- Update rule

$$\theta_{t+1} = \theta_t - \eta_t \nabla \left(\frac{1}{b} \sum_{1 \le i \le b} l(\theta_t, x_{tb+i}, y_{tb+i}) \right)$$

- Other variants: variance reduction etc.

Homework

Homework 1

- Assignment online
  - Course website: http://www.cs.princeton.edu/courses/archive/spring16/cos495/
  - Piazza: https://piazza.com/princeton/spring2016/cos495
- Due date: Feb 17th (one week)
- Submission
  - Math part: hand-written/print; submit to TA (Office: EE, C319B)
  - Coding part: in Matlab/Python; submit the .m/.py file on Piazza

Homework 1

- Grading policy: every late day reduces the attainable credit for the exercise by 10%.
- Collaboration:
  - Discussion on the problem sets is allowed
  - Students are expected to finish the homework by himself/herself
  - The people you discussed with on assignments should be clearly detailed: before the solution to each question, list all people that you discussed with on that particular question.