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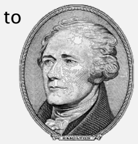
6.5 REDUCTIONS

- *introduction*
- *designing algorithms*
- *establishing lower bounds*
- *classifying problems*
- *intractability*

Last updated on 4/25/16 7:03 AM

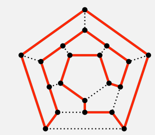
Exercise

Imagine that founding father Alexander Hamilton* has offered to find a **Hamiltonian cycle** in any given graph (if one exists).



Design an efficient algorithm to find a **Hamiltonian path** in a graph (if one exists) by making queries to Hamilton.

Solution. Given graph $G = (V, E)$:



- Query Hamilton with G.
  - If cycle found, remove last vertex and return rest.
- For every *nonexistent* edge $e \notin E$:
  - Query Hamilton with $(V, E \cup \{e\})$
  - If cycle found, "rotate" it so that final two vertices are incident on e; remove final vertex and return the rest.　The cycle *must* traverse e. Why?
- Return "no Hamiltonian path".　Why is this correct?

*Hamiltonian cycle/path are named after William Rowan Hamilton.

Reductions: overview

Main topics.

- Reduction: relationship between two problems.
- Algorithm design: paradigms for solving problems.

Shifting gears.

- From individual problems to problem-solving models.
- From linear/quadratic to polynomial/exponential scale.
- From implementation details to conceptual frameworks.

Goals.

- Place algorithms and techniques we've studied in a larger context.
- Introduce you to important and essential ideas.
- Inspire you to learn more about algorithms!

Reductions: practical tip

Reductions require ingenuity, but a few tricks recur. Practice them.

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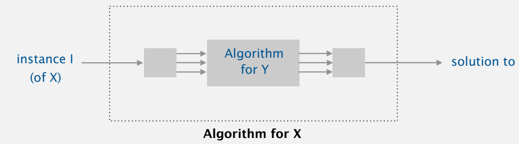
6.5 Reductions

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



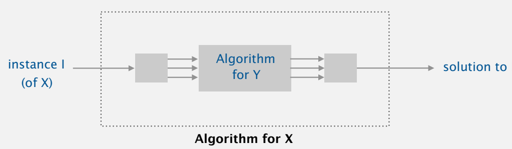
Cost of solving X = total cost of solving Y + cost of reduction.

perhaps many calls to Y on problems of different sizes (typically only 1 call)

preprocessing and postprocessing (typically less than cost of solving Y)

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 1. [finding the median reduces to sorting]

To find the median of N items:

- Sort the N items.
- Return item in the middle.

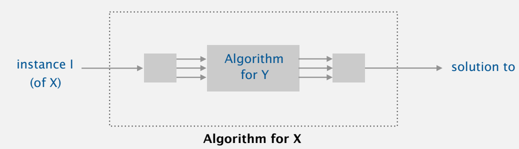
cost of sorting

cost of reduction

Cost of finding the median. $N \log N + 1$.

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 2. [element distinctness reduces to sorting]

To solve element distinctness on N items:

- Sort the N items.
- Check adjacent pairs for equality.

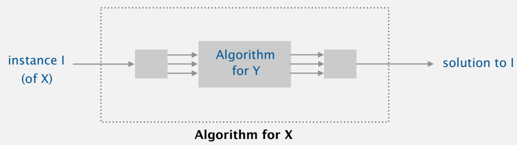
cost of sorting

cost of reduction

Cost of element distinctness. $N \log N + N$.

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Confusing terminology. CS professors often slip up.

~~Novice error.~~ Confusing *X reduces to Y* with *Y reduces to X*.

Reductions: quiz 1

Which of the following reductions have we encountered in this course?

I. MAX-FLOW reduces to MIN-CUT.
II. MIN-CUT reduces to MAX-FLOW.

need to find max st-flow and min st-cut
(not simply compute the value)

A. I only.

B. II only.

C. Both I and II.

D. Neither I nor II.

E. *I don't know.*

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6.5 REDUCTIONS

Reduction: design algorithms

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.

Design algorithm. Given an algorithm for Y, can also solve X.

More familiar reductions.

- Mincut reduces to maxflow.
- Arbitrage reduces to negative cycles.
- Bipartite matching reduces to maxflow.
- Seam carving reduces to shortest paths in a DAG.
- Burrows-Wheeler transform reduces to suffix sort.

...

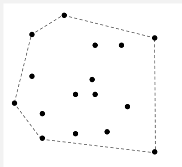
Reasoning. Since I know how to solve Y, can I use that algorithm to solve X?

programmer's version: I have code for Y. Can I use it for X?

Convex hull reduces to sorting

Sorting. Given N distinct integers, rearrange them in ascending order.

Convex hull. Given N points in the plane, identify the extreme points of the convex hull (in counterclockwise order).



convex hull

1251432
2861534
3988818
8111033
13546464
89885444
43434213
34435312

sorting

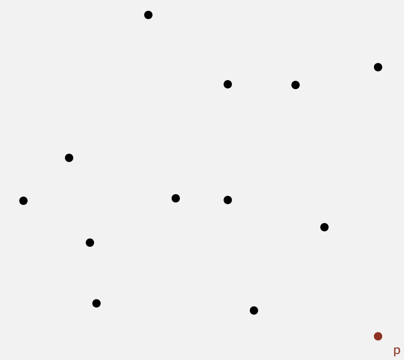
Proposition. Convex hull reduces to sorting.
Pf. Graham scan algorithm.

cost of sorting
cost of reduction

Cost of convex hull. $N \log N \, + \, N$.

Graham scan demo

- Choose point p with smallest y-coordinate.
- Sort points by polar angle with p.
- Consider points in order; discard those that create clockwise turn.



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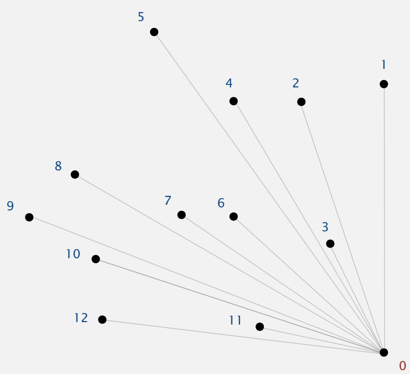

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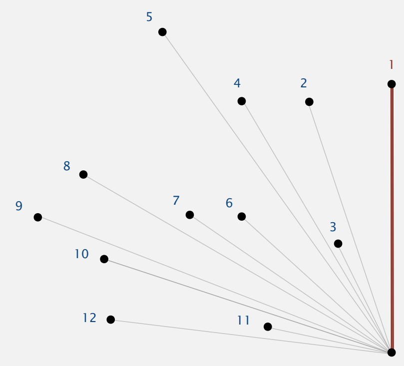

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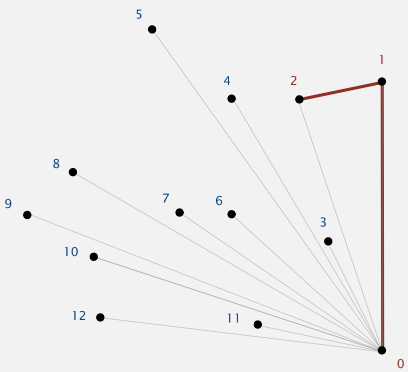

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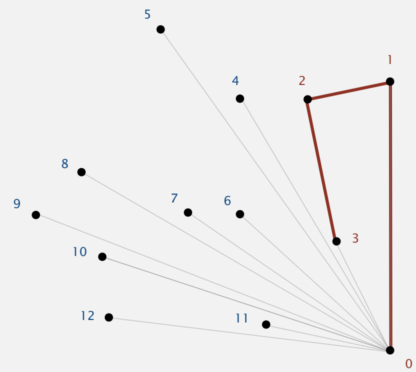

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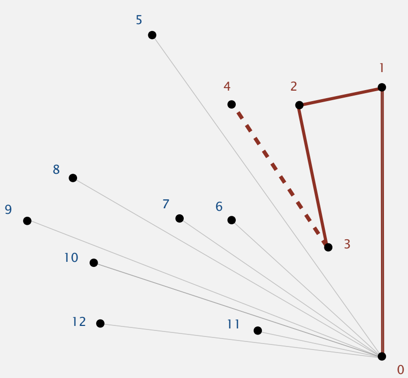

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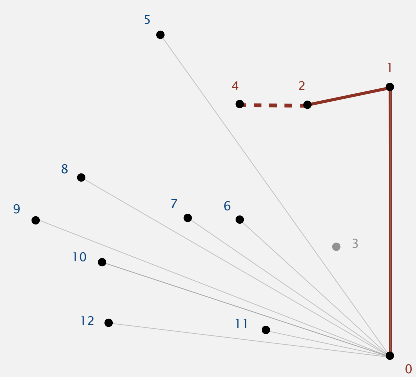

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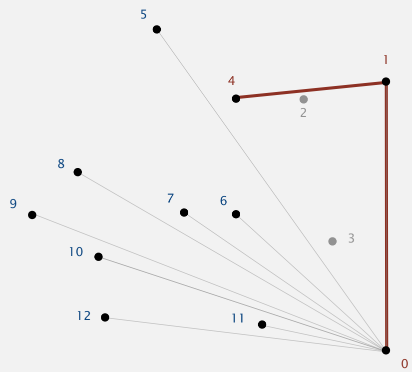

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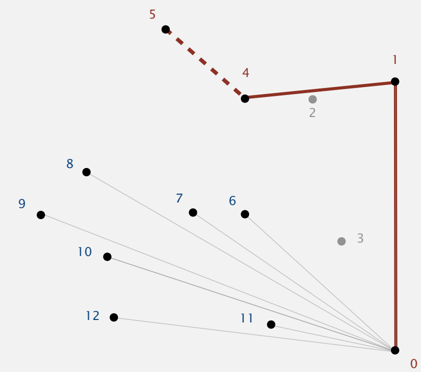

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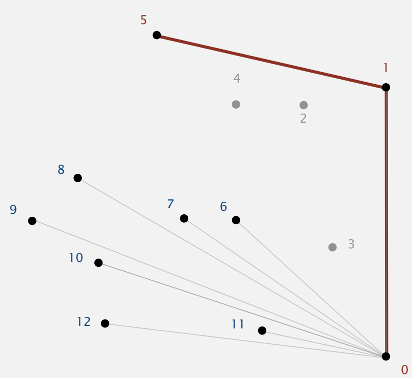

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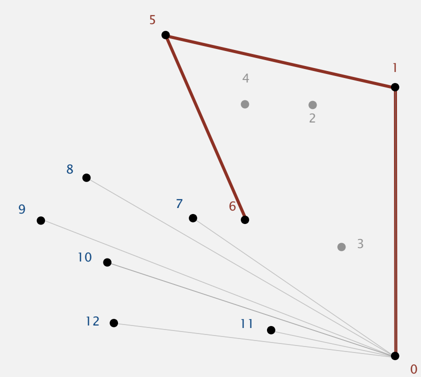

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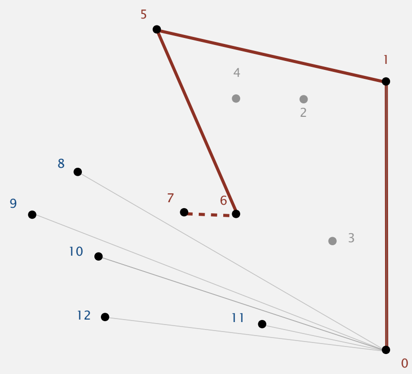

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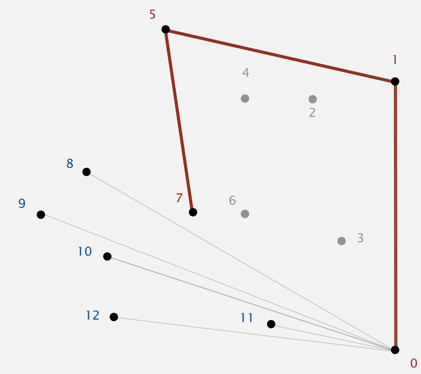

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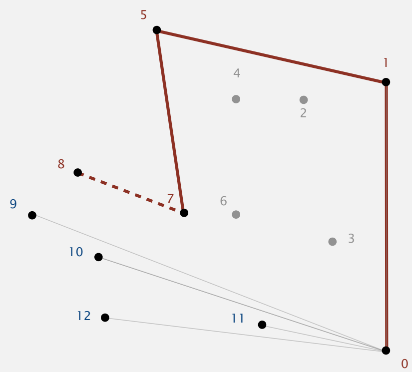

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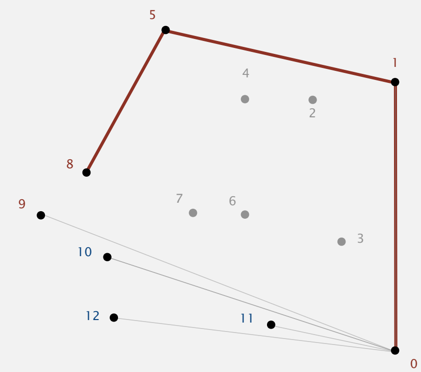

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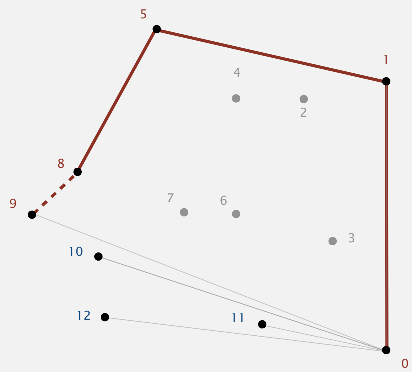

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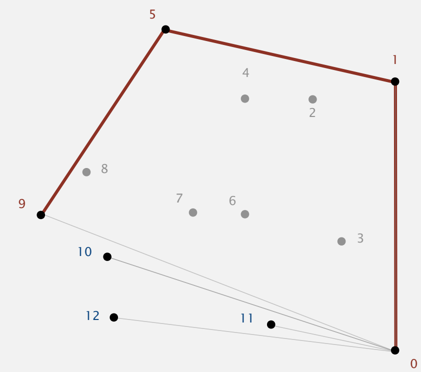

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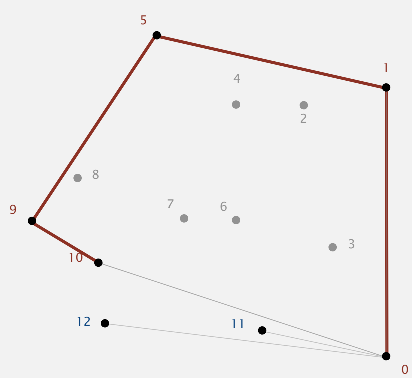

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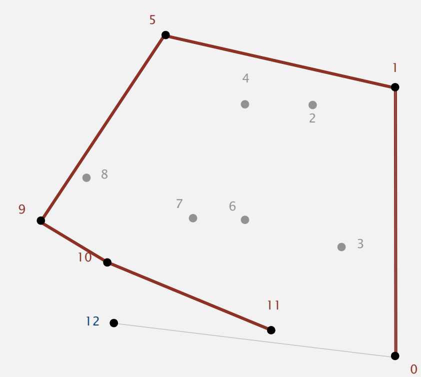

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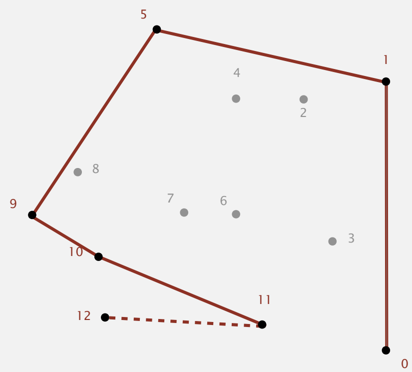

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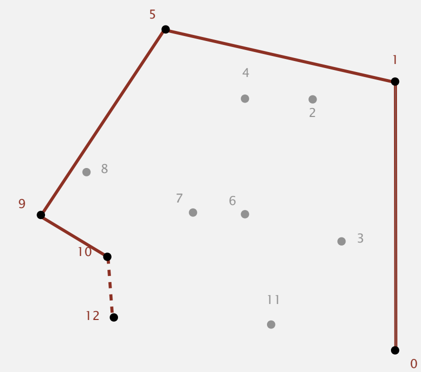

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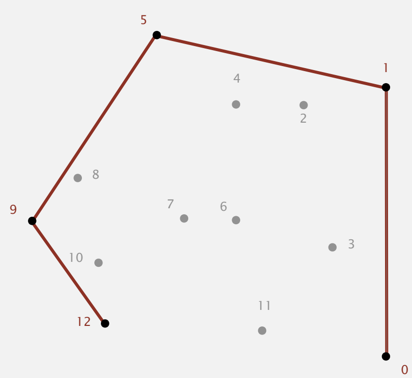

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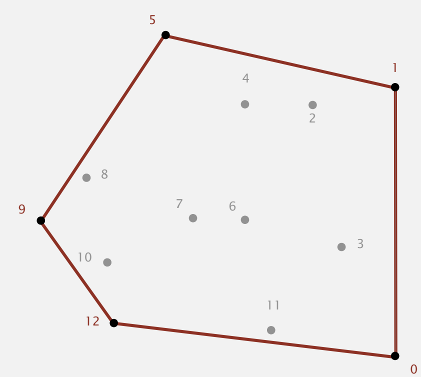
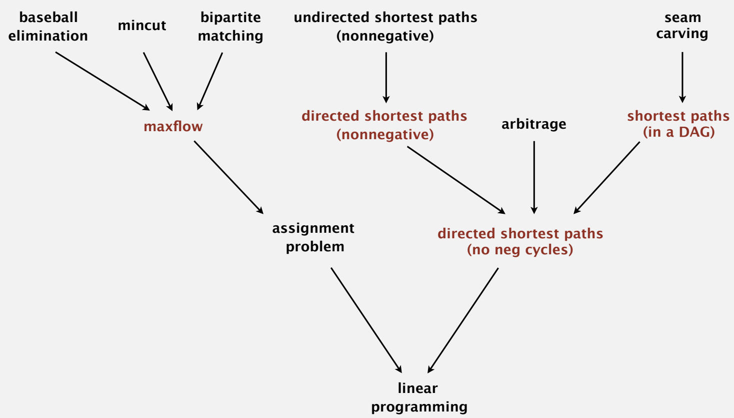


Some reductions in combinatorial optimization



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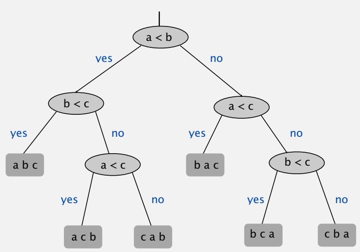
6.5 Reductions

- introduction
- designing algorithms
- *establishing lower bounds*
- classifying problems
- intractability

Bird's-eye view

Goal. Prove that a problem requires a certain number of steps.
Ex. In decision tree model, any compare-based sorting algorithm requires $\Omega(N \log N)$ compares in the worst case.



argument must apply to all conceivable algorithms

Bad news. Very difficult to establish lower bounds from scratch.
Good news. Spread $\Omega(N \log N)$ lower bound to Y by reducing sorting to Y.

assuming cost of reduction is not too high

Linear-time reductions

Def. Problem X linear-time reduces to problem Y if X can be solved with:
- Linear number of standard computational steps.
- Constant number of calls to Y.

Establish lower bound:
- If X takes $\Omega(N \log N)$ steps, then so does Y.
- If X takes $\Omega(N^2)$ steps, then so does Y.
- Intuition: X = known problem; Y = new problem.

Reasoning.
- If I could easily solve Y, then I could easily solve X.
- I can't easily solve X.
- Therefore, I can't easily solve Y.

Reductions: quiz 2

Which of the following reductions is not a linear-time reduction?

A. ELEMENT-DISTINCTNESS reduces to SORTING.

B. MIN-CUT reduces to MAX-FLOW.

C. HAMILTONIAN-PATH reduces to HAMILTONIAN-CYCLE.

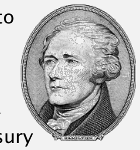
not the one we saw earlier, anyway

D. BURROWS-WHEELER-TRANSFORM reduces to SUFFIX-SORTING.

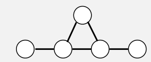
E. *I don't know.*

Exercise: linear-time reduction

Imagine that founding father Alexander Hamilton has offered to find a **Hamiltonian cycle** in any given graph (if one exists).



Design an efficient algorithm to find a **Hamiltonian path** in a graph (if one exists) by making queries to Hamilton. The Treasury Secretary's time is valuable, so you must minimize the number of queries.



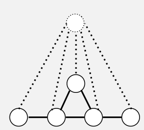
Exercise: linear-time reduction

Imagine that founding father Alexander Hamilton has offered to find a **Hamiltonian cycle** in any given graph (if one exists).

Design an efficient algorithm to find a **Hamiltonian path** in a graph (if one exists) by making queries to Hamilton. The Treasury Secretary's time is valuable, so you must minimize the number of queries.

Solution. Given graph $G = (V, E)$:

- Add a virtual vertex v and connect it to all vertices.
- Query Hamilton with the resulting graph:



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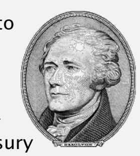
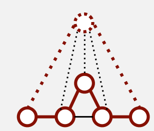


Design an efficient algorithm to find a **Hamiltonian path** in a graph (if one exists) by making queries to Hamilton. The Treasury Secretary's time is valuable, so you must minimize the number of queries.

Solution. Given graph $G = (V, E)$:

- Add a virtual vertex v and connect it to all vertices.
- Query Hamilton with the resulting graph:
  - If cycle found, rotate so that v is first/last vertex; remove v and return the rest.
  - Else return "no Hamiltonian path".

Why is this correct?



Lower bound for convex hull

Proposition. Sorting linear-time reduces to convex hull.
Pf. [see next slide]

lower-bound reasoning: I can't sort in linear time, so I can't solve convex hull in linear time either

1251432
2861534
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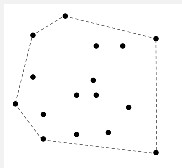


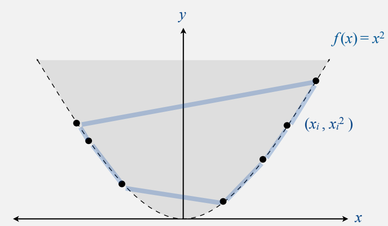
convex hull

Implication. Any convex hull algorithm requires $\Omega(N \log N)$ ops.

Sorting linear-time reduces to convex hull

Proposition. Sorting linear-time reduces to convex hull.

- Sorting instance: $x_1, x_2, \ldots, x_N$.
- Convex hull instance: $(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_N, x_N^2)$.



Pf.

- Region $\{ (x, y) : y \geq x^2 \}$ is convex $\Rightarrow$ all N points are on hull.
- Starting at point with most negative x, counterclockwise order of hull points yields integers in ascending order.

Establishing lower bounds: summary

Establishing lower bounds through reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself no linear-time CONVEX-HULL algorithm exists?
A1. [hard way] Long futile search for a linear-time algorithm.
A2. [easy way] Linear-time reduction from sorting.

Reductions: quiz 3

Our lower bound proof strategy for CONVEX-HULL would work even if:

A. Our reduction invoked CONVEX-HULL $\Theta(\log N)$ times instead of once.

B. Our pre-/post-processing was linearithmic instead of linear.

C. Both **A.** and **B.**

D. Neither **A.** nor **B.**

E. *I don't know.*

Cost of solving SORTING = total cost of CONVEX-HULL + cost of reduction.

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6.5 Reductions

Bird's-eye view

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	*min, max, median, Burrows-Wheeler transform, ...*
linearithmic	$N \log N$	*sorting, element distinctness, closest pair, Euclidean MST, ...*
quadratic	N^2	?
⋮	⋮	⋮
exponential	c^N	?

Frustrating news. Huge number of problems have defied classification.

Bird's-eye view

Desiderata. Classify problems according to computational requirements.

Desiderata'. Suppose we could (could not) solve problem X efficiently. What else could (could not) we solve efficiently?

" Give me a lever long enough and a fulcrum on which to place it, and I shall move the world. " — *Archimedes*

Classifying problems: summary

Desiderata. Problem with algorithm that matches lower bound.
Ex. Sorting and element distinctness have complexity $N \log N$.

Desiderata'. Prove that two problems X and Y have the same complexity.

- First, show that problem X linear-time reduces to Y.
- Second, show that Y linear-time reduces to X.
- Conclude that X has complexity $T(N)$ iff Y has complexity $T(N)$.

even if we don't know what it is

X = sorting ⇅ **Y = element distinctness**

integer multiplication ⇅ **integer division**

Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.
Brute force. N^2 bit operations.

```
                1 1 0 1 0 1 0 1
              × 0 1 1 1 1 1 0 1
                ---------------
                1 1 0 1 0 1 0 1
              0 0 0 0 0 0 0 0
            1 1 0 1 0 1 0 1
          1 1 0 1 0 1 0 1
        1 1 0 1 0 1 0 1
      1 1 0 1 0 1 0 1
    1 1 0 1 0 1 0 1
  0 0 0 0 0 0 0 0
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0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1
```


Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.
Brute force. N^2 bit operations.

problem	arithmetic	order of growth
integer multiplication	$a \times b$	$M(N)$
integer division	$a / b,\ a \bmod b$	$M(N)$
integer square	a^2	$M(N)$
integer square root	$\lfloor \sqrt{a} \rfloor$	$M(N)$

integer arithmetic problems with the same complexity as integer multiplication

Q. Is brute-force algorithm optimal?

History of complexity of integer multiplication

year	algorithm	order of growth
?	**brute force**	N^2
1962	**Karatsuba**	$N^{1.585}$
1963	**Toom-3, Toom-4**	$N^{1.465}$, $N^{1.404}$
1966	**Toom-Cook**	$N^{1+\varepsilon}$
1971	**Schönhage–Strassen**	$N \log N \log \log N$
2007	**Fürer**	$N \log N\, 2^{\log^* N}$
?	?	N

number of bit operations to multiply two N-bit integers

used in Maple, Mathematica, gcc, cryptography, ...

Remark. GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.

GMP
«Arithmetic without limitations»

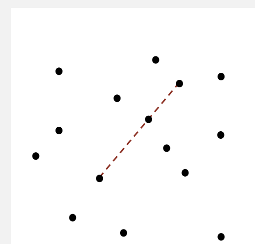
Lower bound for 3-Collinear

3-Sum. Given N distinct integers, are there three that sum to 0?

3-Collinear. Given N distinct points in the plane, are there 3 (or more) that lie on the same line?

```
   590584
-23439854
  1251432
 -2861534
  3988818
 -4190745
   333255
 13546464
 89885444
-43434213
 11998833
```

3-sum



3-collinear

Lower bound for 3-COLLINEAR

3-SUM. Given N distinct integers, are there three that sum to 0 ?

3-COLLINEAR. Given N distinct points in the plane, are there 3 (or more) that lie on the same line?

Proposition. 3-SUM linear-time reduces to 3-COLLINEAR.

Pf. [next two slides]

lower-bound reasoning:
if I can't solve 3-SUM in $N^{1.99}$ time,
I can't solve 3-COLLINEAR
in $N^{1.99}$ time either

Conjecture. No sub-quadratic algorithm for 3-SUM.

Implication. No sub-quadratic algorithm for 3-COLLINEAR likely.

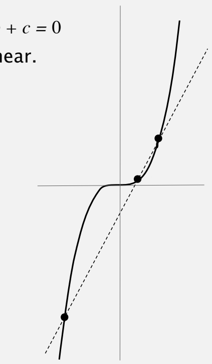
our $N^2 \log N$ algorithm was pretty good

3-SUM linear-time reduces to 3-COLLINEAR

Reduction. 3-SUM linear-time reduces to 3-COLLINEAR.

- 3-SUM instance: $x_1, x_2, \ldots, x_N$.
- 3-COLLINEAR instance: $(x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_N, f(x_N))$.

We hope to prove: If $a, b,$ and c are distinct, then $a + b + c = 0$ if and only if $(a_1, f(a_1)), (b_2, f(b_2)), \ldots, (c_N, f(c_N))$ are collinear.



3-SUM linear-time reduces to 3-COLLINEAR

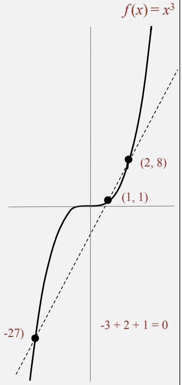
Proposition. 3-SUM linear-time reduces to 3-COLLINEAR.

- 3-SUM instance: $x_1, x_2, \ldots, x_N$.
- 3-COLLINEAR instance: $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$.

Lemma. If $a, b,$ and c are distinct, then $a + b + c = 0$ if and only if $(a, a^3), (b, b^3),$ and (c, c^3) are collinear.

Pf. Three distinct points $(a, a^3), (b, b^3), (c, c^3)$ are collinear iff:

$$\begin{aligned} 0 &= \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} \\ &= a(b^3 - c^3) - b(a^3 - c^3) + c(a^3 - b^3) \\ &= (a-b)(b-c)(c-a)(a+b+c) \end{aligned}$$



Complexity of 3-SUM

Some recent (2014) evidence that the complexity might be $N^{3/2}$.

Threesomes, Degenerates, and Love Triangles*

Allan Grønlund
MADALGO, Aarhus University

Seth Pettie
University of Michigan

April 4, 2014

Abstract

The 3SUM problem is to decide, given a set of n real numbers, whether any three sum to zero. We prove that the decision tree complexity of 3SUM is $O(n^{3/2}\sqrt{\log n})$, that there is a randomized 3SUM algorithm running in $O(n^2(\log\log n)^2/\log n)$ time, and a deterministic algorithm running in $O(n^2(\log\log n)^{5/3}/(\log n)^{2/3})$ time. These results refute the strongest version of the 3SUM conjecture, namely that its decision tree (and algorithmic) complexity is $\Omega(n^2)$.

Reductions: summary

Reduction: relationship between two problems.

How to apply:

- Reduction to solved problem: paradigm for designing algorithms.
- Reduction **from** solved problem: technique for proving lower bounds.
- Putting the two together: classify **problems** into complexity classes.
  - Especially useful for proving NP-completeness.

Reductions require ingenuity, but a few tricks recur.

http://algs4.cs.princeton.edu

6.5 Reductions

Bird's-eye view

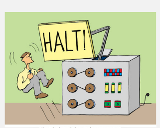
Def. A problem is intractable if it can't be solved in polynomial time.
Desiderata. Prove that a problem is intractable.

input size = c + lg K

Two problems that provably require exponential time.

- Given a constant-size program, does it halt in at most K steps?
- Given N-by-N checkers board position, can the first player force a win?

using forced capture rule



Frustrating news. Very few successes.

A core problem: satisfiability

SAT. Given a system of boolean equations, find a solution.

Ex.

$\neg x_1$	*or*	x_2	*or*	x_3			=	*true*
x_1	*or*	$\neg x_2$	*or*	x_3			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*	$\neg x_3$			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*		*or*	x_4	=	*true*
		$\neg x_2$	*or*	x_3	*or*	x_4	=	*true*

instance I

x_1	x_2	x_3	x_4
T	T	F	T

solution S

3-SAT. All equations of this form (with three variables per equation).

Key applications.

- Automatic verification systems for software.
- Mean field diluted spin glass model in physics.
- Electronic design automation (EDA) for hardware.
- ...

Satisfiability is conjectured to be intractable

Q. How to solve an instance of 3-SAT with N variables?
A. Exhaustive search: try all 2^N truth assignments.

Q. Can we do anything substantially more clever?

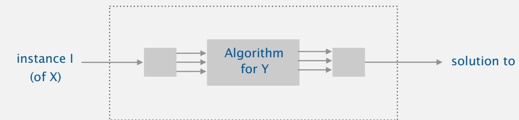
Conjecture (**P** $\neq$ **NP**). 3-SAT is intractable (no poly-time algorithm).

consensus opinion

Polynomial-time reductions

Problem X poly-time (Cook) reduces to problem Y if X can be solved with:

- Polynomial number of standard computational steps.
- Polynomial number of calls to Y.



Algorithm for X

Establish intractability. If 3-SAT poly-time reduces to Y, then Y is intractable. (assuming 3-SAT is intractable)

Reasoning.

- If I could solve Y in poly-time, then I could also solve 3-SAT in poly-time.
- 3-SAT is believed to be intractable.
- Therefore, so is Y.

Integer linear programming

ILP. Given a system of linear inequalities, find an integral solution.

$$3x_1 + 5x_2 + 2x_3 + x_4 + 4x_5 \geq 10$$
$$5x_1 + 2x_2 + 4x_4 + 1x_5 \leq 7$$
$$x_1 + x_3 + 2x_4 \leq 2$$
$$3x_1 + 4x_3 + 7x_4 \leq 7$$
$$x_1 + x_4 \leq 1$$
$$x_1 + x_3 + x_5 \leq 1$$
$$\text{all } x_i = \{ 0, 1 \}$$

linear inequalities

integer variables

instance I

x_1	x_2	x_3	x_4	x_5
0	1	0	1	1

solution S

Context. Cornerstone problem in operations research.
Remark. Finding a real-valued solution is tractable (linear programming).

3-SAT poly-time reduces to ILP

3-SAT. Given a system of boolean equations, find a solution.

$\neg x_1$	*or*	x_2	*or*	x_3			=	*true*
x_1	*or*	$\neg x_2$	*or*	x_3			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*	$\neg x_3$			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*		*or*	x_4	=	*true*
		$\neg x_2$	*or*	x_3	*or*	x_4	=	*true*

ILP. Given a system of linear inequalities, find a 0-1 solution.

$(1 - x_1)$	+	x_2	+	x_3			$\geq$	1
x_1	+	$(1 - x_2)$	+	x_3			$\geq$	1
$(1 - x_1)$	+	$(1 - x_2)$	+	$(1 - x_3)$			$\geq$	1
$(1 - x_1)$	+	$(1 - x_2)$	+		+	x_4	$\geq$	1
		$(1 - x_2)$	+	x_3	+	x_4	$\geq$	1

solution to this ILP instance gives solution to original 3-SAT instance

Reductions: quiz 3

Suppose that Problem X poly-time reduces to Problem Y. Which of the following can you infer?

A. If X can be solved in poly-time, then so can Y.

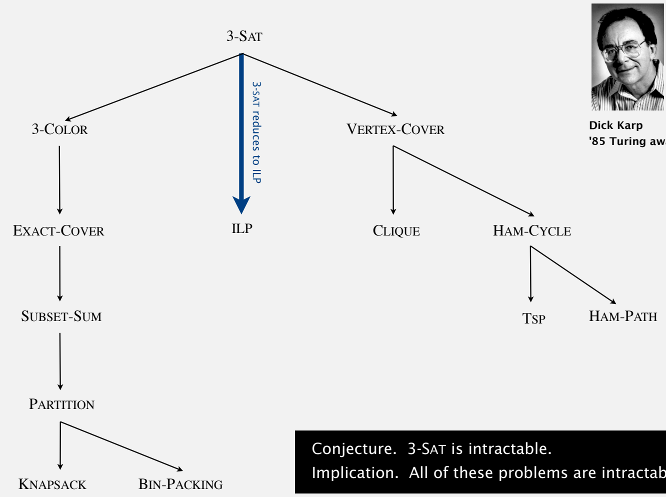
B. If X cannot be solved in cubic time, Y cannot be solved in poly-time.

C. If Y can be solved in cubic time, then X can be solved in poly-time.

D. If Y cannot be solved in poly-time, then neither can X.

E. *I don't know.*

More poly-time reductions from 3-satisfiability



Dick Karp
'85 Turing award

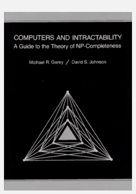
Conjecture. 3-SAT is intractable.
Implication. All of these problems are intractable.

Implications of poly-time reductions from 3-satisfiability

Establishing intractability through poly-time reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself that a new problem is (probably) intractable?
A1. [hard way] Long futile search for an efficient algorithm (as for 3-SAT).
A2. [easy way] Reduction from 3-SAT.

Caveat. Intricate reductions are common.



Search problems

Search problem. Problem where you can check a solution in poly-time.

Ex 1. 3-SAT.

$$\begin{array}{ccccccccc}
\neg x_1 & or & x_2 & or & x_3 & & & = & true \\
x_1 & or & \neg x_2 & or & x_3 & & & = & true \\
\neg x_1 & or & \neg x_2 & or & \neg x_3 & & & = & true \\
\neg x_1 & or & \neg x_2 & or & & or & x_4 & = & true \\
 & & \neg x_2 & or & x_3 & or & x_4 & = & true
\end{array}$$

instance I

x_1	x_2	x_3	x_4
T	T	F	T

solution S

Ex 2. FACTOR. Given an N-bit integer x, find a nontrivial factor.

147573952589676412927

instance I

193707721

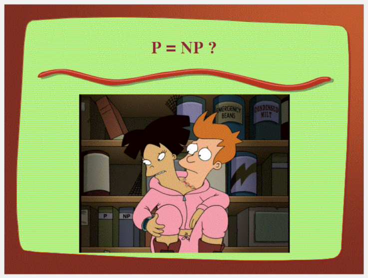
solution S

P vs. NP

P. Set of search problems solvable in poly-time.
Importance. What scientists and engineers can compute feasibly.

NP. Set of search problems (checkable in poly-time).
Importance. What scientists and engineers aspire to compute feasibly.

Fundamental question.



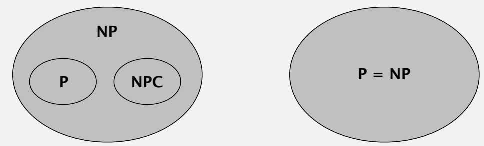
Consensus opinion. No.

Cook-Levin theorem

A problem is **NP-COMPLETE** if
- It is in **NP**.
- All problems in **NP** poly-time to reduce to it.

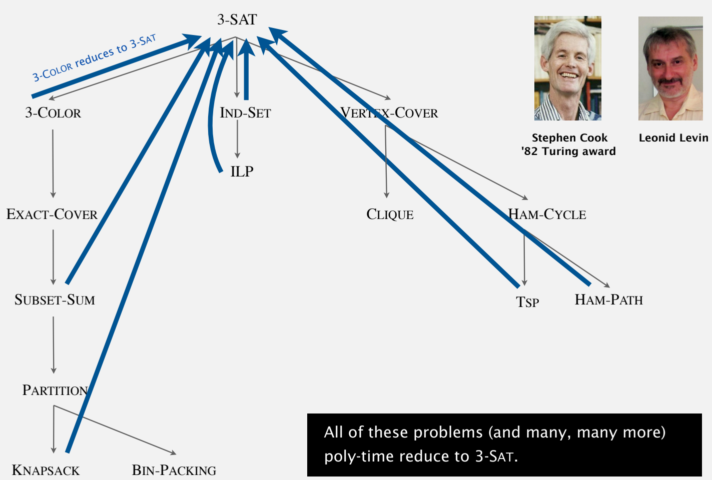
Cook-Levin theorem. 3-SAT is **NP-COMPLETE**.
Corollary. 3-SAT is tractable if and only if **P = NP**.

Two worlds.



P ≠ NP   P = NP

Implications of Cook-Levin theorem

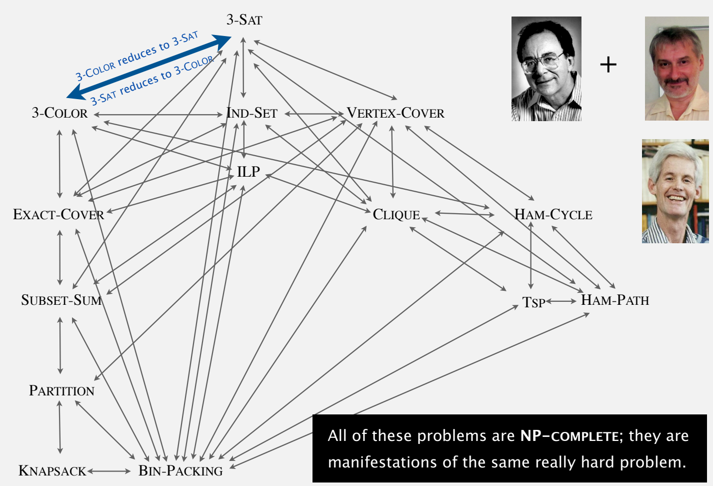


Stephen Cook
'82 Turing award

Leonid Levin

All of these problems (and many, many more) poly-time reduce to 3-SAT.

Implications of Karp + Cook-Levin



+

All of these problems are **NP-COMPLETE**; they are manifestations of the same really hard problem.

Reductions: quiz 4

Suppose that X is **NP-Complete**, Y is in **NP**, and X poly-time reduces to Y. Which of the following statements can you infer?

I. Y is NP-Complete.
II. If Y cannot be solved in poly-time, then P ≠ NP.
III. If P ≠ NP, then neither X nor Y can be solved in poly-time.

A. I only.

B. II only.

C. I and II only.

D. I, II, and III.

E. *I don't know.*

Birds-eye view: review

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	*min, max, median, Burrows-Wheeler transform, ...*
linearithmic	$N \log N$	*sorting, element distinctness, ...*
quadratic	N^2	?
⋮	⋮	⋮
exponential	c^N	?

Frustrating news. Huge number of problems have defied classification.

Birds-eye view: revised

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	*min, max, median, Burrows-Wheeler transform, ...*
linearithmic	$N \log N$	*sorting, element distinctness, ...*
M(N)	?	*integer multiplication, division, square root, ...*
MM(N)	?	*matrix multiplication, Ax = b, least square, determinant, ...*
⋮	⋮	⋮
NP-complete	*probably not* N^b	3-Sat, Ind-Set, ILP, ...

Good news. Can put many problems into equivalence classes.

Complexity zoo

Complexity class. Set of problems sharing some computational property.

https://complexityzoo.uwaterloo.ca

Bad news. Lots of complexity classes (496 animals in zoo).

Summary

Reductions are important in theory to:

- Design algorithms.
- Establish lower bounds.
- Classify problems according to their computational requirements.

Reductions are important in practice to:

- Design algorithms.
- Design reusable software modules.
  - stacks, queues, priority queues, symbol tables, sets, graphs
  - sorting, regular expressions, suffix arrays
  - MST, shortest paths, maxflow, linear programming
- Determine difficulty of your problem and choose the right tool.