http://algs4.cs.princeton.edu

4.1 Undirected Graphs

- *introduction*
- *graph API*
- *depth-first search*
- *breadth-first search*
- *challenges*
- *flipped lecture experiment*

Last updated on 3/21/16 6:14 PM

http://algs4.cs.princeton.edu

4.1 Undirected Graphs

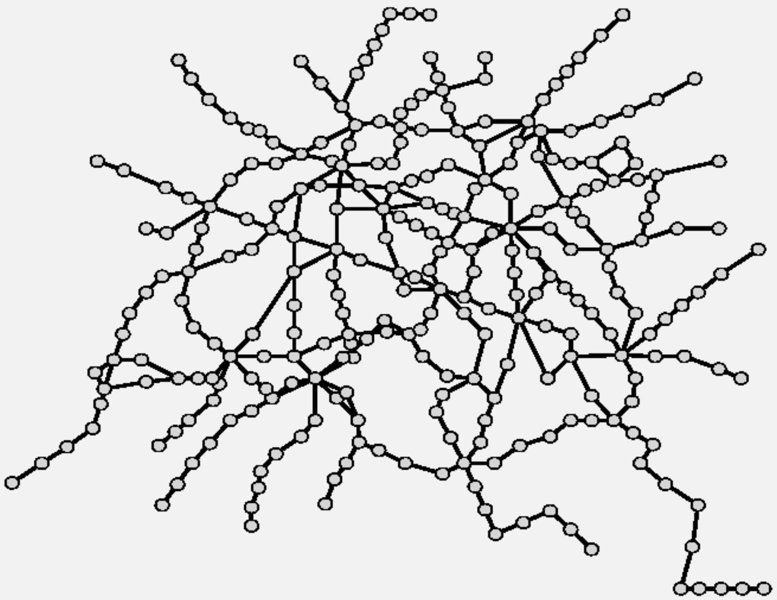
- *introduction*
- *graph API*
- *depth-first search*
- *breadth-first search*
- *challenges*
- *flipped lecture experiment*

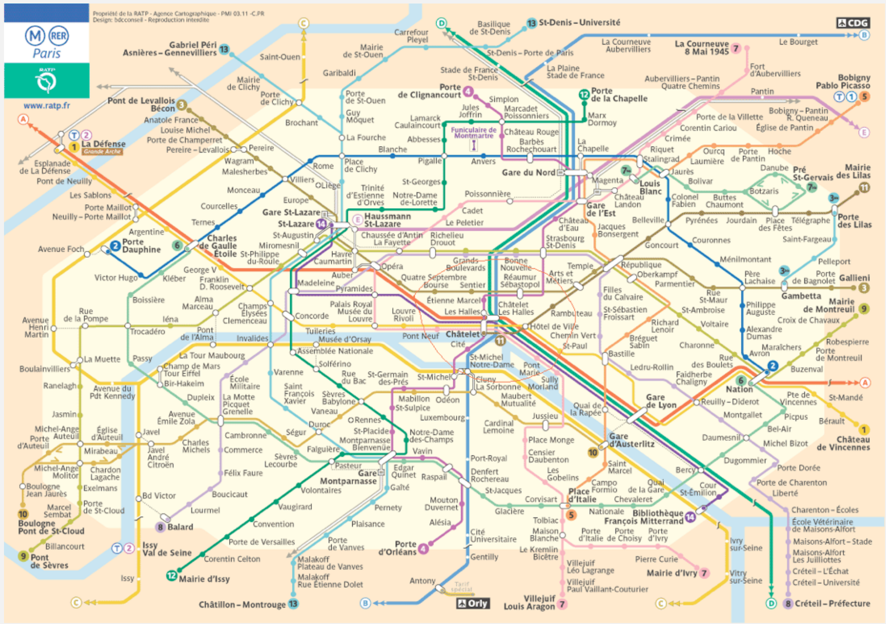
Undirected graphs

Graph. Set of vertices connected pairwise by edges.

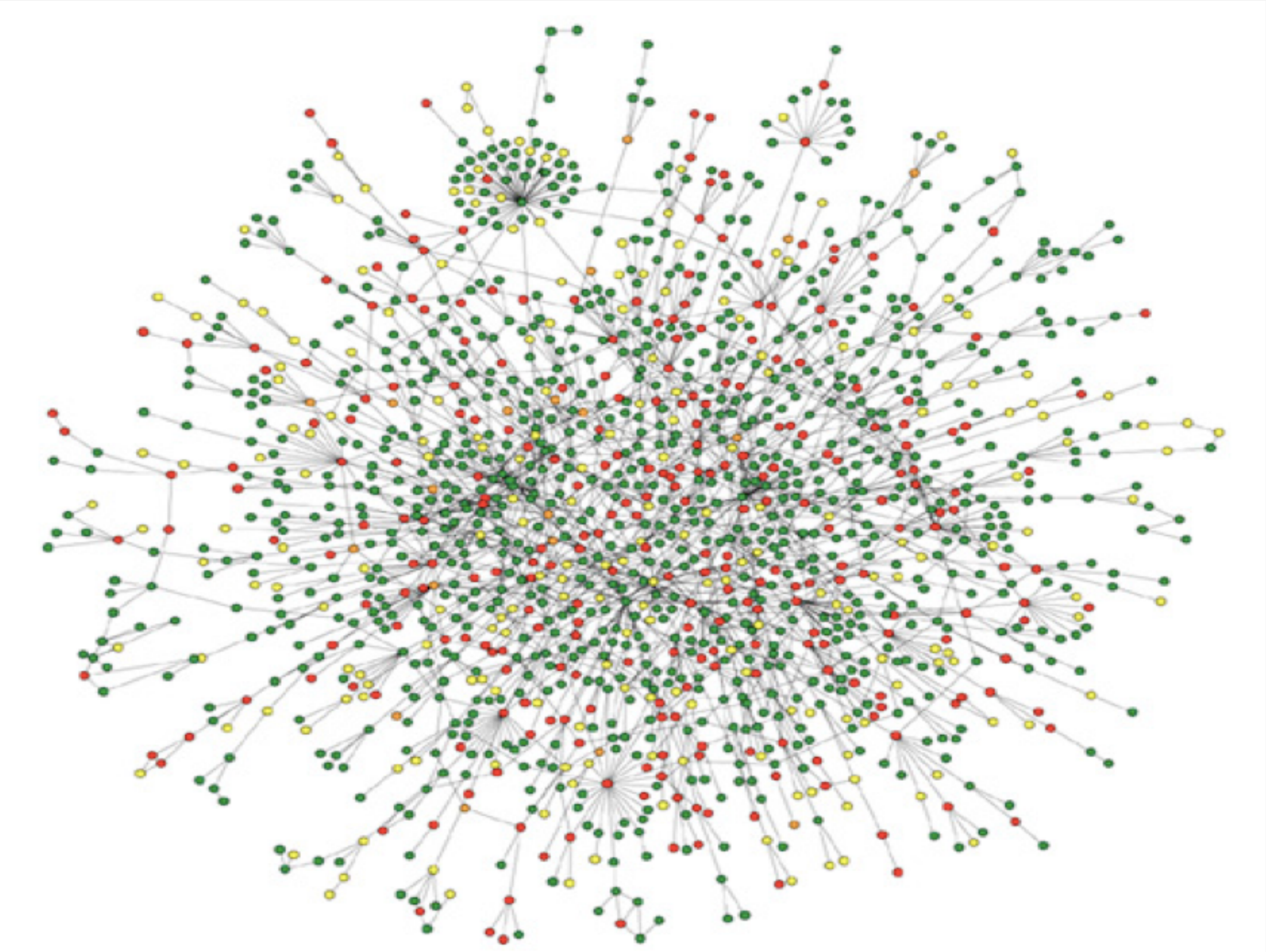
Why study graph algorithms?

- Thousands of practical applications.
- Hundreds of graph algorithms known.
- Interesting and broadly useful abstraction.
- Challenging branch of computer science and discrete math.





Protein-protein interaction network



Reference: Jeong et al, Nature Review | Genetics

Framingham heart study

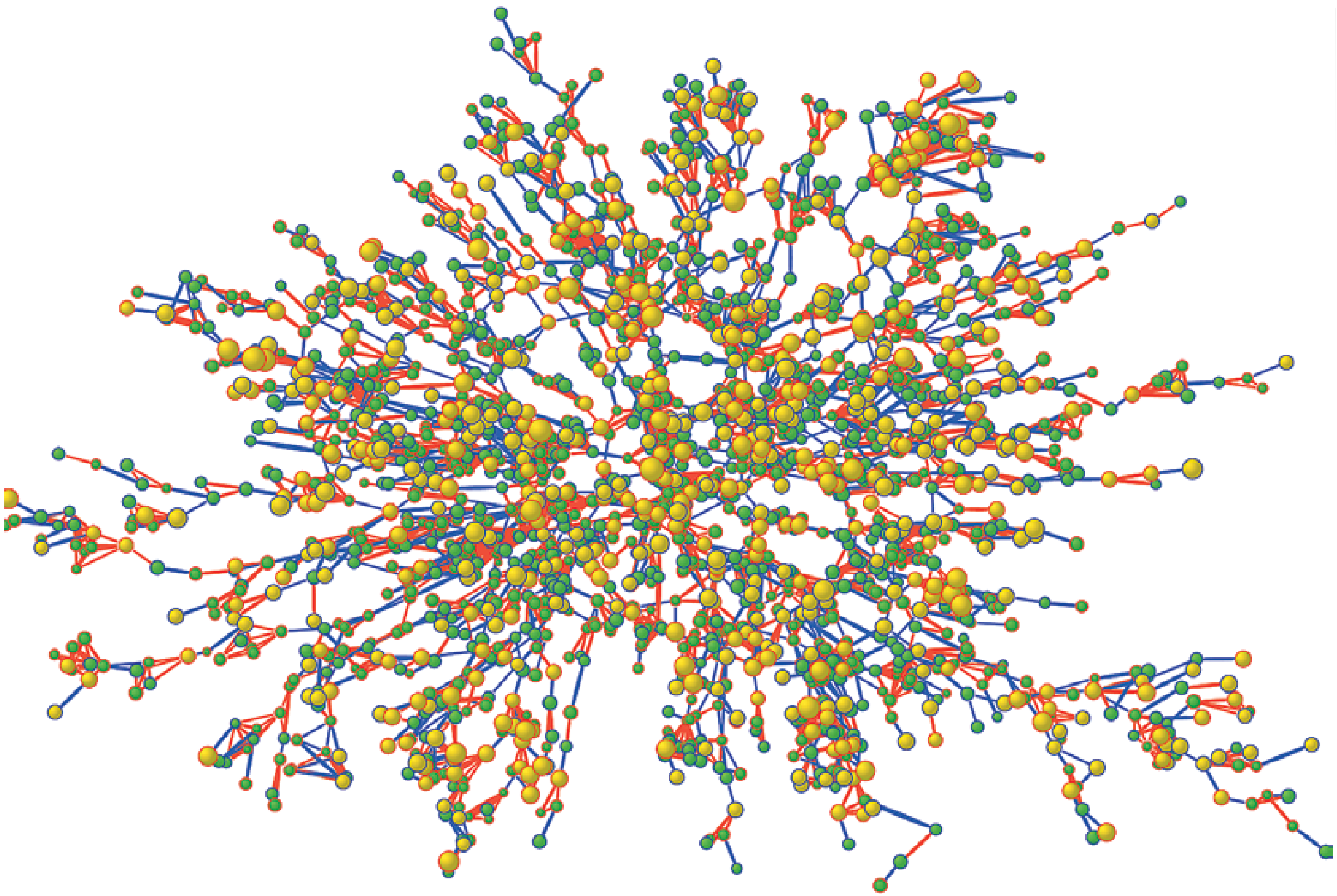
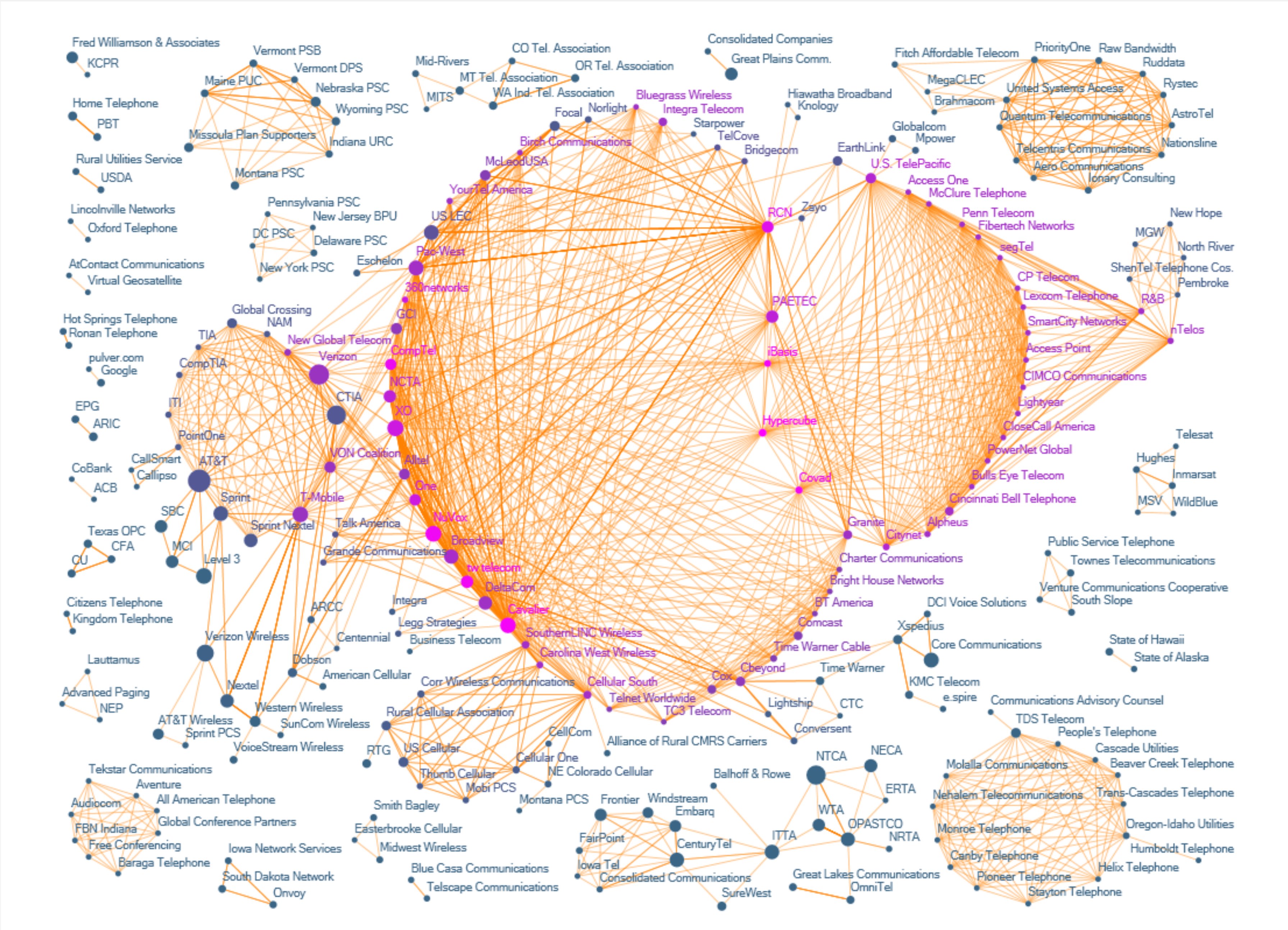


Figure 1. Largest Connected Subcomponent of the Social Network in the Framingham Heart Study in the Year 2000.
Each circle (node) represents one person in the data set. There are 2200 persons in this subcomponent of the social network. Circles with red borders denote women, and circles with blue borders denote men. The size of each circle is proportional to the person's body-mass index. The interior color of the circles indicates the person's obesity status: yellow denotes an obese person (body-mass index, ≥30) and green denotes a nonobese person. The colors of the ties between the nodes indicate the relationship between them: purple denotes a friendship or marital tie and orange denotes a familial tie.

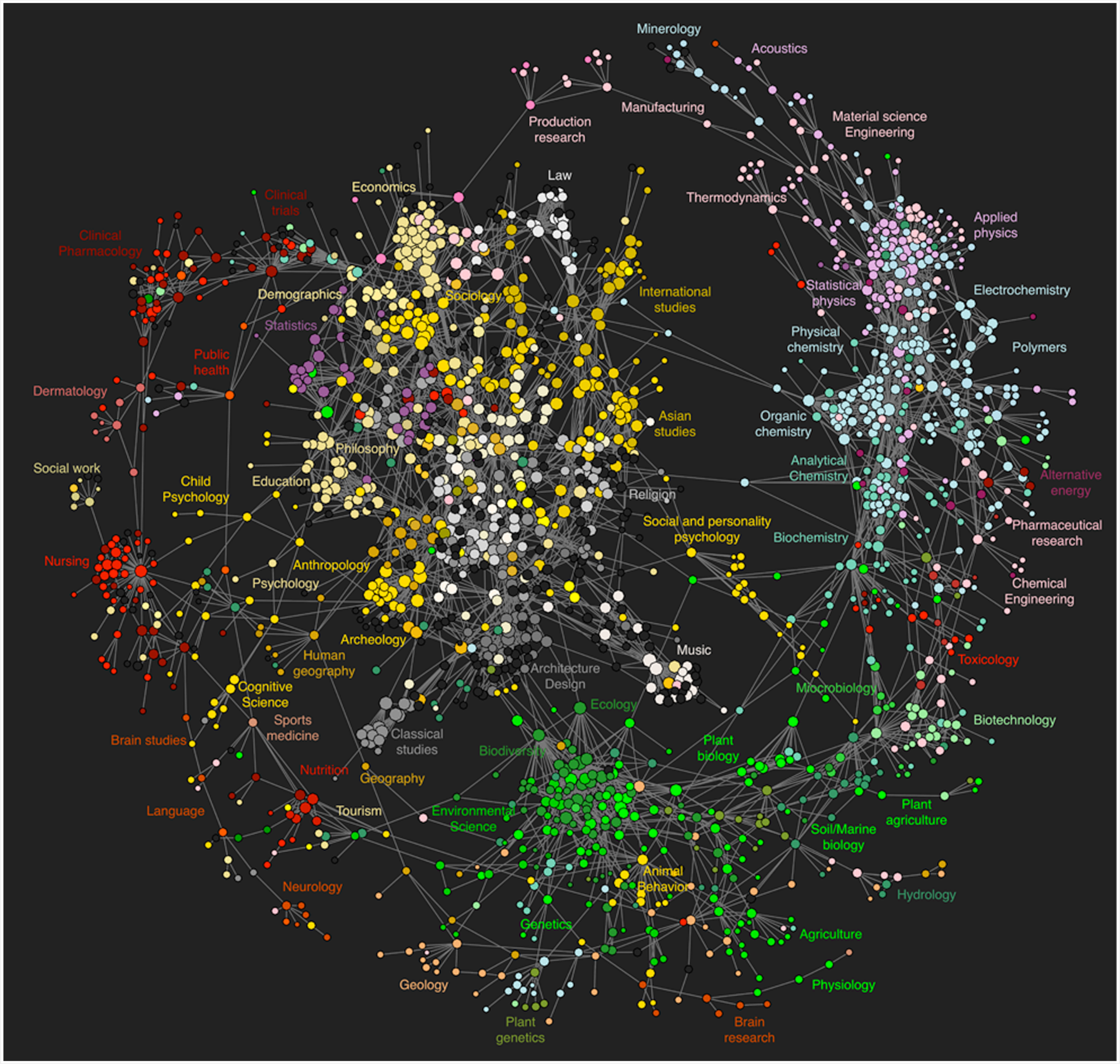
"The Spread of Obesity in a Large Social Network over 32 Years" by Christakis and Fowler in New England Journal of Medicine, 2007

The evolution of FCC lobbying coalitions



"The Evolution of FCC Lobbying Coalitions" by Pierre de Vries in JoSS Visualization Symposium 2010

Map of science clickstreams

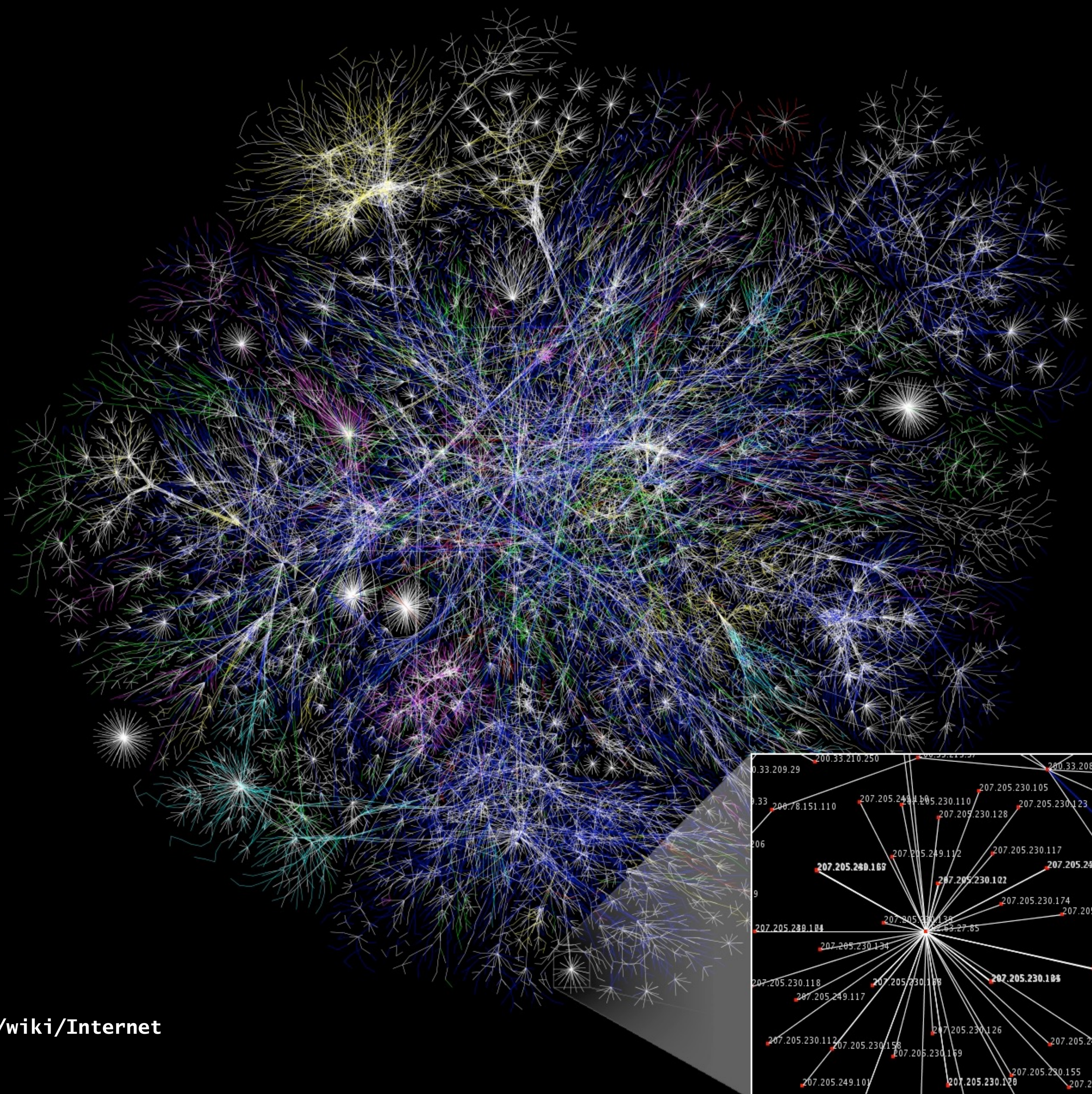


http://www.plosone.org/article/info:doi/10.1371/journal.pone.0004803

10 million Facebook friends

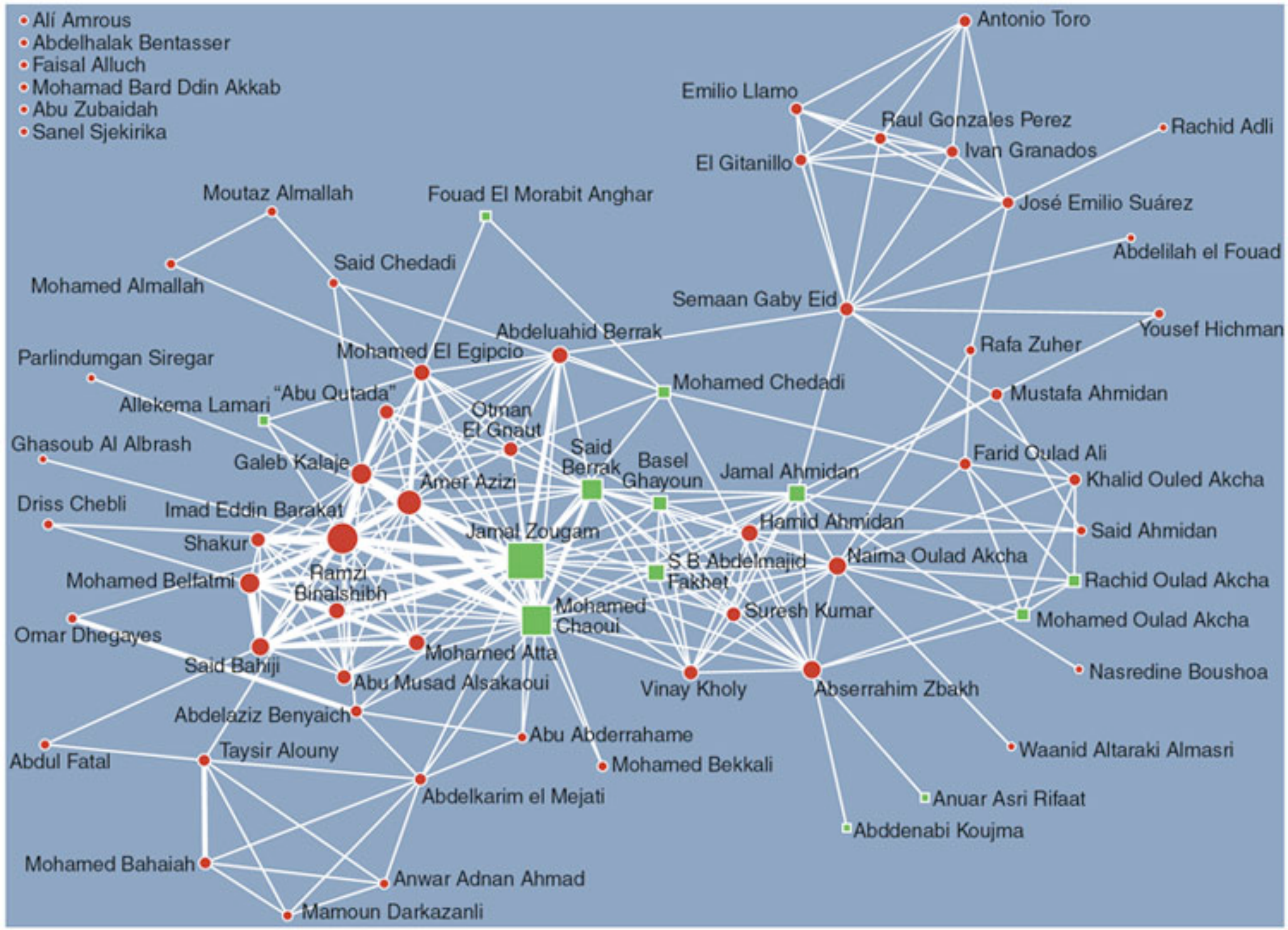
"Visualizing Friendships" by Paul Butler

The Internet as mapped by the Opte Project



http://en.wikipedia.org/wiki/Internet

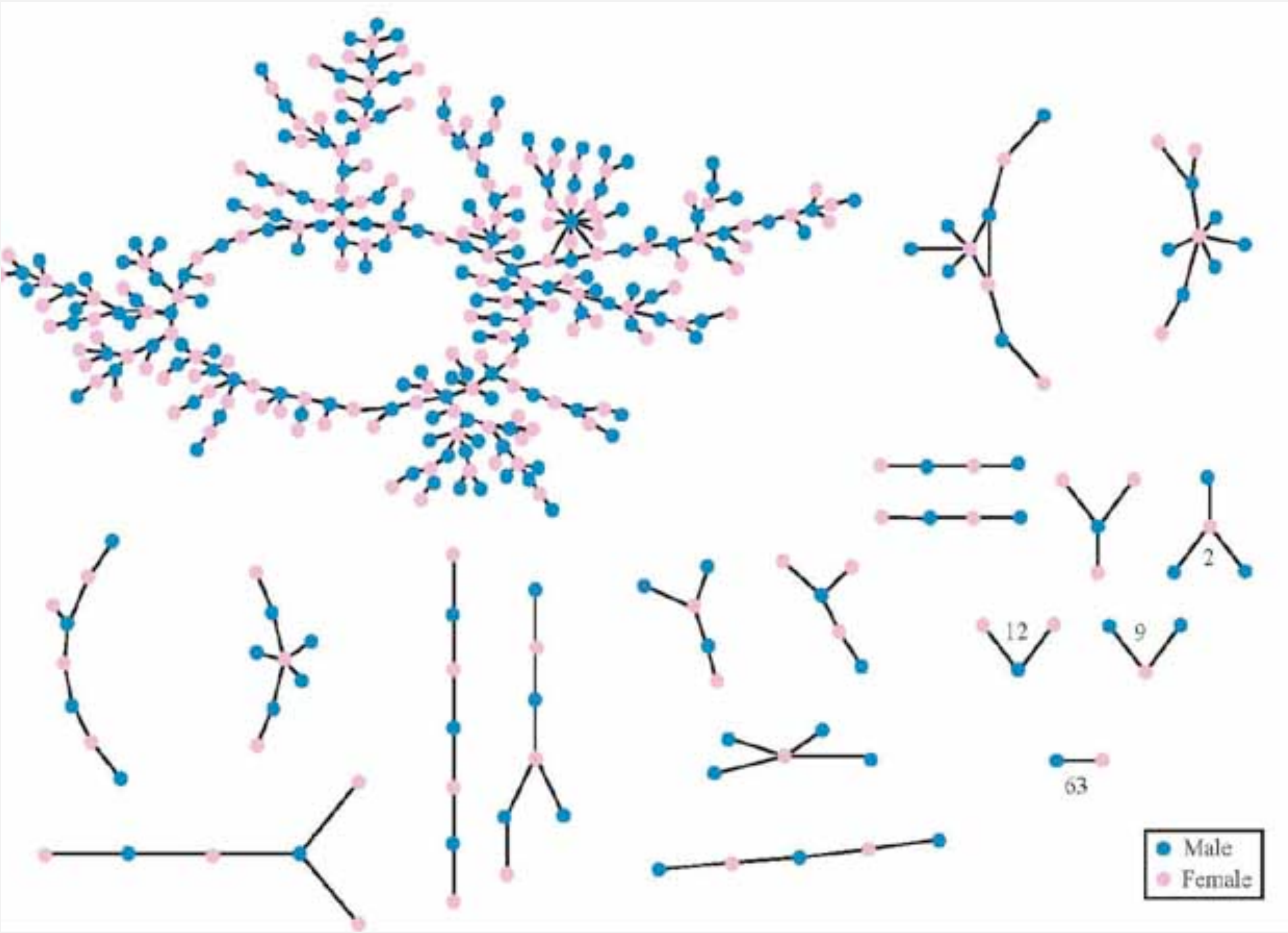
Terrorist networks



Relationships among individuals associated with the 2004 Madrid bombings

Connecting the Dots: Can the tools of graph theory and social-network studies unravel the next big plot?
http://www.americanscientist.org/issues/pub/connecting-the-dots

Sexual network



Structure of romantic and sexual relations at "Jefferson High School"

Researchers Map The Sexual Network Of An Entire High School
http://researchnews.osu.edu/archive/chains.htm and http://www.soc.duke.edu/~jmoody77/chains.pdf

Graph applications

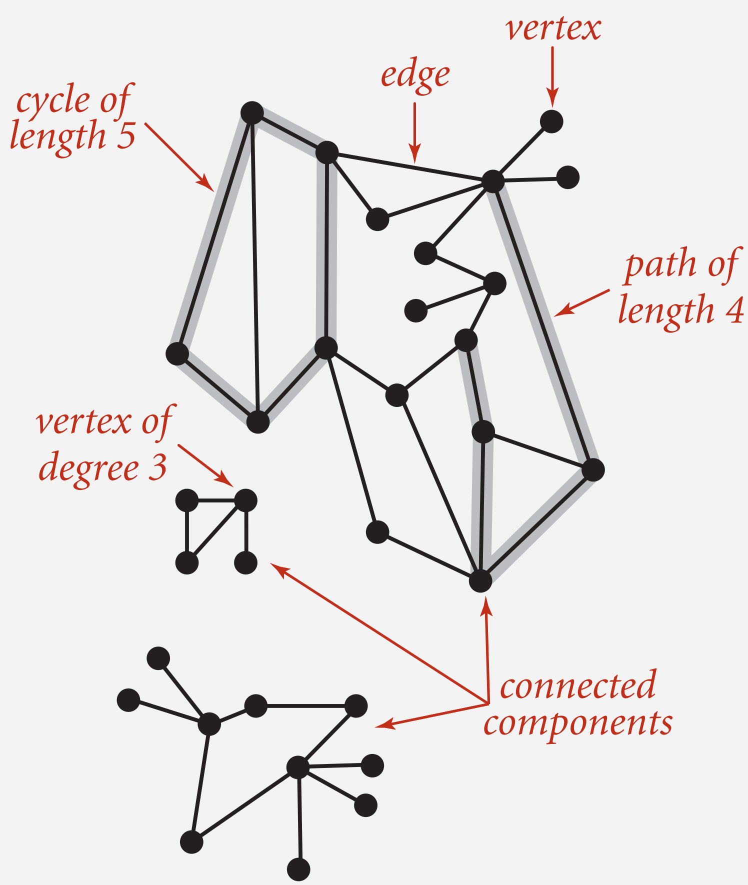
graph	vertex	edge
communication	telephone, computer	fiber optic cable
circuit	gate, register, processor	wire
mechanical	joint	rod, beam, spring
financial	stock, currency	transactions
transportation	intersection	street
internet	class C network	connection
game	board position	legal move
social relationship	person	friendship
neural network	neuron	synapse
protein network	protein	protein-protein interaction
molecule	atom	bond

Graph terminology

Path. Sequence of vertices connected by edges.

Cycle. Path whose first and last vertices are the same.

Two vertices are connected if there is a path between them.



Some graph-processing problems

problem	description
s-t path	*Is there a path between s and t ?*
shortest s-t path	*What is the shortest path between s and t ?*
cycle	*Is there a cycle in the graph ?*
Euler cycle	*Is there a cycle that uses each edge exactly once ?*
Hamilton cycle	*Is there a cycle that uses each vertex exactly once ?*
connectivity	*Is there a path between every pair of vertices ?*
biconnectivity	*Is there a vertex whose removal disconnects the graph ?*
planarity	*Can the graph be drawn in the plane with no crossing edges ?*
graph isomorphism	*Are two graphs isomorphic?*

Challenge. Which graph problems are easy? difficult? intractable?

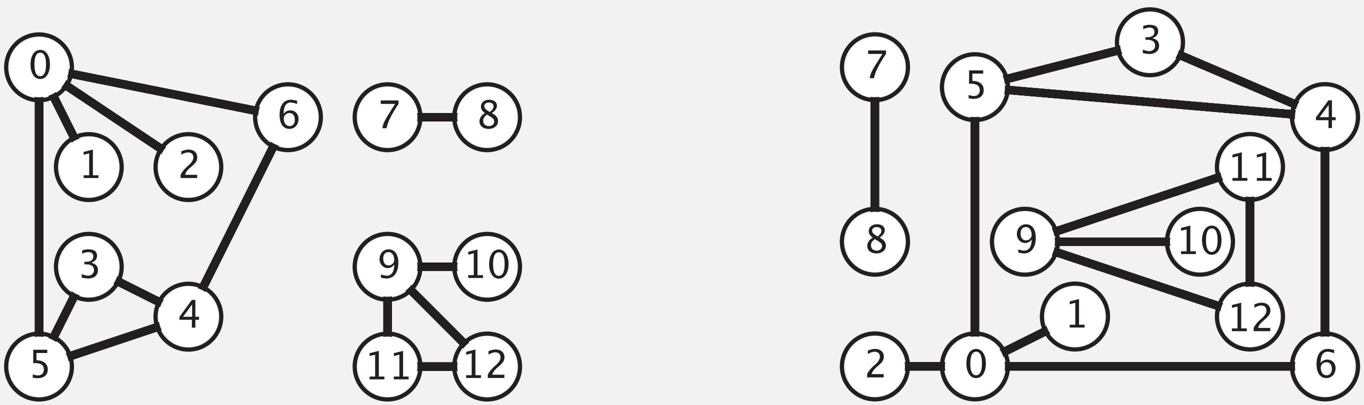
http://algs4.cs.princeton.edu

4.1 Undirected Graphs

- introduction
- graph API
- depth-first search
- breadth-first search
- challenges

Graph representation

Graph drawing. Provides intuition about the structure of the graph.



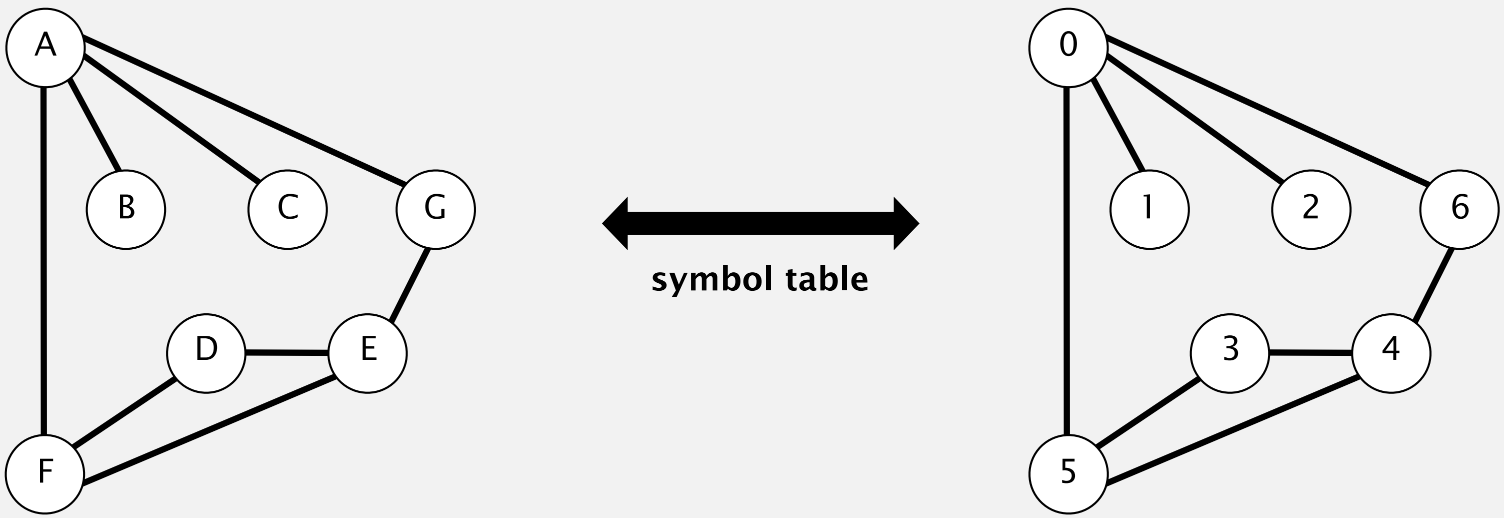
two drawings of the same graph

Caveat. Intuition can be misleading.

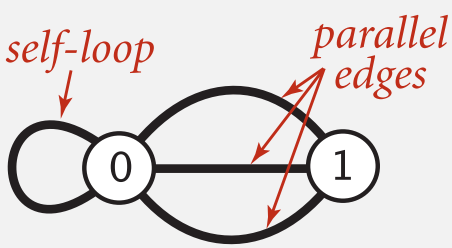
Graph representation

Vertex representation.

- This lecture: use integers between 0 and $V-1$.
- Applications: convert between names and integers with symbol table.



Anomalies.



Graph API

public class Graph		
	`Graph(int V)`	*create an empty graph with V vertices*
	`Graph(In in)`	*create a graph from input stream*
`void`	`addEdge(int v, int w)`	*add an edge v-w*
`Iterable<Integer>`	`adj(int v)`	*vertices adjacent to v*
`int`	`V()`	*number of vertices*
`int`	`E()`	*number of edges*

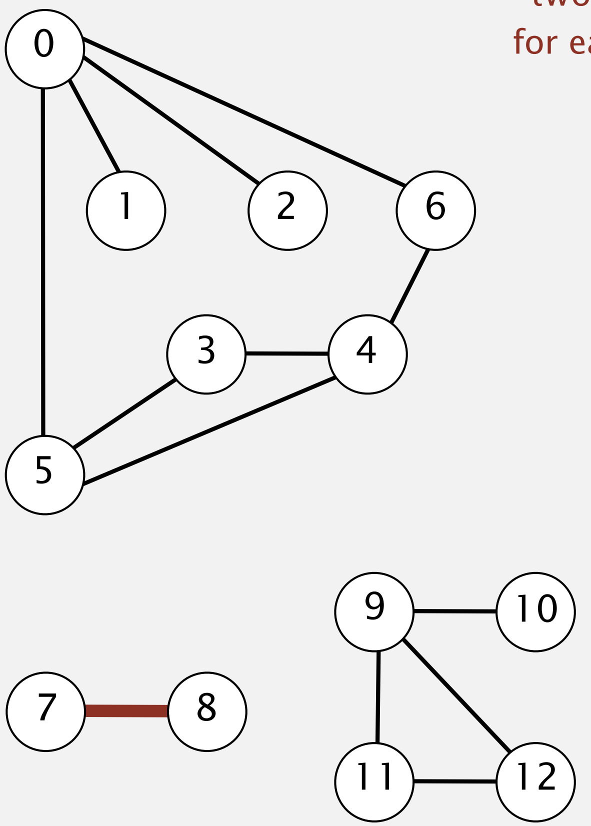
```
// degree of vertex v in graph G
public static int degree(Graph G, int v)
{
    int degree = 0;
    for (int w : G.adj(v))
       degree++;
    return degree;
}
```

Toy API. No efficient way to compute degree, check if edge exists, etc.

Graph representation: adjacency matrix

Maintain a two-dimensional V-by-V boolean array;

for each edge $v–w$ in graph: `adj[v][w] = adj[w][v] = true`.



two entries
for each edge

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0
4	0	0	0	1	0	1	1	0	0	0	0	0	0
5	1	0	0	1	1	0	0	0	0	0	0	0	0
6	1	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	1
10	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	0	1
12	0	0	0	0	0	0	0	0	0	1	0	1	0

Undirected graphs: quiz 1

Which is order of growth of running time of the following code fragment if the graph uses the adjacency-matrix representation, where V is the number of vertices and E is the number of edges?

```
for (int v = 0; v < G.V(); v++)
   for (int w : G.adj(v))
      StdOut.println(v + "-" + w);
```

prints edges

A. V

B. $E + V$

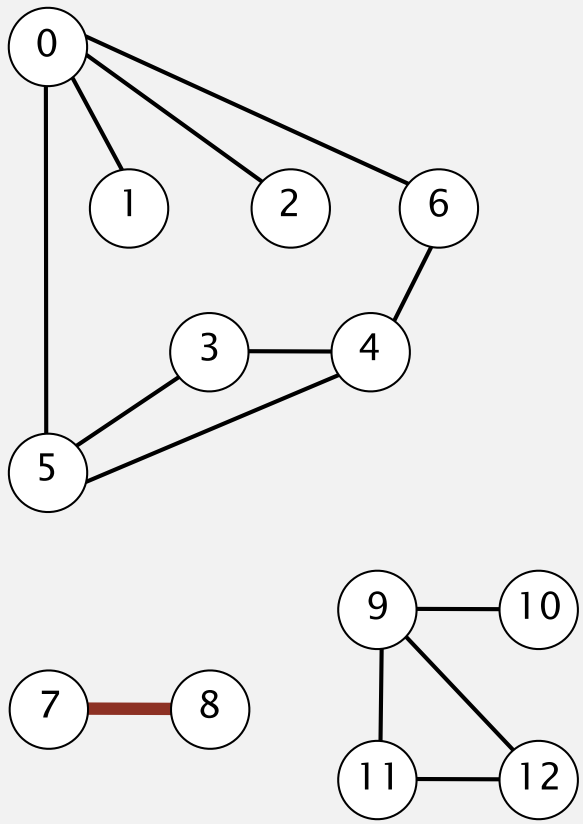
C. V^2

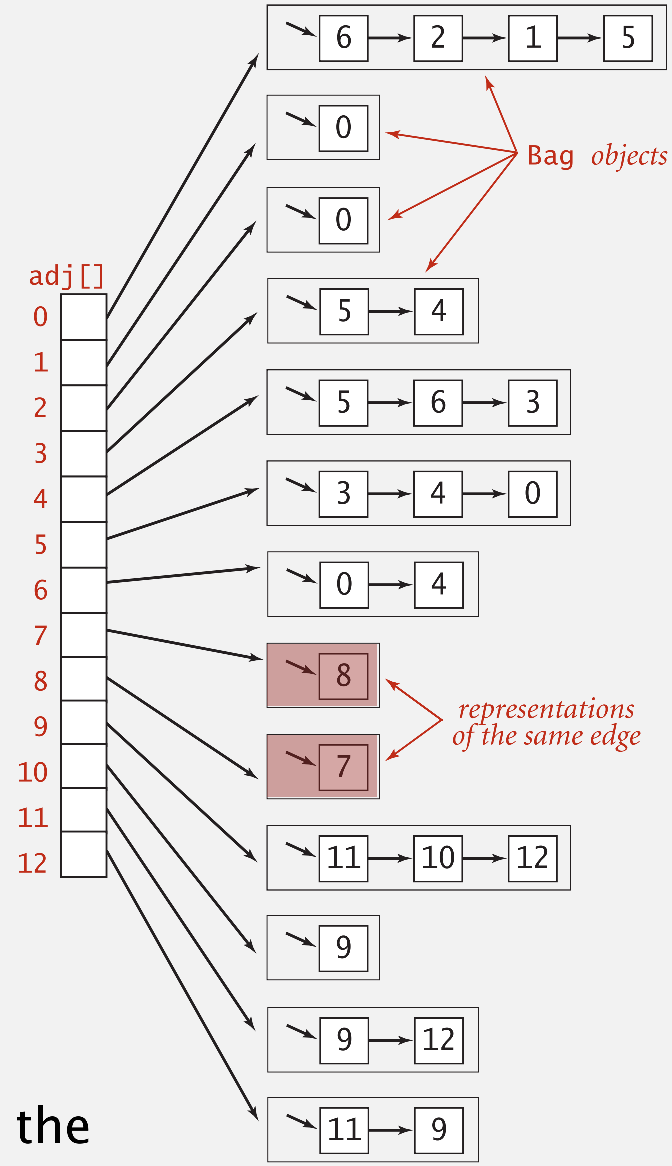
D. $V E$

E. *I don't know.*

Graph representation: adjacency lists

Maintain vertex-indexed array of lists.





We use Bag objects because we don't care about the order in which we iterate over the adjacent vertices.

Undirected graphs: quiz 2

Which is order of growth of running time of the following code fragment if the graph uses the adjacency-lists representation, where V is the number of vertices and E is the number of edges?

```
for (int v = 0; v < G.V(); v++)
   for (int w : G.adj(v))
      StdOut.println(v + "-" + w);
```

prints edges

A. V

B. $E + V$

C. V^2

D. $V E$

E. *I don't know.*

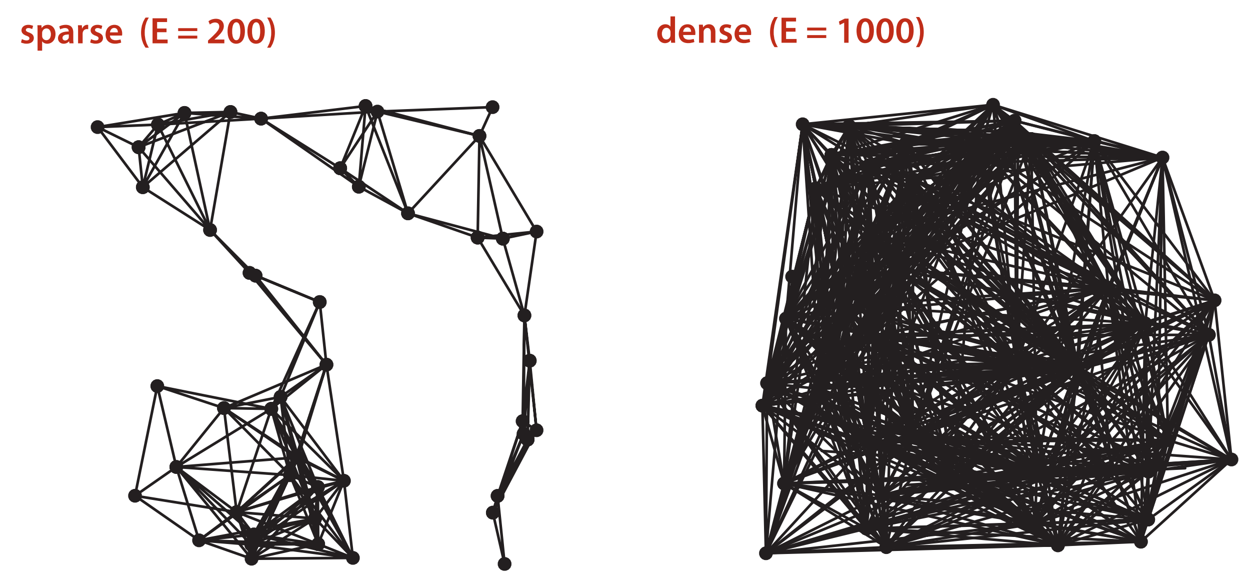
Homework:
verify answer

Graph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent to v.
- Real-world graphs tend to be sparse.

huge number of vertices,
small average vertex degree



Two graphs (V = 50)

Graph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent to v.
- Real-world graphs tend to be sparse.

huge number of vertices,
small average vertex degree

representation	space	add edge	edge between v and w?	iterate over vertices adjacent to v?
list of edges	E	1	E	E
adjacency matrix	V^2	1 †	1	V
adjacency lists	$E + V$	1	$degree(v)$	$degree(v)$

† disallows parallel edges

Homework. Design a representation that improves *degree(v)* bound for checking if edge exists, and is as good as adjacency lists for all other ops

Adjacency-list graph representation: Java implementation

```
public class Graph
{
   private final int V;
   private Bag<Integer>[] adj;

   public Graph(int V)
   {
     this.V = V;
     adj = (Bag<Integer>[]) new Bag[V];
     for (int v = 0; v < V; v++)
        adj[v] = new Bag<Integer>();
   }

   public void addEdge(int v, int w)
   {
     adj[v].add(w);
     adj[w].add(v);
   }

   public Iterable<Integer> adj(int v)
   {  return adj[v];   }
}
```

adjacency lists
(using Bag data type)

create empty graph
with V vertices

Skipped in class

add edge v–w
(parallel edges and
self-loops allowed)

iterator for vertices adjacent to v

Algorithms

Robert Sedgewick | Kevin Wayne

http://algs4.cs.princeton.edu

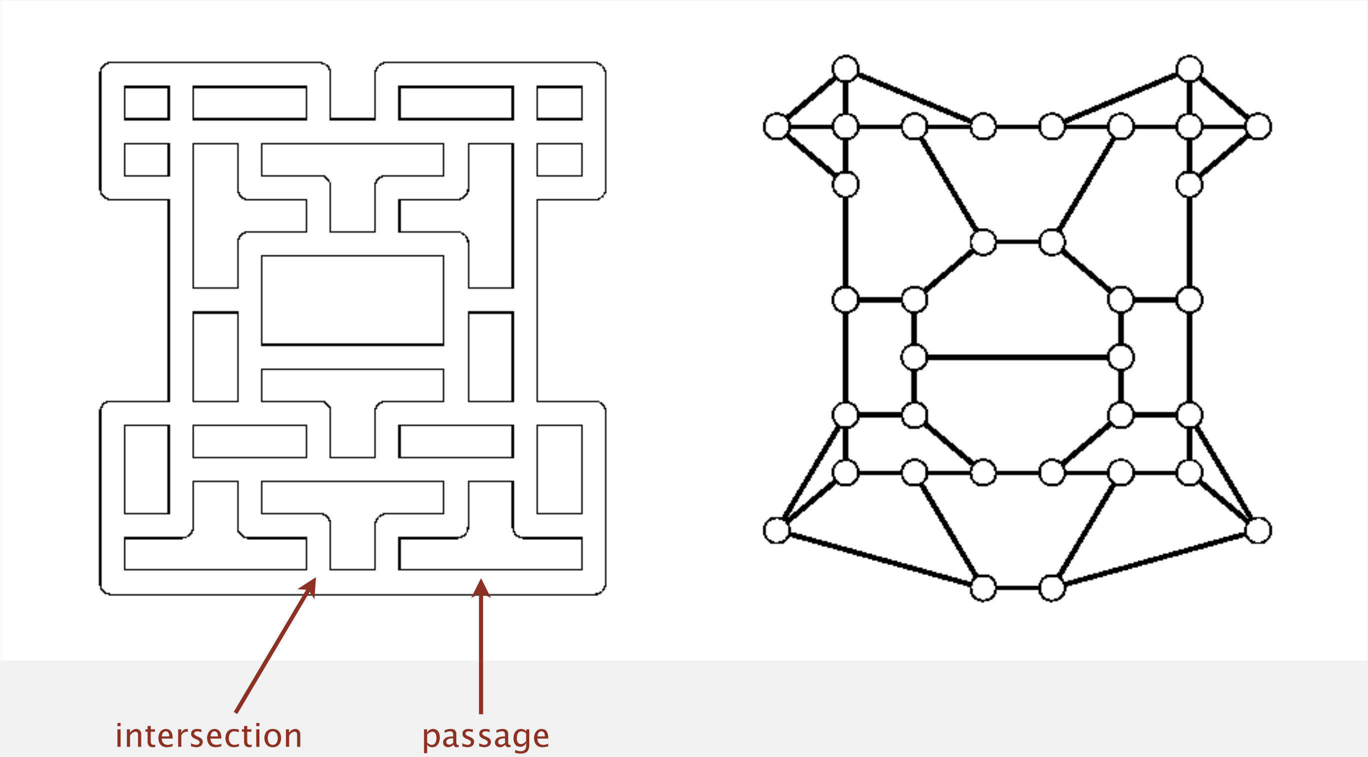
4.1 Undirected Graphs

- *introduction*
- *graph API*
- *depth-first search*
- *breadth-first search*
- *challenges*

Maze exploration

Maze graph.

- Vertex = intersection.
- Edge = passage.



Goal. Explore every intersection in the maze.

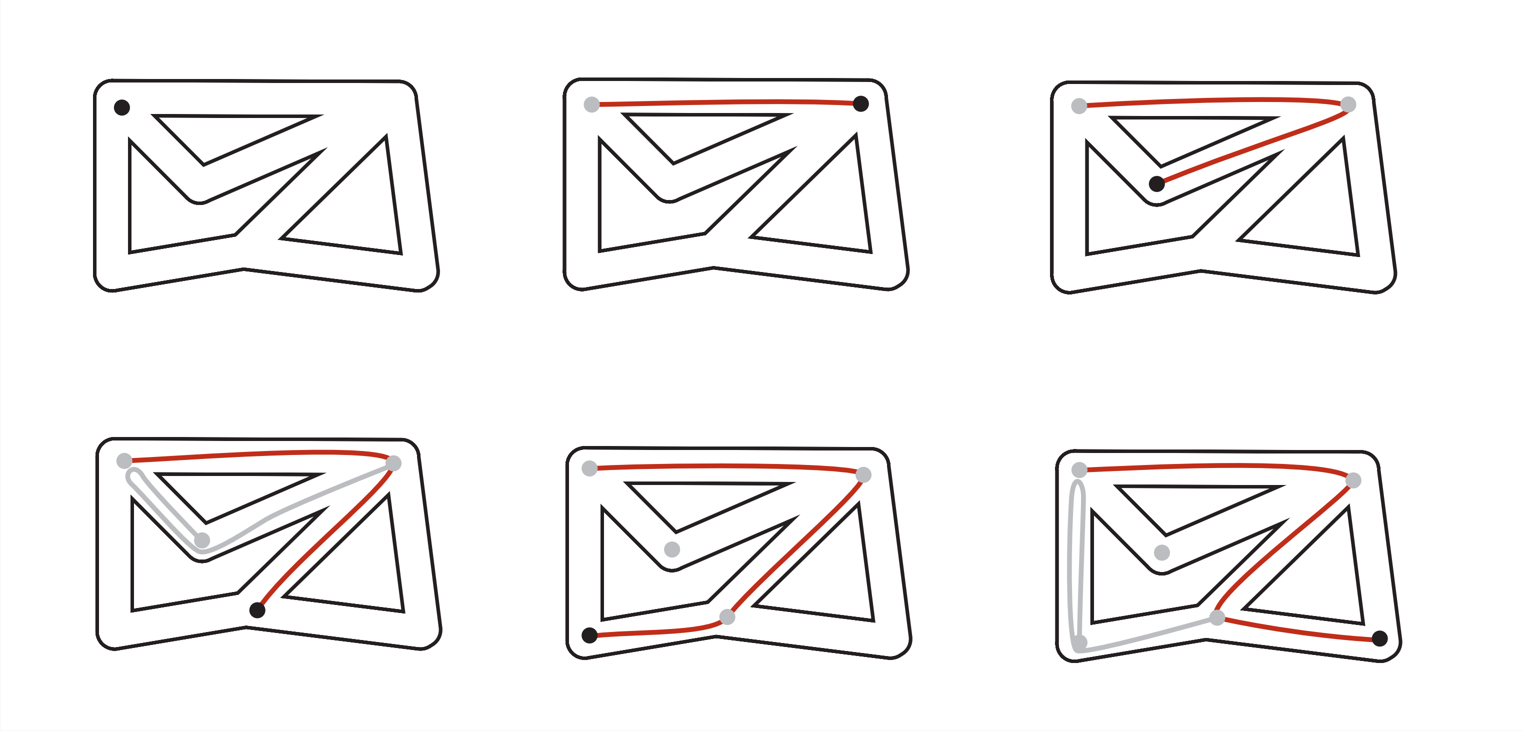
Maze exploration: National Building Museum

http://www.smithsonianmag.com/travel/winding-history-maze-180951998/?no-ist

Trémaux maze exploration

Algorithm.

- Unroll a ball of string behind you.
- Mark each newly discovered intersection and passage.
- Retrace steps when no unmarked options.



Trémaux maze exploration

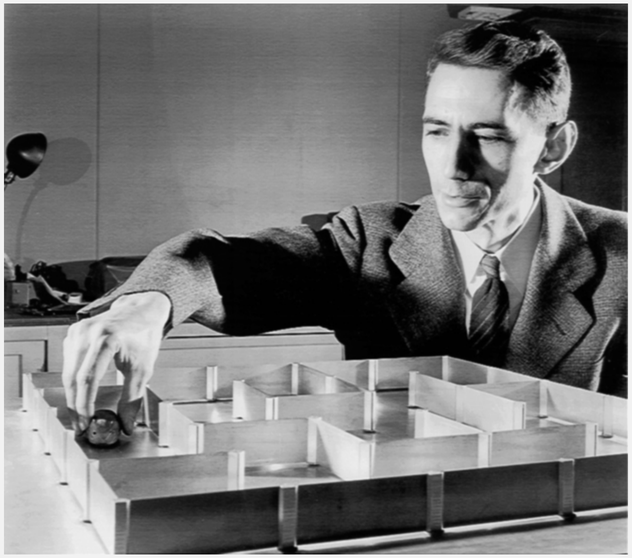
Algorithm.

- Unroll a ball of string behind you.
- Mark each newly discovered intersection and passage.
- Retrace steps when no unmarked options.

First use? Theseus entered Labyrinth to kill the monstrous Minotaur; Ariadne instructed Theseus to use a ball of string to find his way back out.

The Cretan Labyrinth (with Minotaur)

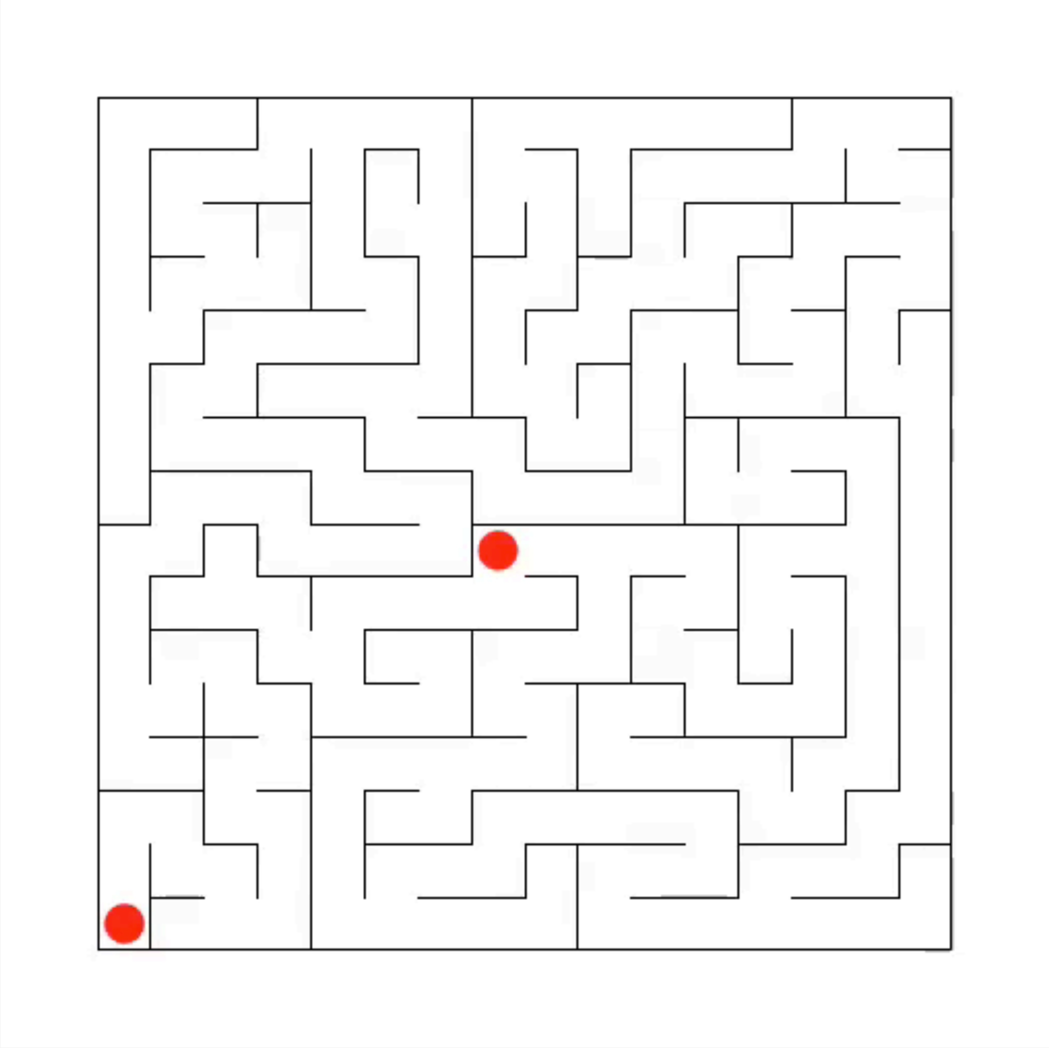
http://commons.wikimedia.org/wiki/File:Minotaurus.gif



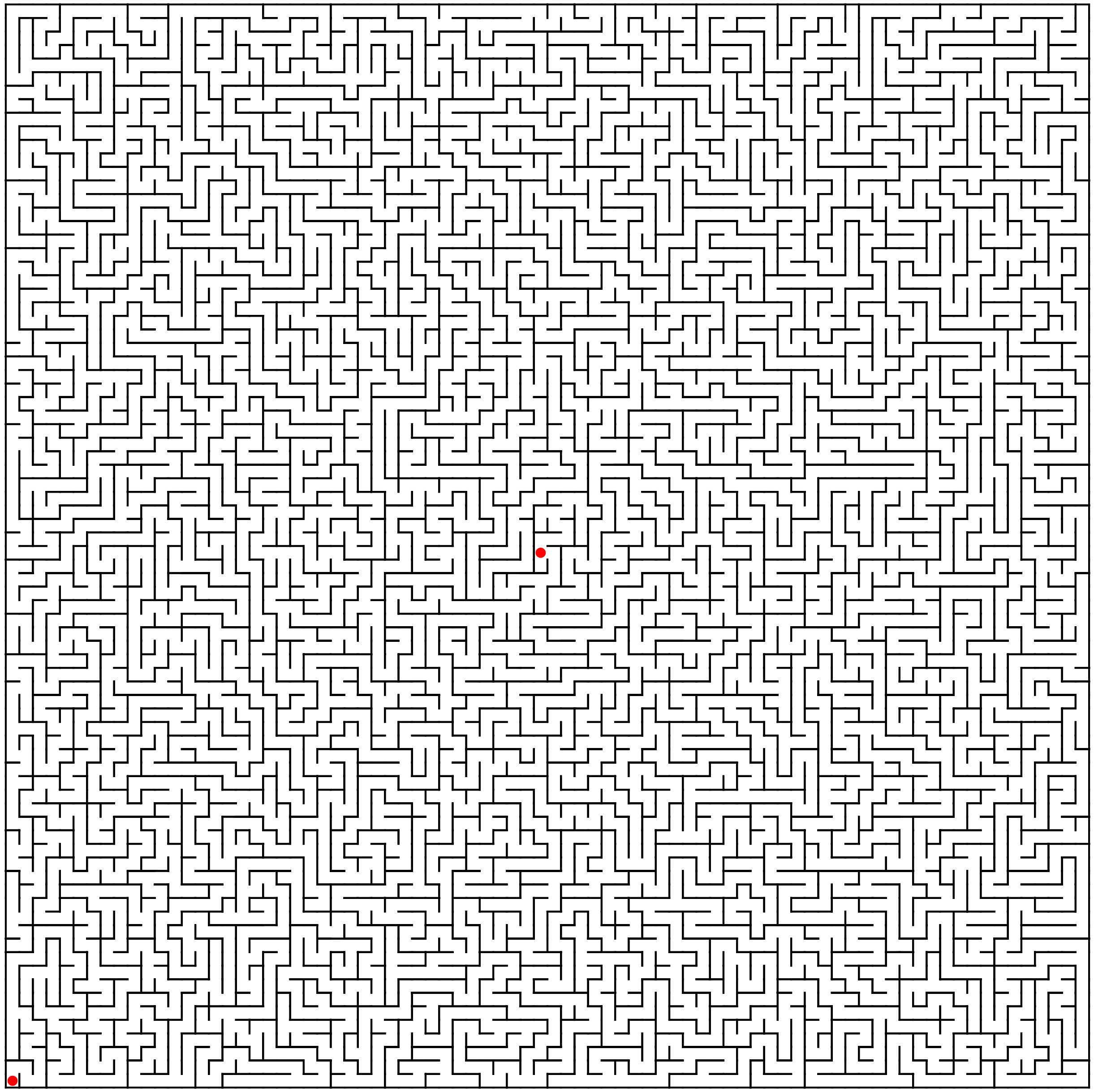
Claude Shannon (with electromechanical mouse)

http://www.corp.att.com/attlabs/reputation/timeline/16shannon.html

Maze exploration



Maze exploration: challenge for the bored



Depth-first search

Goal. Systematically traverse a graph.

Idea. Mimic maze exploration. ← function-call stack acts as ball of string

DFS (to visit a vertex v)

Mark vertex v.
Recursively visit all unmarked
vertices w adjacent to v.

Typical applications.

- Find all vertices connected to a given source vertex.
- Find a path between two vertices.

Undirected graphs: quiz 3

DFS of a tree (starting at the root) corresponds to which traversal?

A. In–order

B. Pre–order

C. Post–order

D. Level–order

E. *I don't know.*

DFS (to visit a vertex v)

Mark vertex v.

Recursively visit all unmarked vertices w adjacent to v.

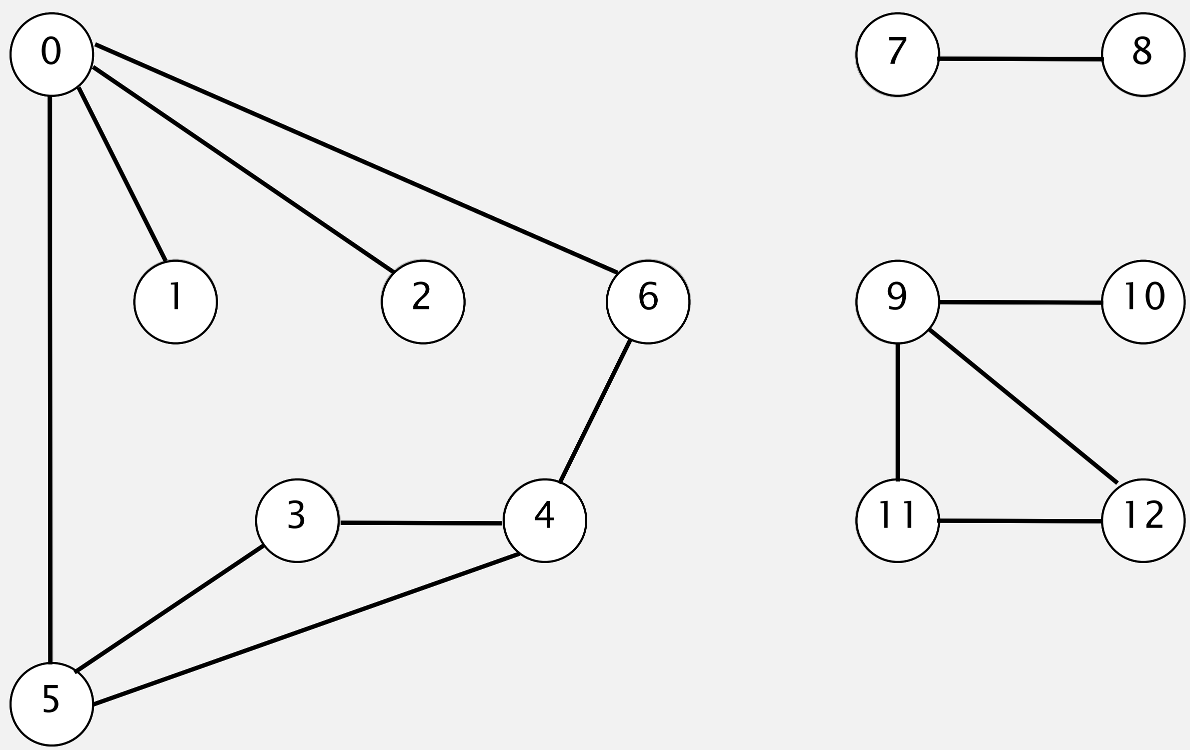
Trick question! DFS doesn't care about order of visiting adjacent nodes.

May correspond to pre-order or to none of the orders.

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



tinyG.txt

$V \rightarrow$ 13
13 $\leftarrow E$

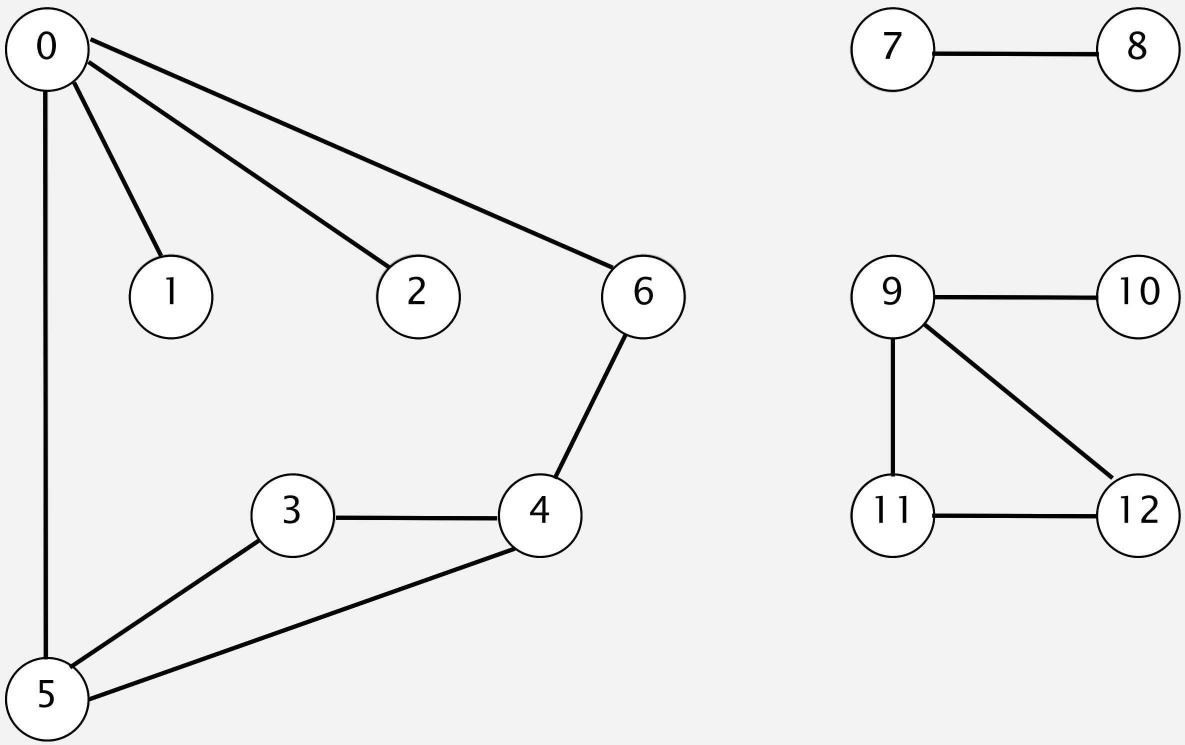
```
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3
```

graph G

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



graph G

tinyG.txt

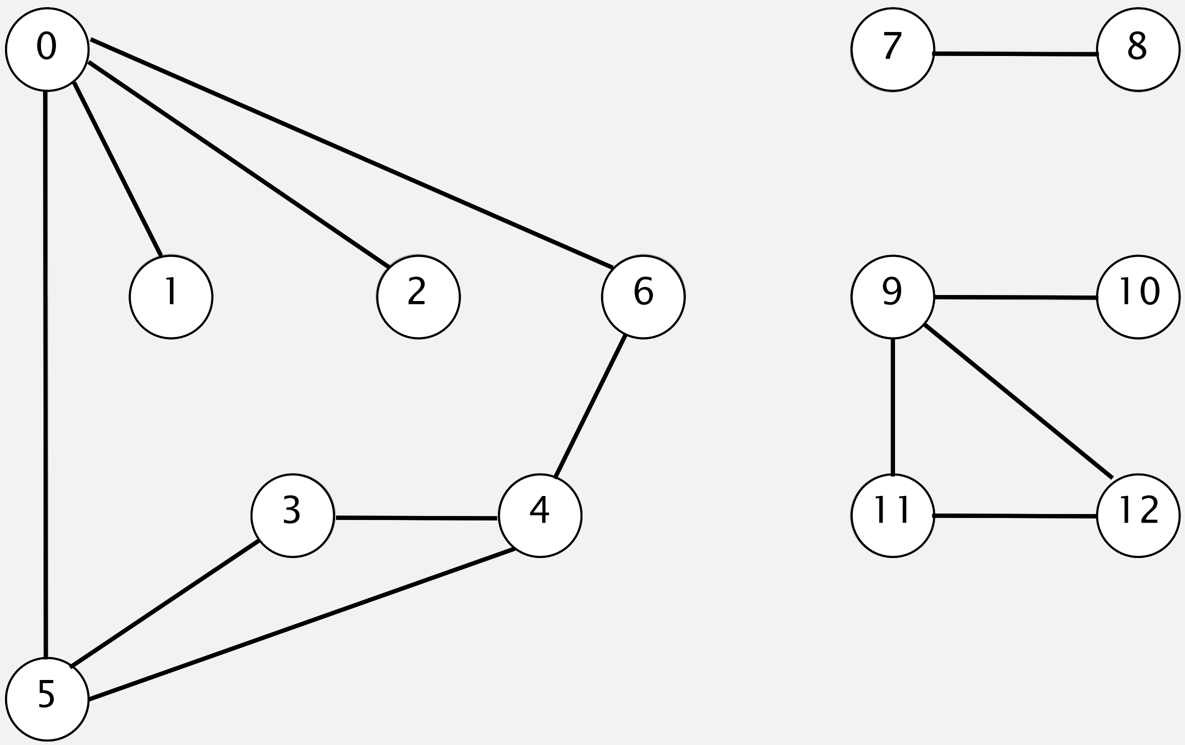
V E

```
13
13
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3
```

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



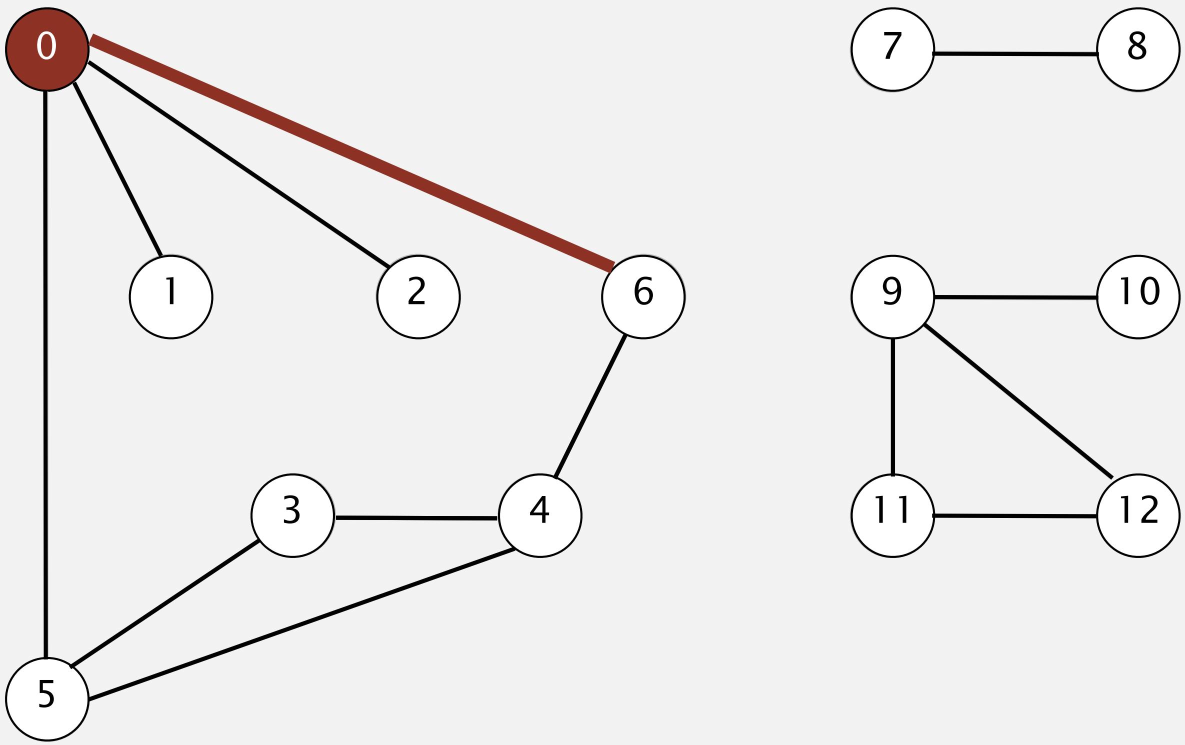
graph G

v	marked[]	edgeTo[]
0	F	–
1	F	–
2	F	–
3	F	–
4	F	–
5	F	–
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



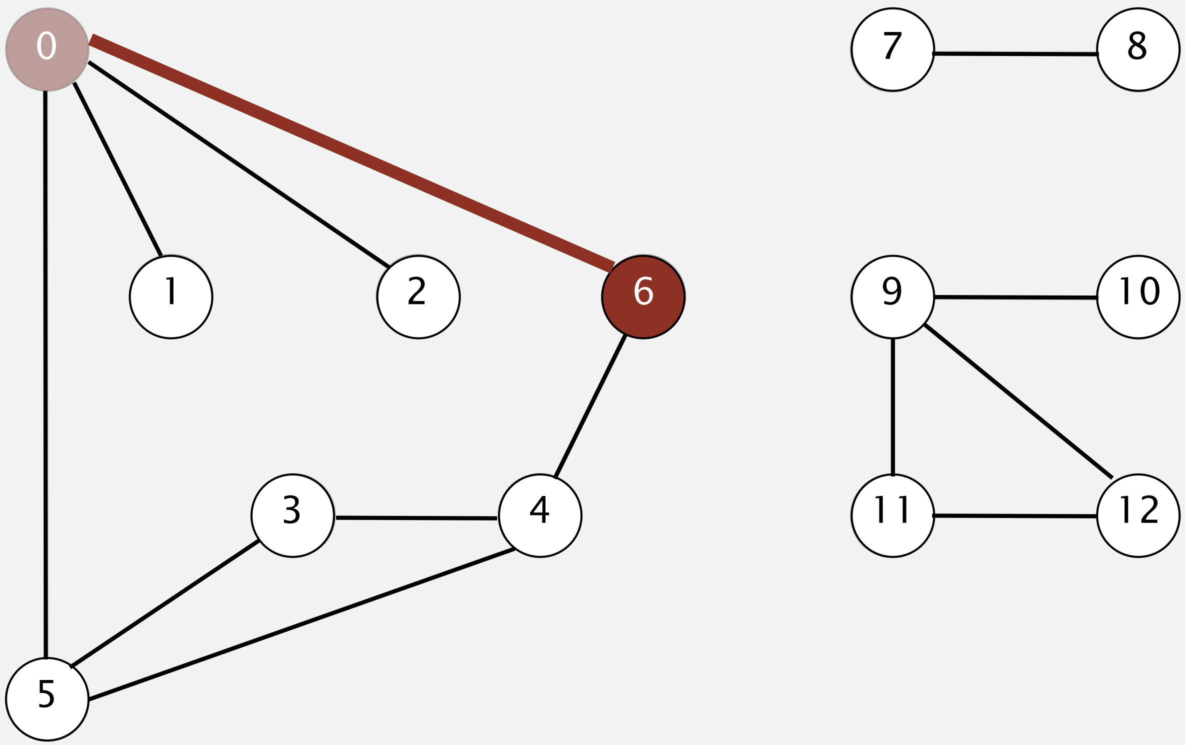
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	F	–
4	F	–
5	F	–
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 0: check 6, check 5, check 2, check 1, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



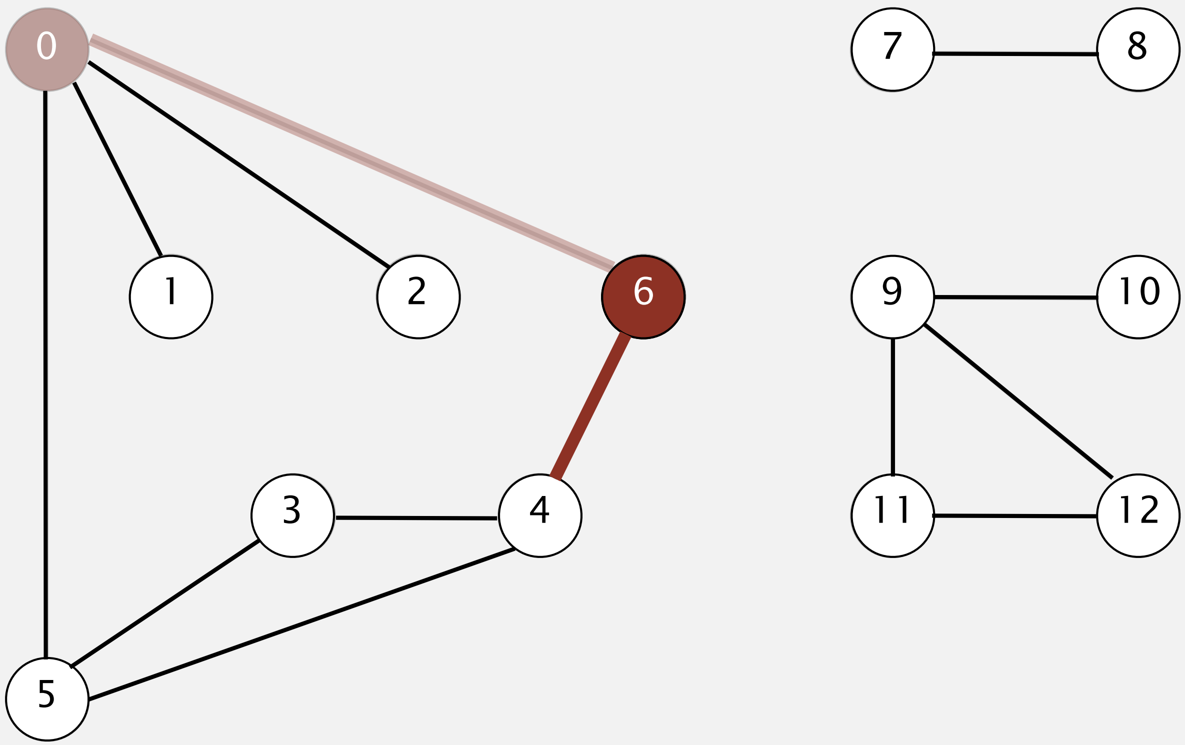
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	F	–
4	F	–
5	F	–
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 6: check 0, check 4, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



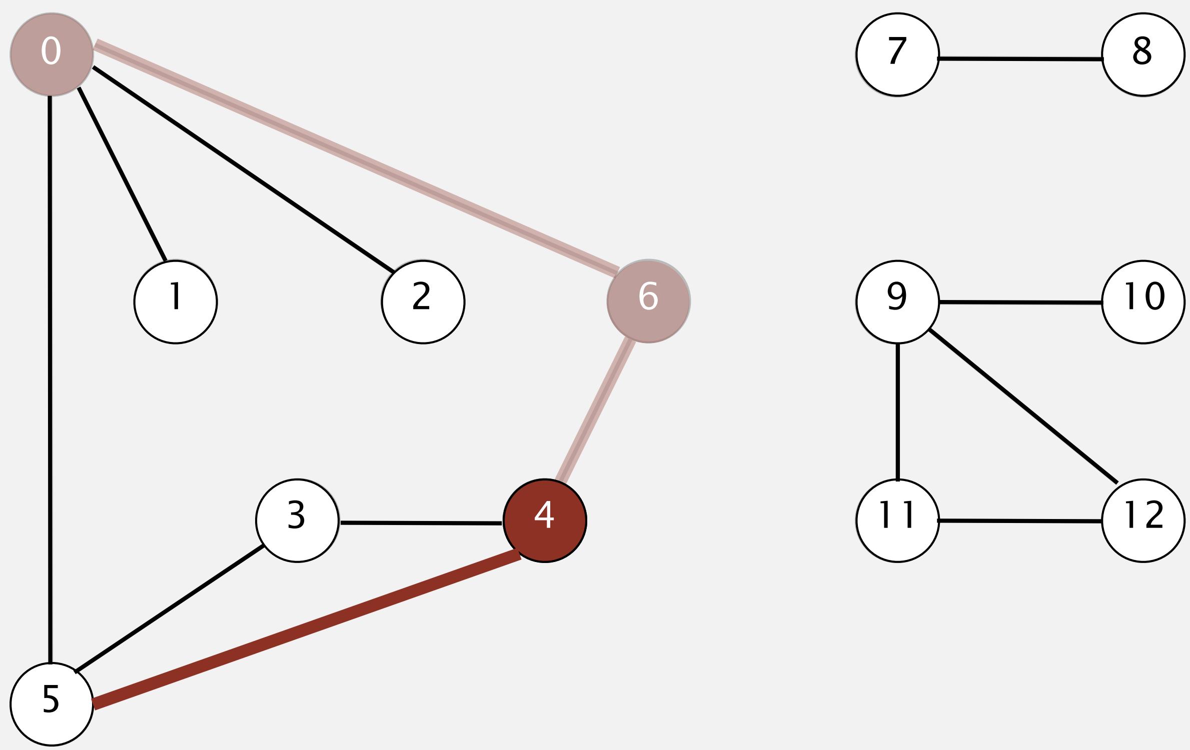
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	F	–
4	F	–
5	F	–
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 6: check 0, **check 4,** done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



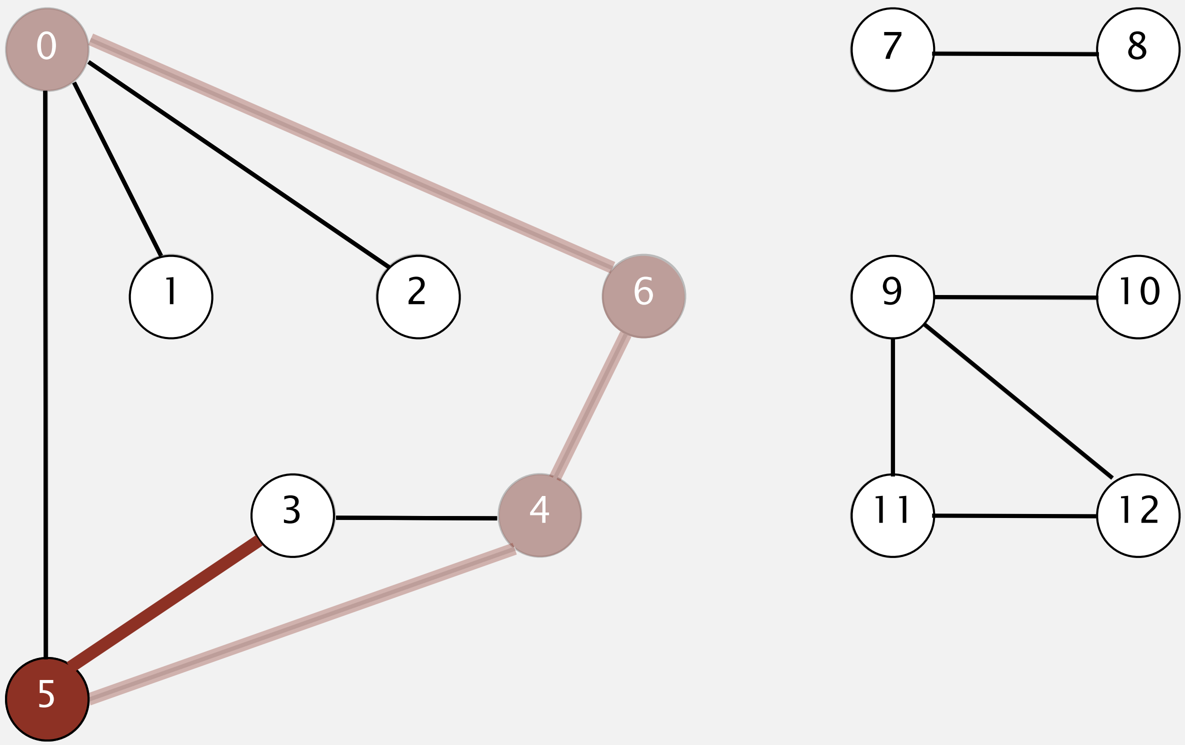
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	F	–
4	T	6
5	F	–
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 4: check 5, check 6, check 3, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



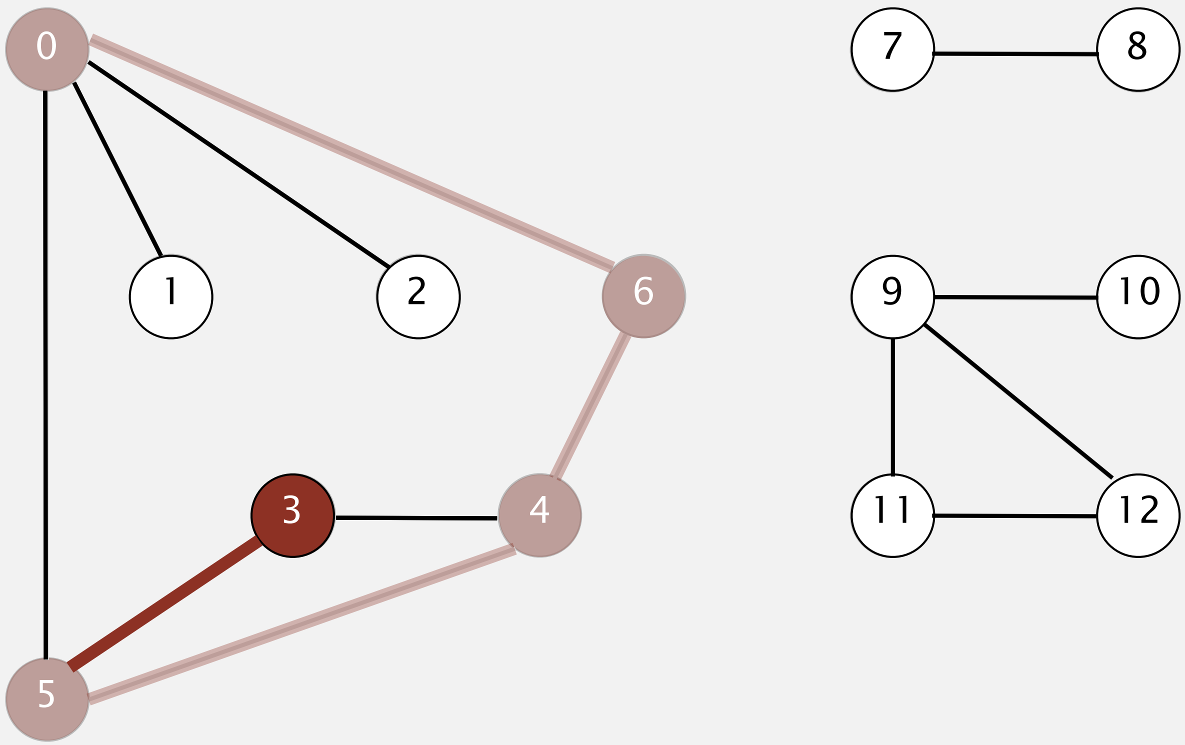
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	F	–
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 5: check 3, check 4, check 0, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



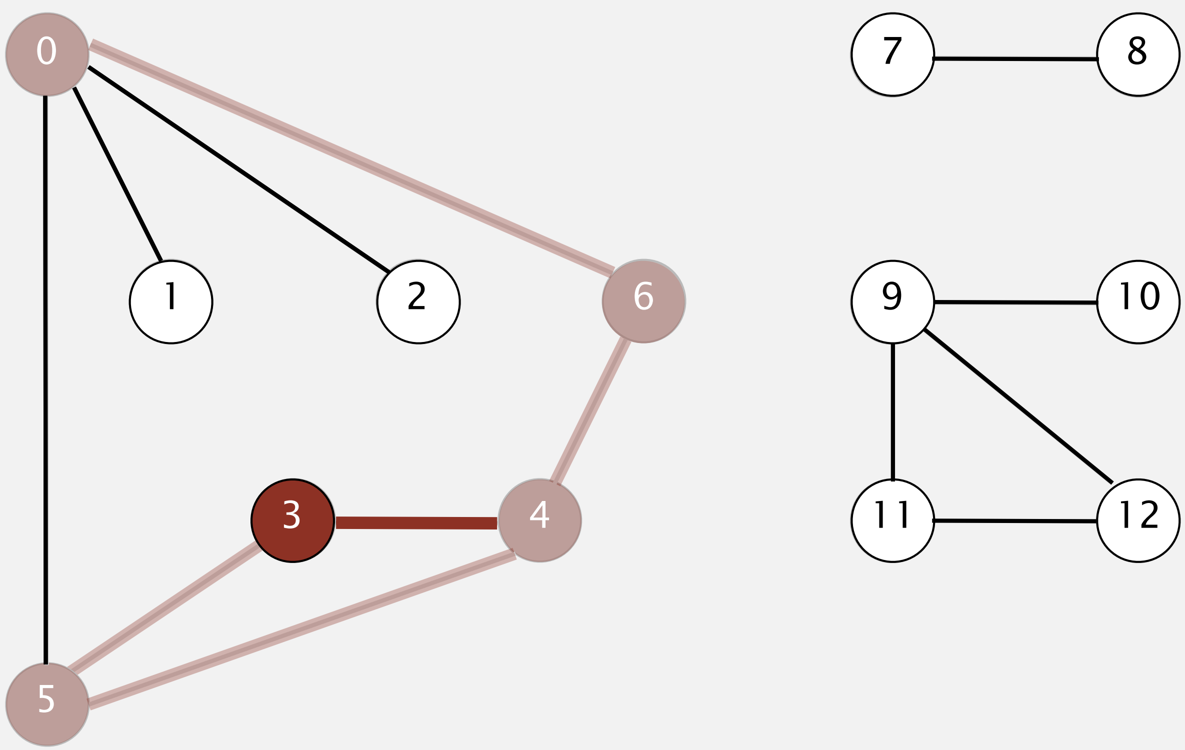
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 3: check 5, check 4, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



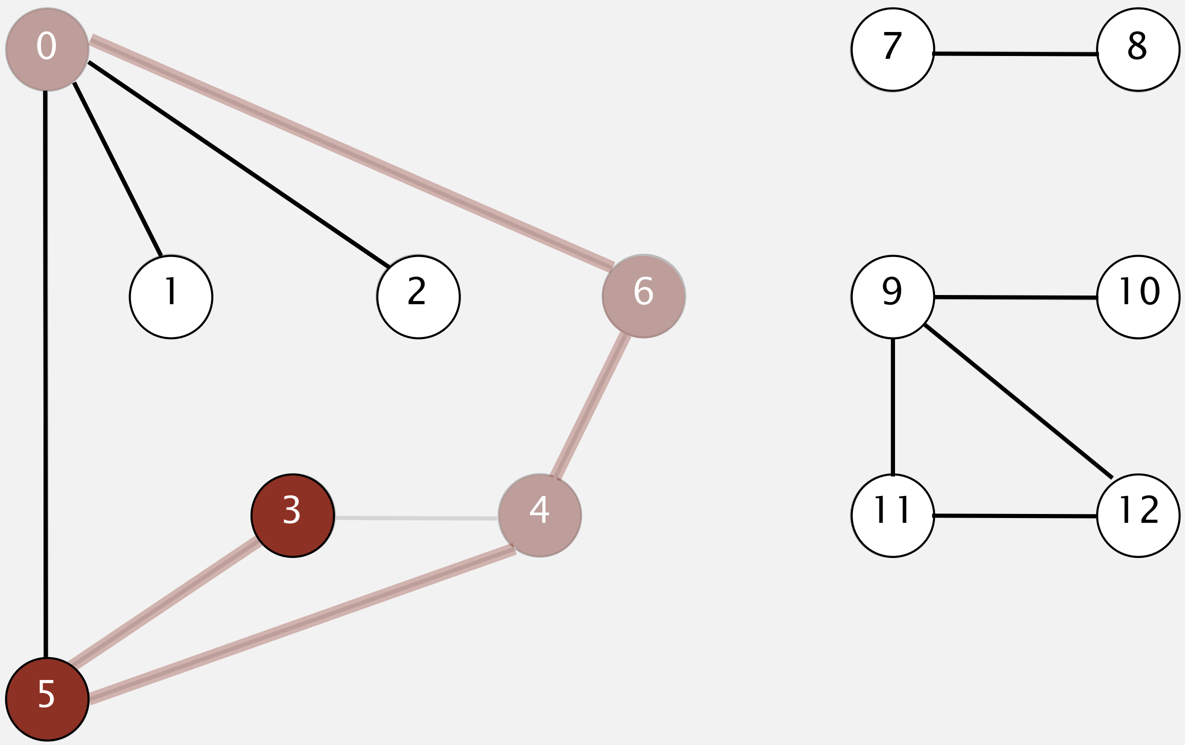
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 3: check 5, **check 4,** done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



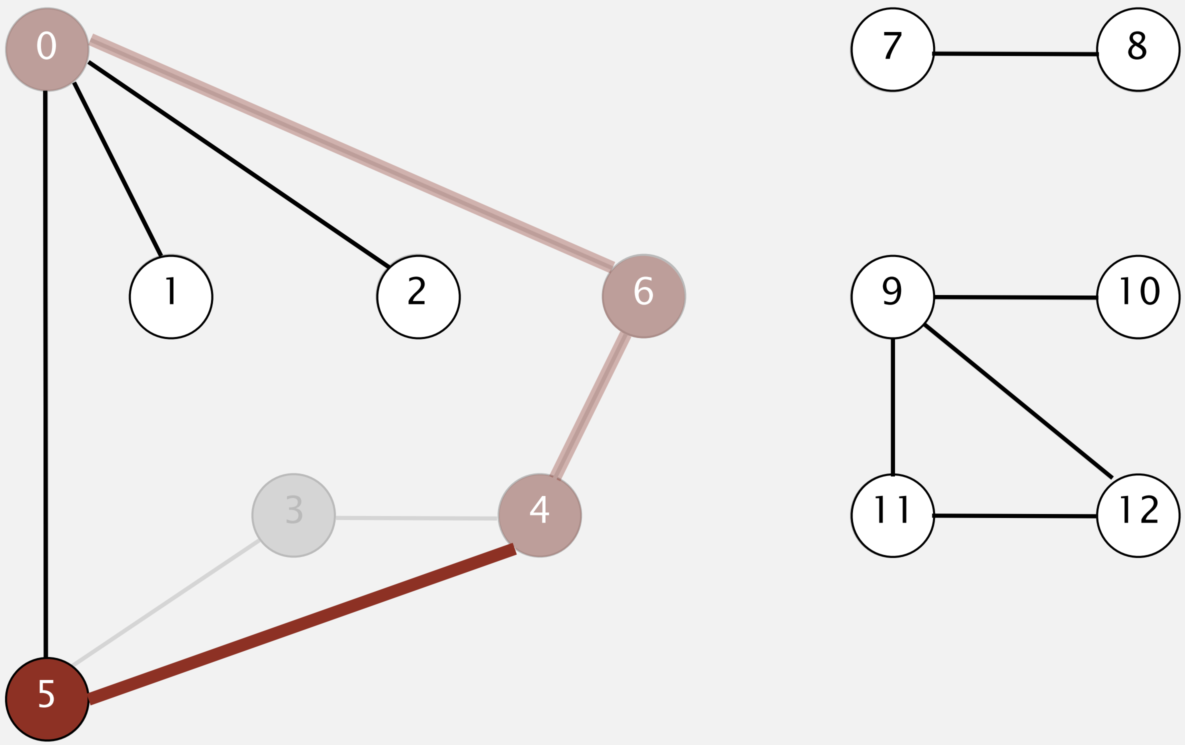
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 3: check 5, check 4, **done**

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



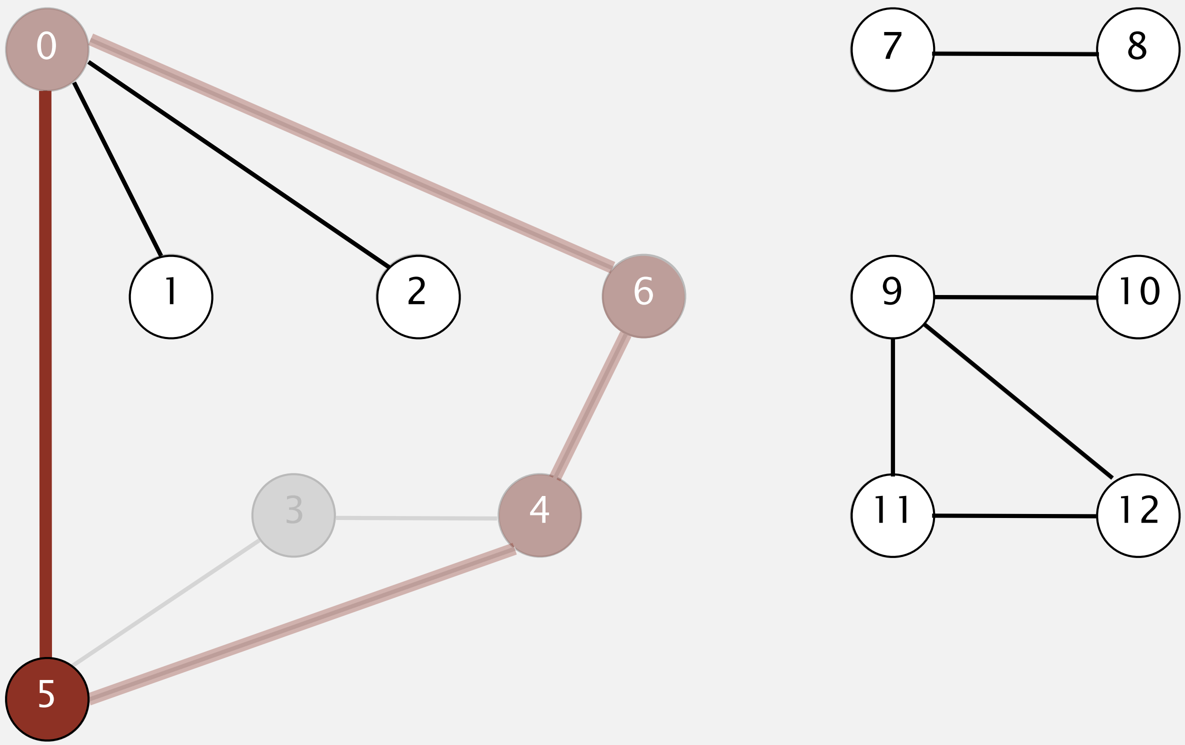
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 5: check 3, check 4, check 0, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



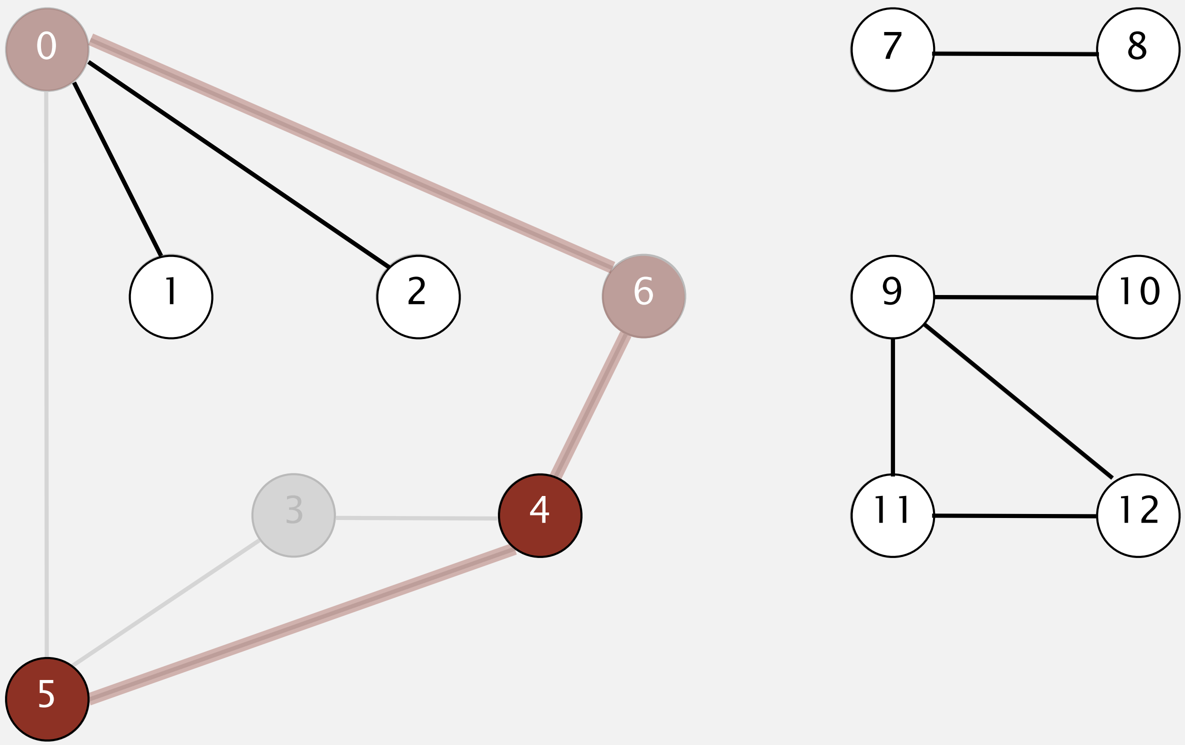
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 5: check 3, check 4, **check 0**, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



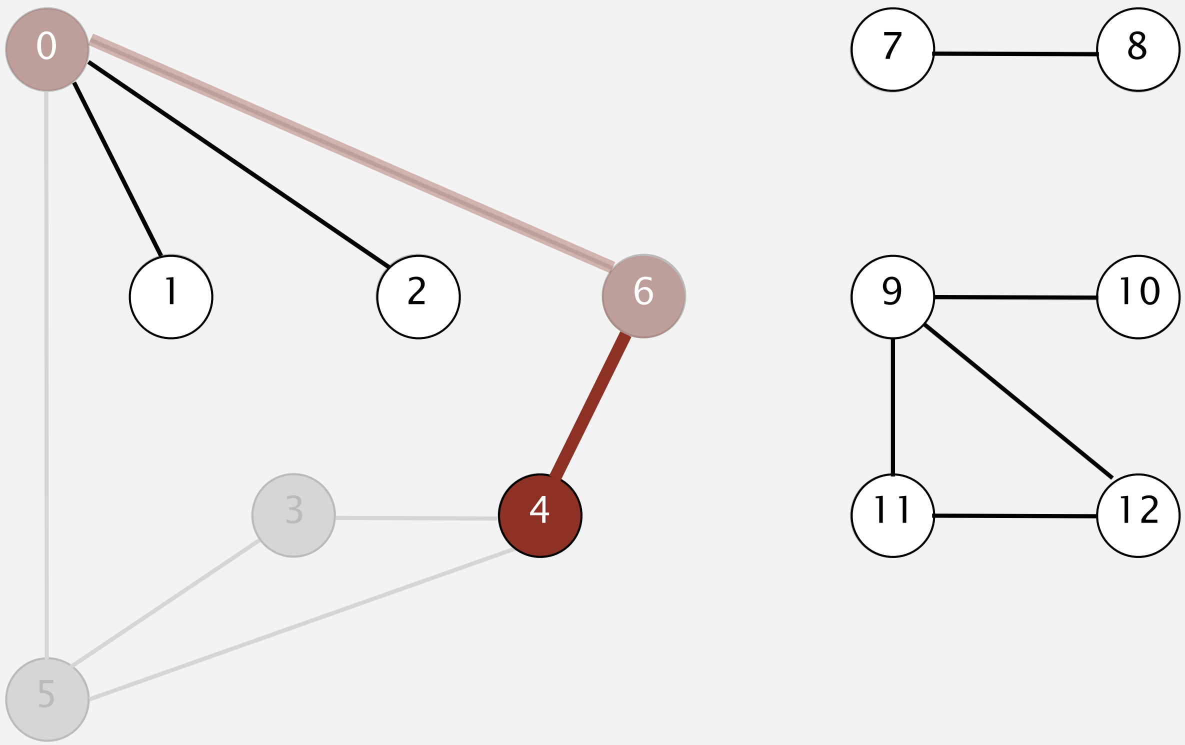
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 5: check 3, check 4, check 0, **done**

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



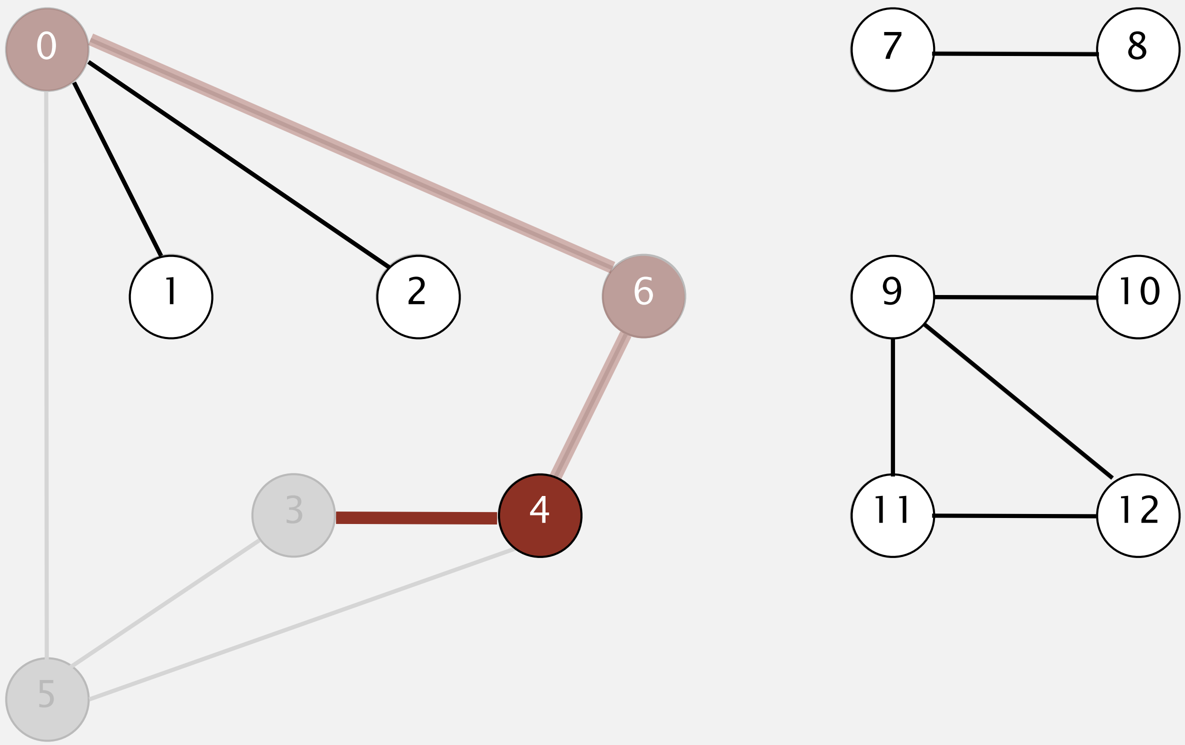
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 4: check 5, check 6, check 3, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



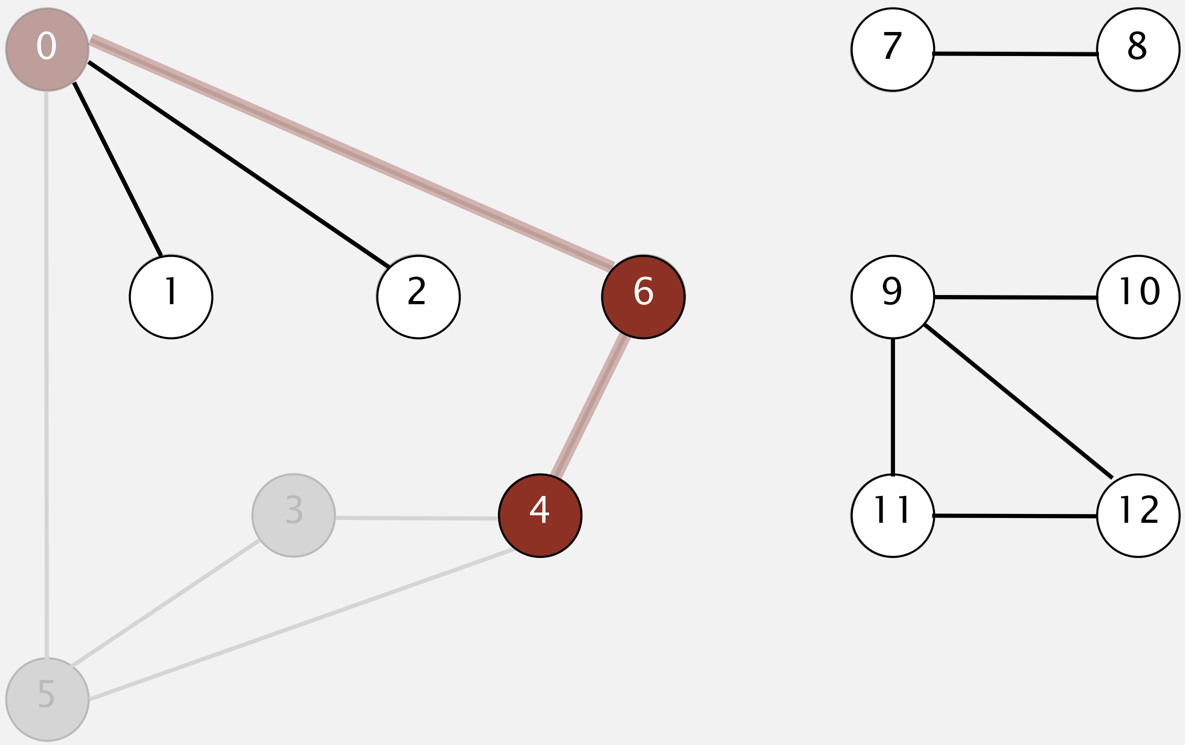
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 4: check 5, check 6, **check 3,** done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



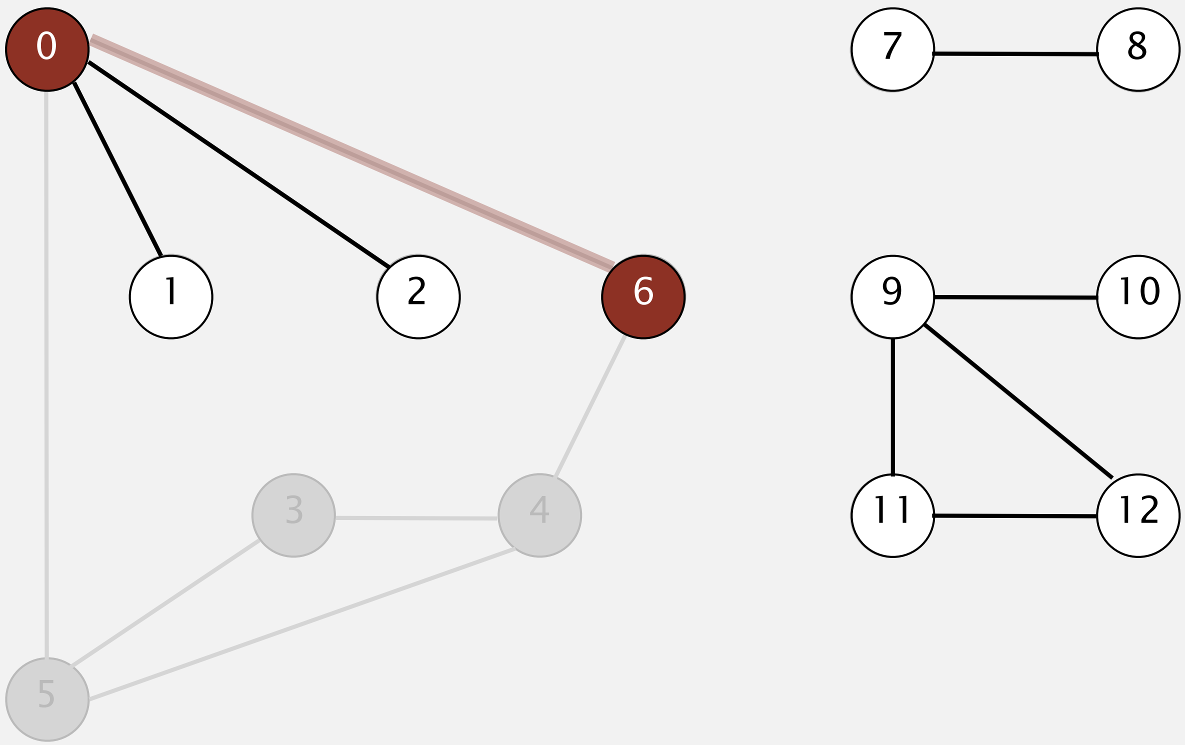
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 4: check 5, check 6, check 3, **done**

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



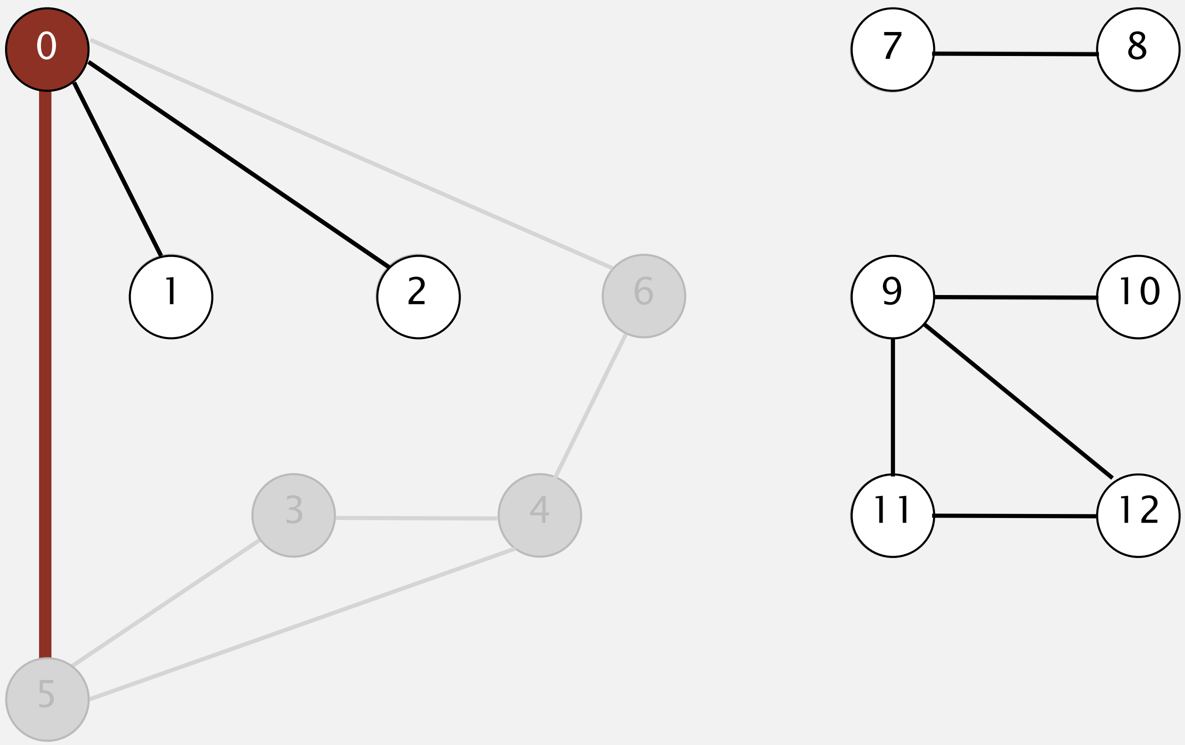
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 6: check 0, check 4, **done**

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



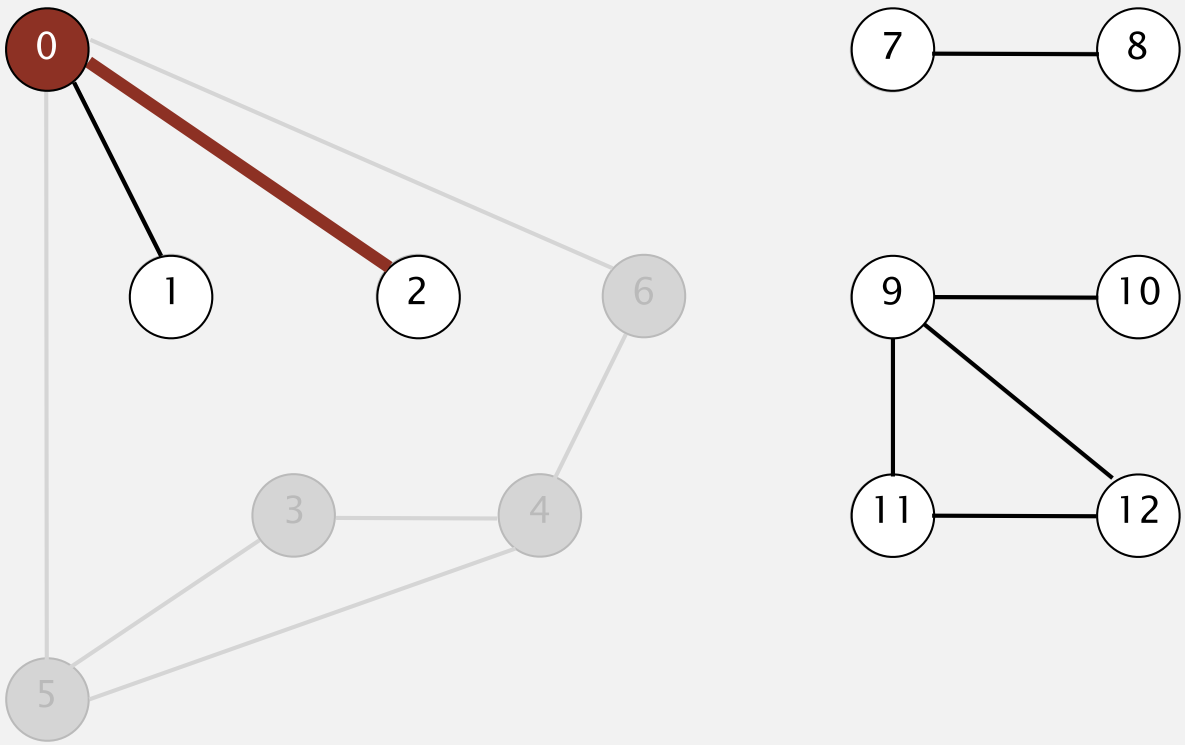
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 0: check 6, **check 5,** check 2, check 1, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



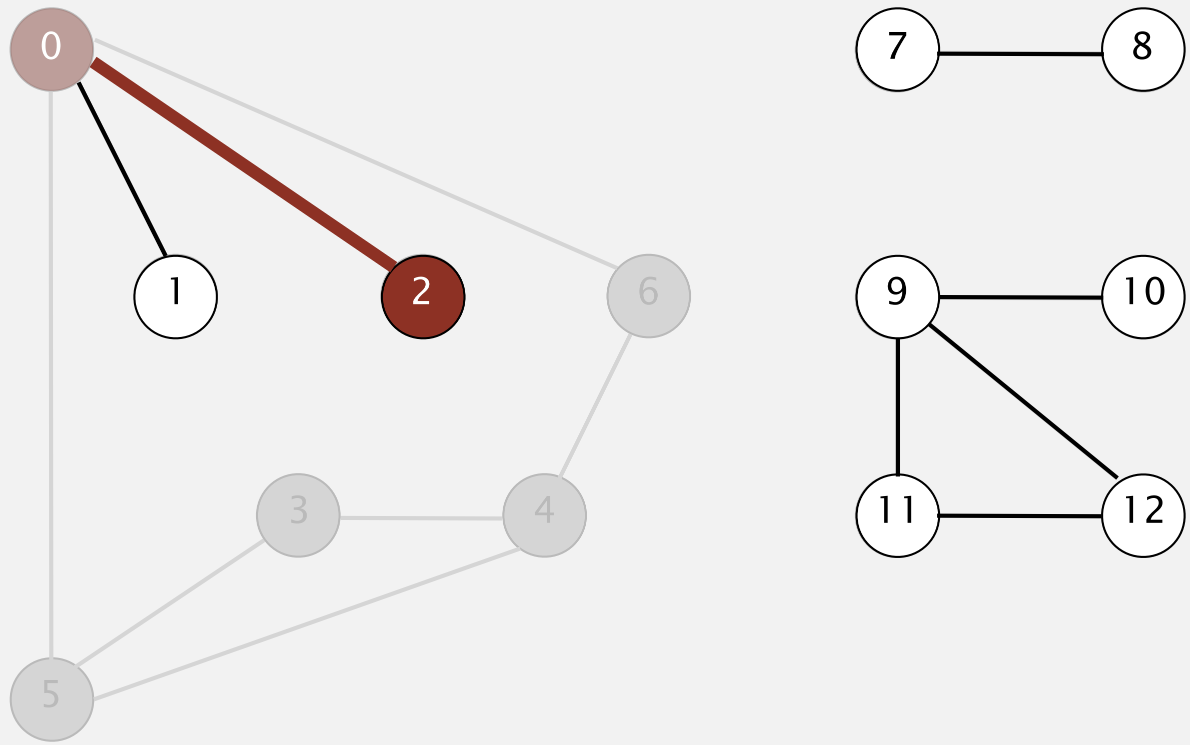
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 0: check 6, check 5, **check 2,** check 1, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



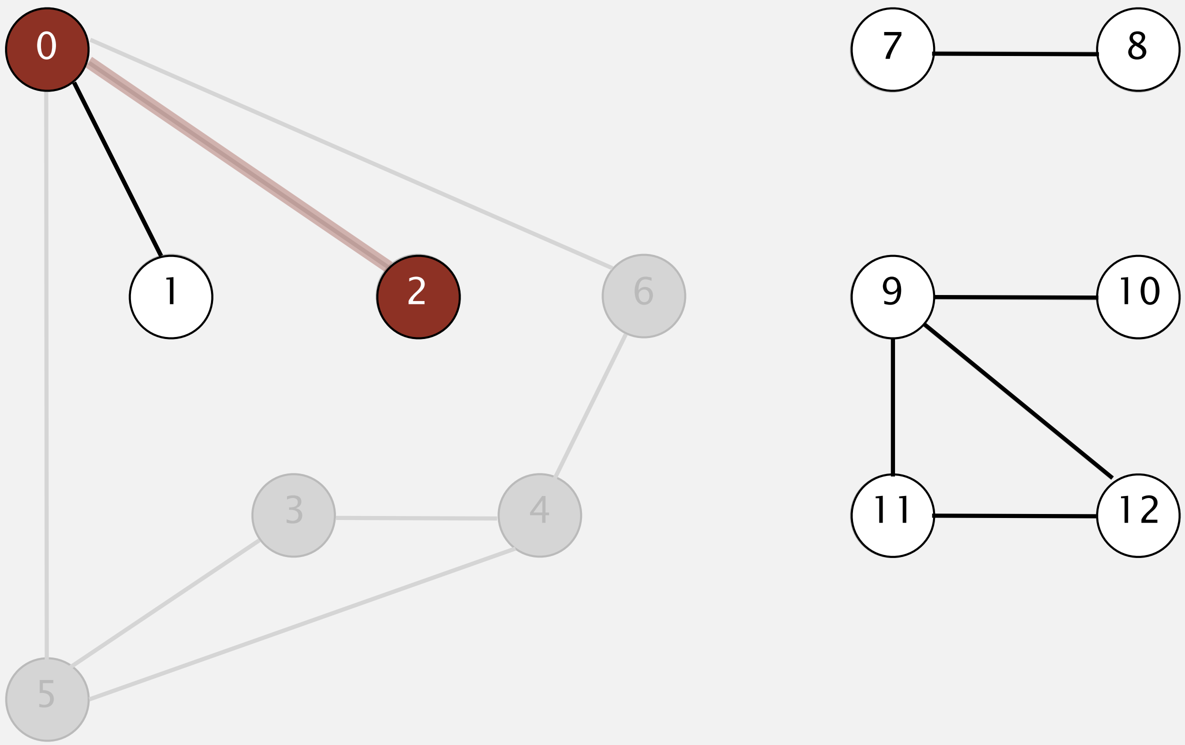
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 2: check 0, done

Depth-first search demo

To visit a vertex *v* :

- Mark vertex *v*.
- Recursively visit all unmarked vertices adjacent to *v*.



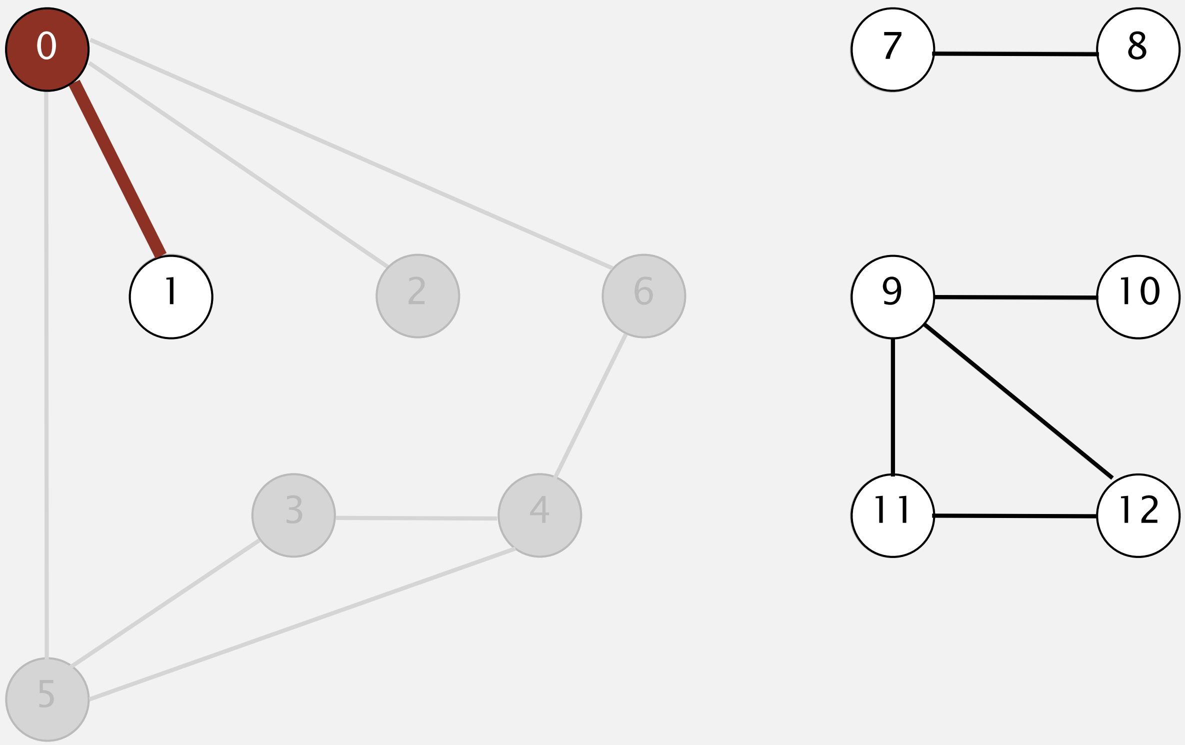
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 2: check 0, **done**

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



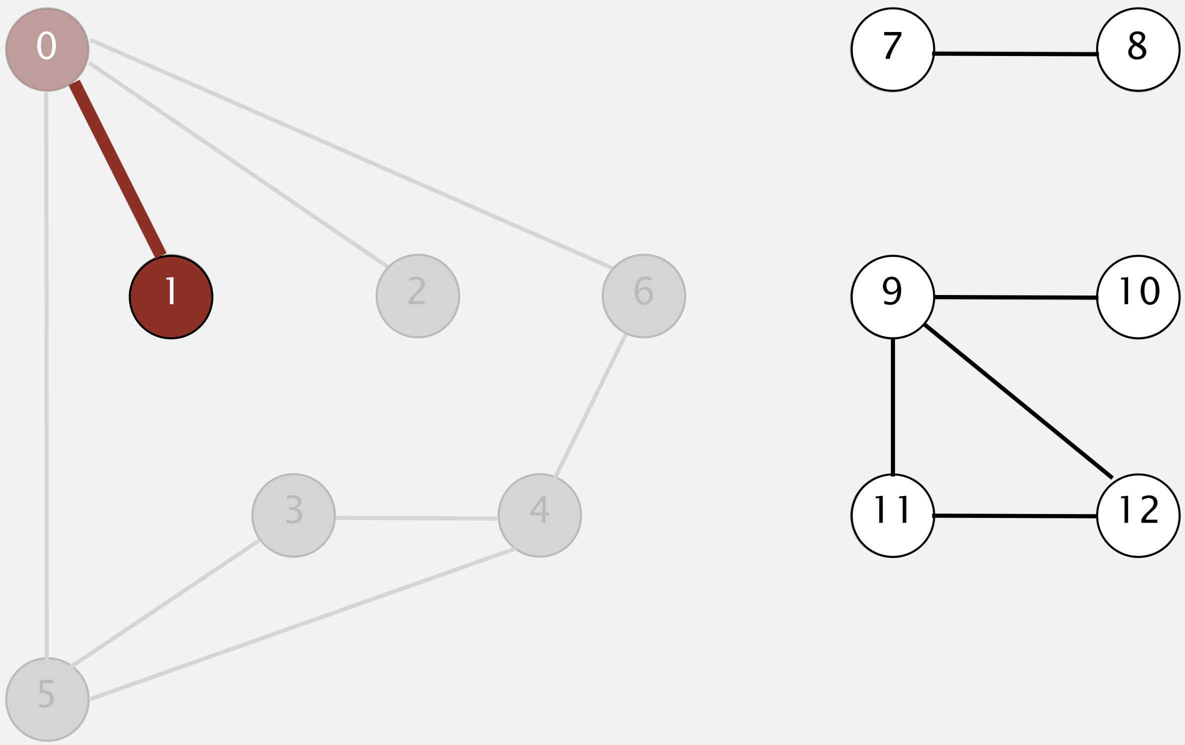
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 0: check 6, check 5, check 2, **check 1,** done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



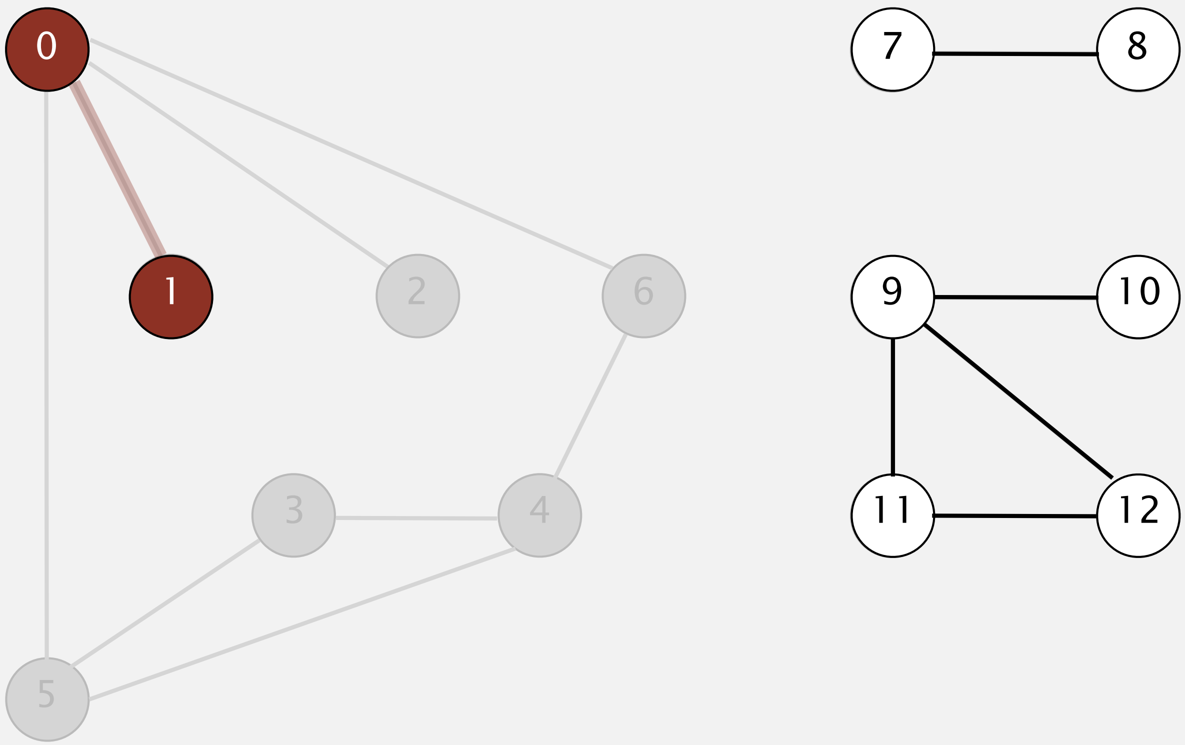
v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 1: check 0, done

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



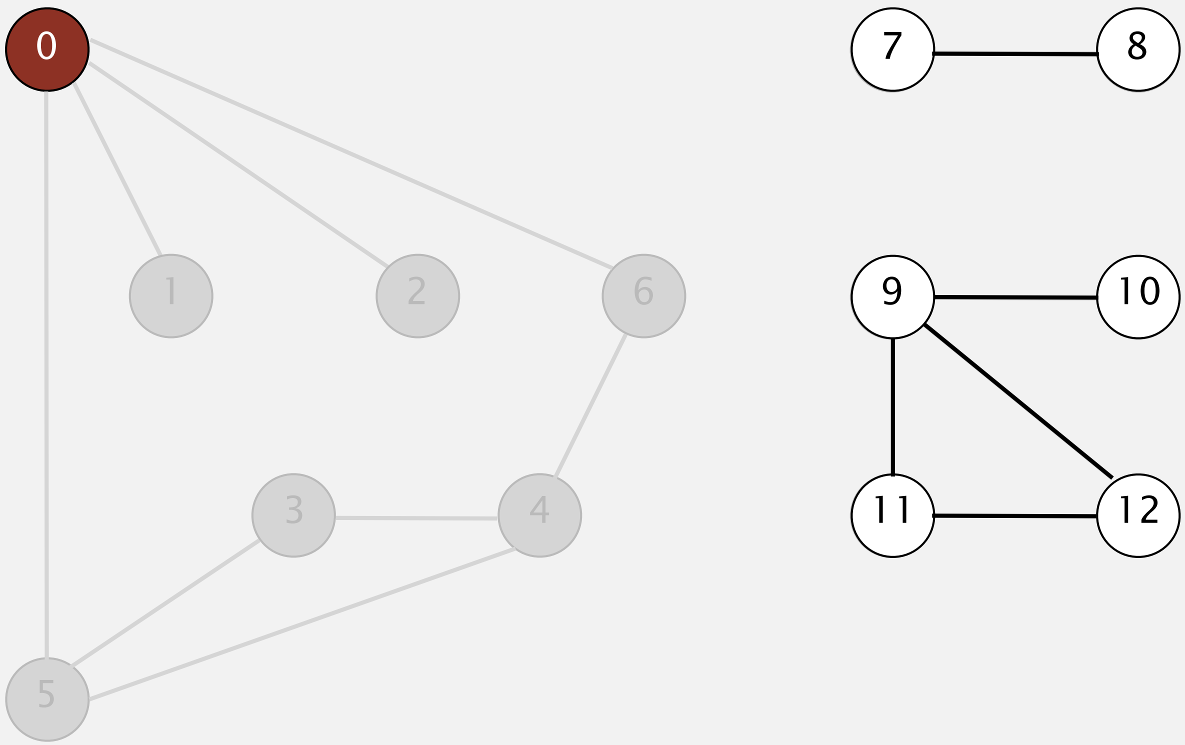
v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 1: check 0, **done**

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



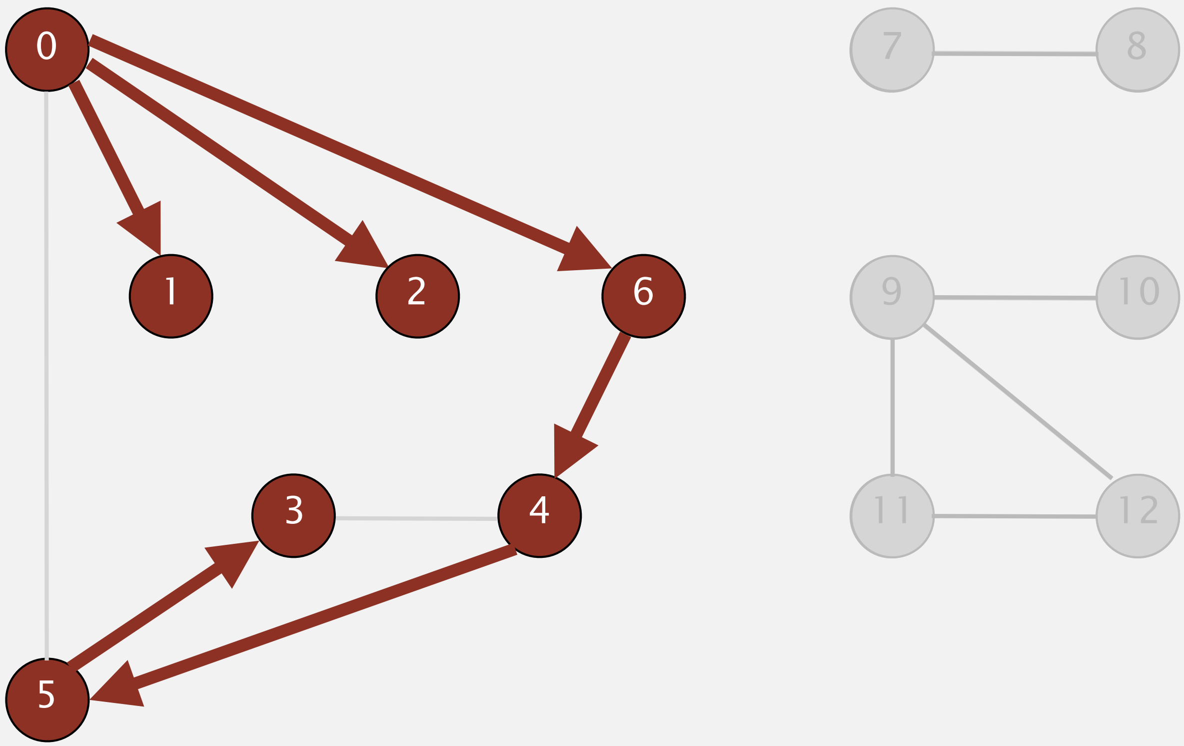
v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 0: check 6, check 5, check 2, check 1, **done**

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



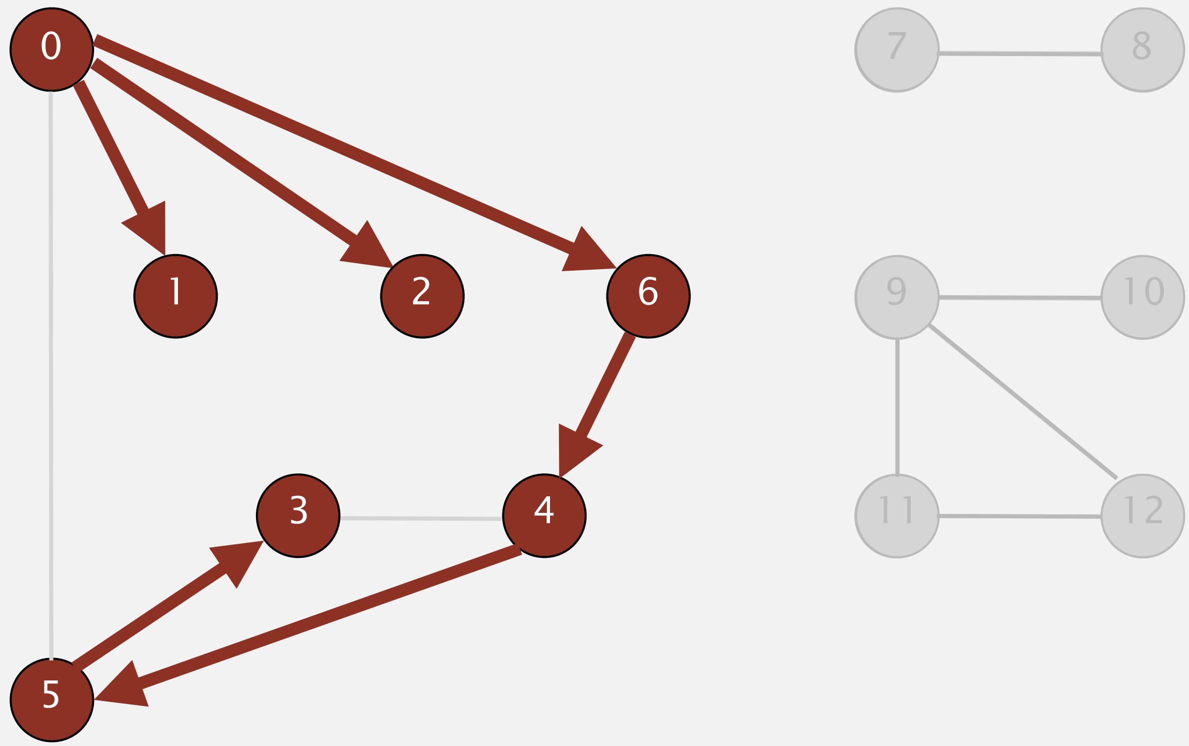
vertices reachable from 0

v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

vertices reachable from 0

Design pattern for graph processing

Design pattern. Decouple graph data type from graph processing.

- Create a `Graph` object.
- Pass the `Graph` to a graph-processing routine.
- Query the graph-processing routine for information.

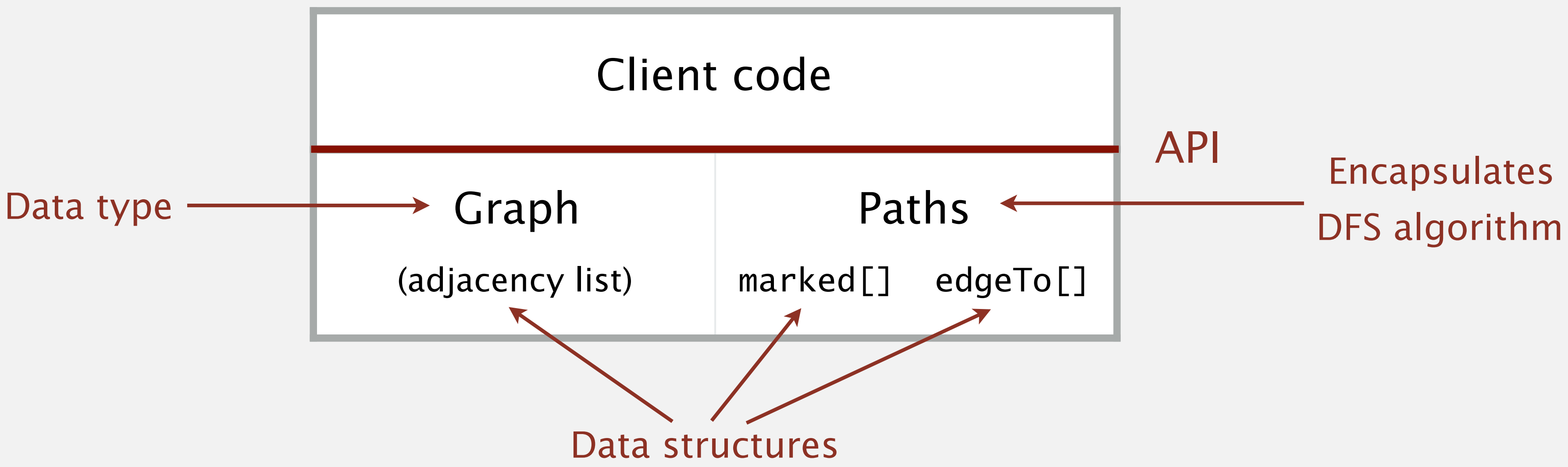
	`public class Paths`	
	`Paths(Graph G, int s)`	*find paths in G from source s*
`boolean`	`hasPathTo(int v)`	*is there a path from s to v?*
`Iterable<Integer>`	`pathTo(int v)`	*path from s to v; null if no such path*

```
Paths paths = new Paths(G, s);
for (int v = 0; v < G.V(); v++)
   if (paths.hasPathTo(v))
      StdOut.println(v);
```

print all vertices connected to s

Modularity

As usual, client doesn't care about implementation details, including data structures used



Depth-first search: data structures

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.

Data structures.

- Boolean array `marked[]` to mark vertices.
- Integer array `edgeTo[]` to keep track of paths.
  `(edgeTo[w] == v)` means that edge `v-w` taken to discover vertex `w`
- Function-call stack for recursion.

Depth-first search: Java implementation

```
public class DepthFirstPaths
{

   private boolean[] marked;
   private int[] edgeTo;
   private int s;

   public DepthFirstPaths(Graph G, int s)
   {
     ...
     dfs(G, s);
   }

   private void dfs(Graph G, int v)
   {
     marked[v] = true;
     for (int w : G.adj(v))
        if (!marked[w])
        {
           edgeTo[w] = v;
           dfs(G, w);
        }
   }
}
```

marked[v] = true if v connected to s

edgeTo[v] = previous vertex on path from s to v

initialize data structures

find vertices connected to s

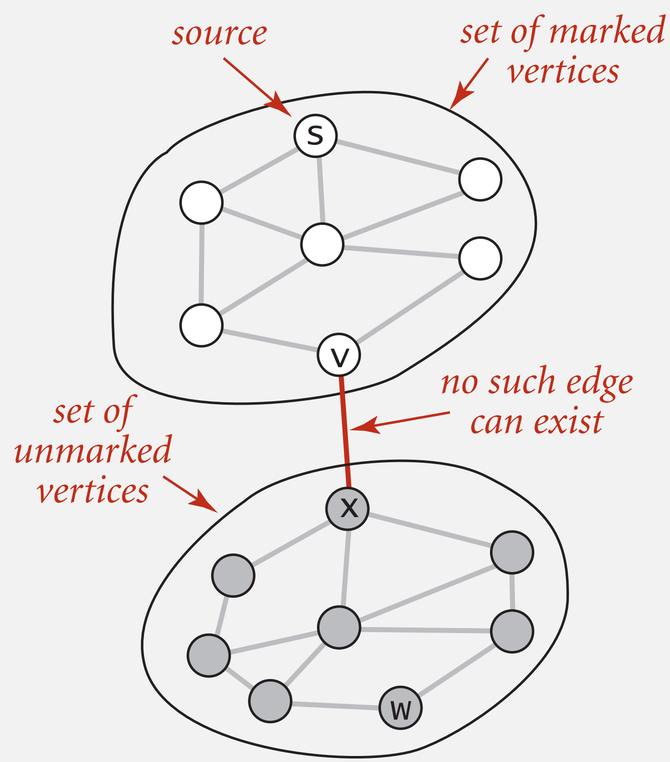
recursive DFS does the work

Depth-first search: properties

Proposition. DFS marks all vertices connected to s in time proportional to the sum of their degrees (plus time to initialize the `marked[]` array).

Pf. [correctness]

- If w marked, then w connected to s (why?)
- If w connected to s, then w marked.
  (if w unmarked, then consider last edge on a path from s to w that goes from a marked vertex to an unmarked one).



Pf. [running time]

Each vertex connected to s is visited once.

Depth-first search: properties

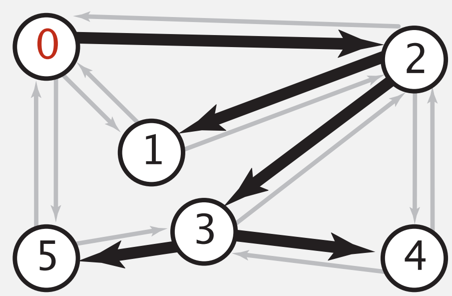
Proposition. After DFS, can check if vertex v is connected to s in constant time and can find v–s path (if one exists) in time proportional to its length.

Pf. `edgeTo[]` is parent-link representation of a tree rooted at vertex `s`.

```
public boolean hasPathTo(int v)
{  return marked[v];  }

public Iterable<Integer> pathTo(int v)
{
   if (!hasPathTo(v)) return null;
   Stack<Integer> path = new Stack<Integer>();
   for (int x = v; x != s; x = edgeTo[x])
      path.push(x);
   path.push(s);
   return path;
}
```

Skipped in class



edgeTo[]

0	
1	2
2	0
3	2
4	3
5	3

Algorithms

Robert Sedgewick | Kevin Wayne

http://algs4.cs.princeton.edu

4.1 Undirected Graphs

Breadth-first search

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.

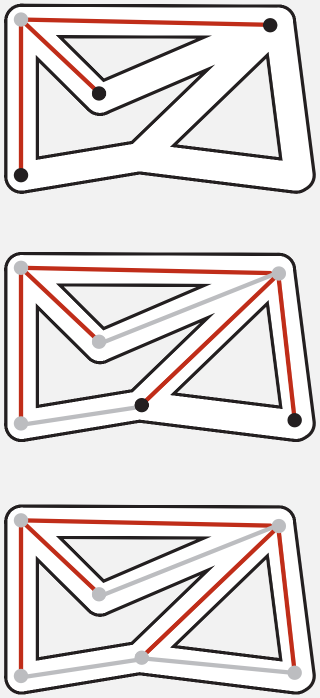
BFS (from source vertex s)

Enqueue s, mark s as visited.

While queue is not empty:

- dequeue v

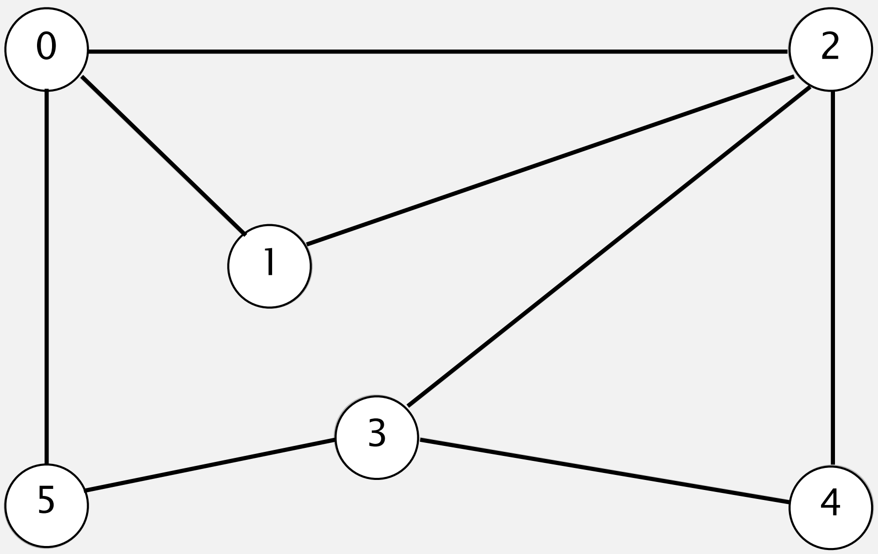
- enqueue each of v's unmarked neighbors, and mark them.



Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



graph G

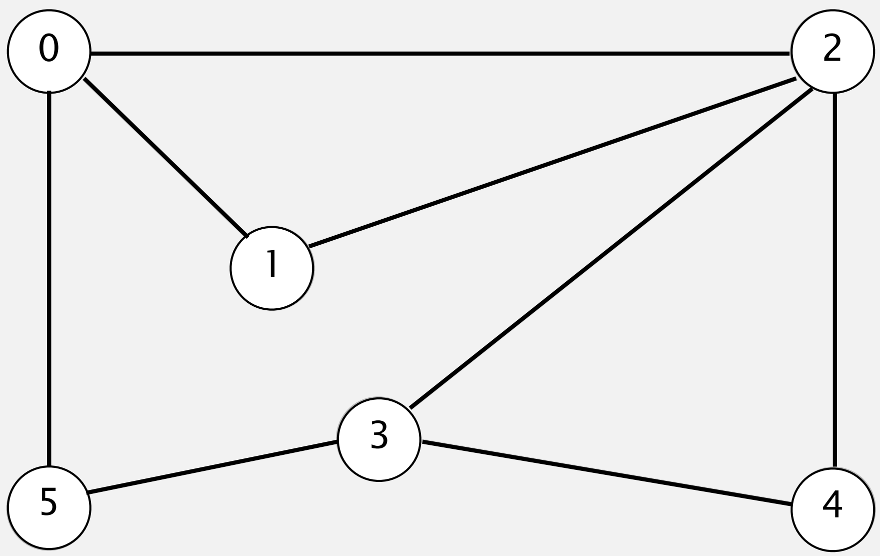
`tinyCG.txt`

$V \rightarrow$ 6
8 $\leftarrow E$
0 5
2 4
2 3
1 2
0 1
3 4
3 5
0 2

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



graph G

tinyCG.txt

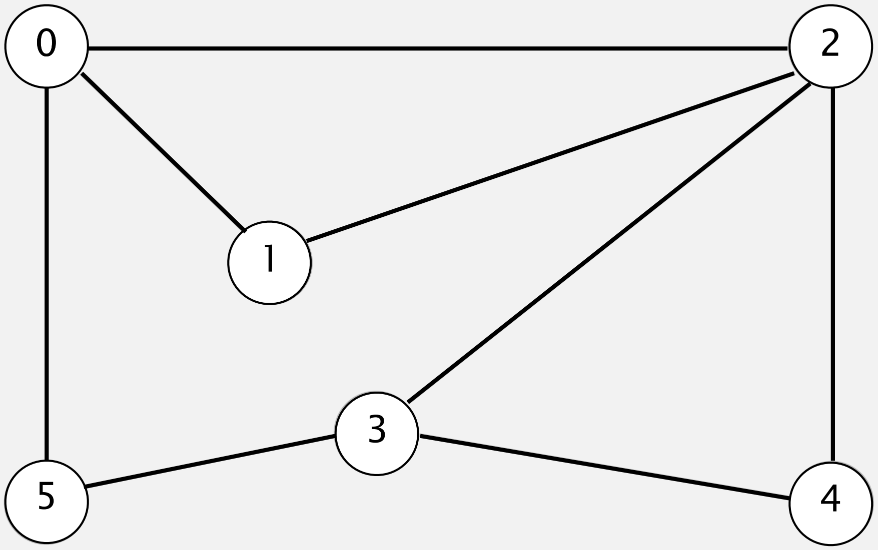
V → 6
E → 8

```
6
8
0 5
2 4
2 3
1 2
0 1
3 4
3 5
0 2
```

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

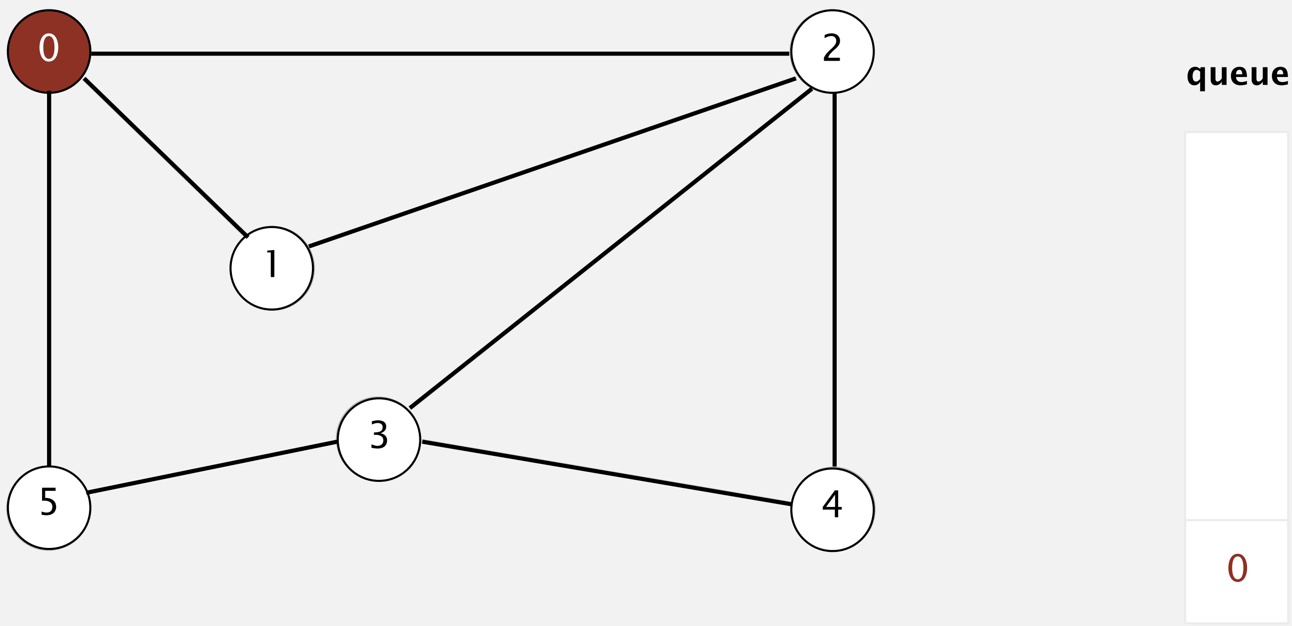
v	edgeTo[]	distTo[]
0	–	0
1	–	–
2	–	–
3	–	–
4	–	–
5	–	–

add 0 to queue

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



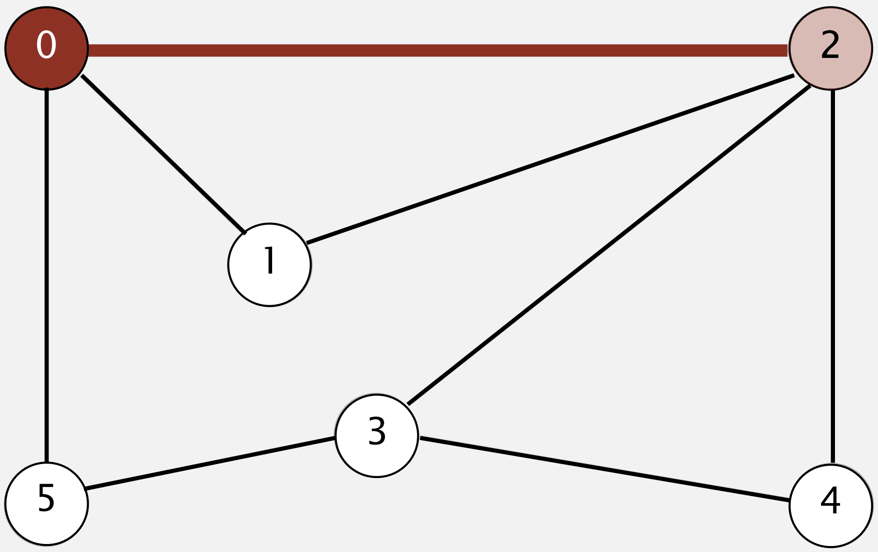
v	edgeTo[]	distTo[]
0	–	0
1	–	–
2	–	–
3	–	–
4	–	–
5	–	–

dequeue 0

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

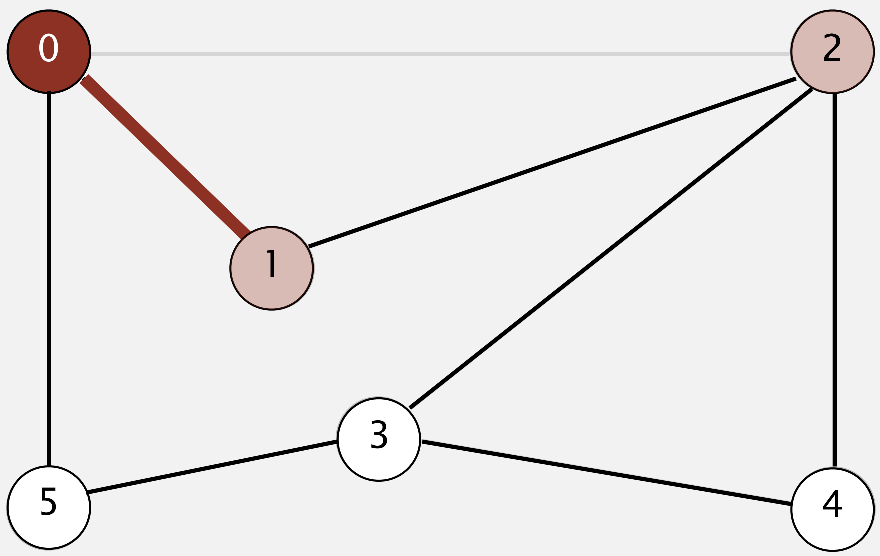
v	edgeTo[]	distTo[]
0	–	0
1	–	–
2	0	1
3	–	–
4	–	–
5	–	–

dequeue 0

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

2

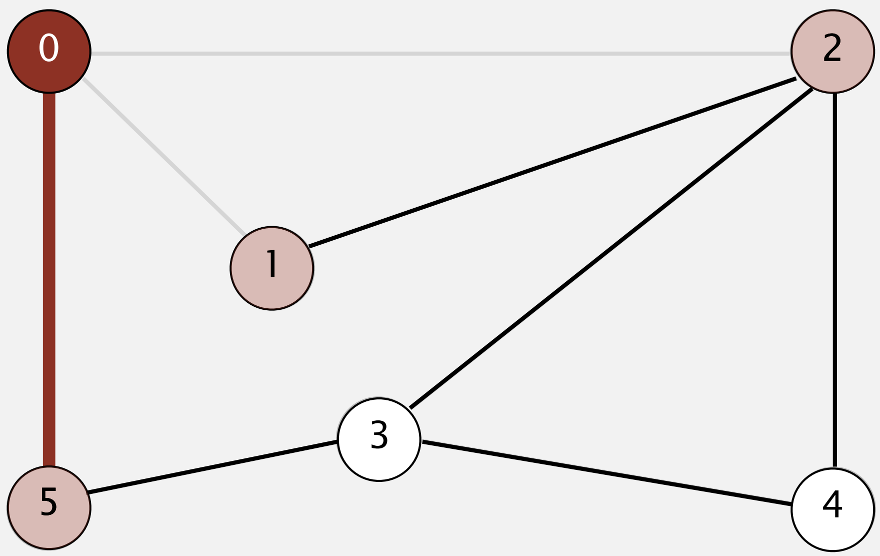
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	–	–
4	–	–
5	–	–

dequeue 0

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

1
2

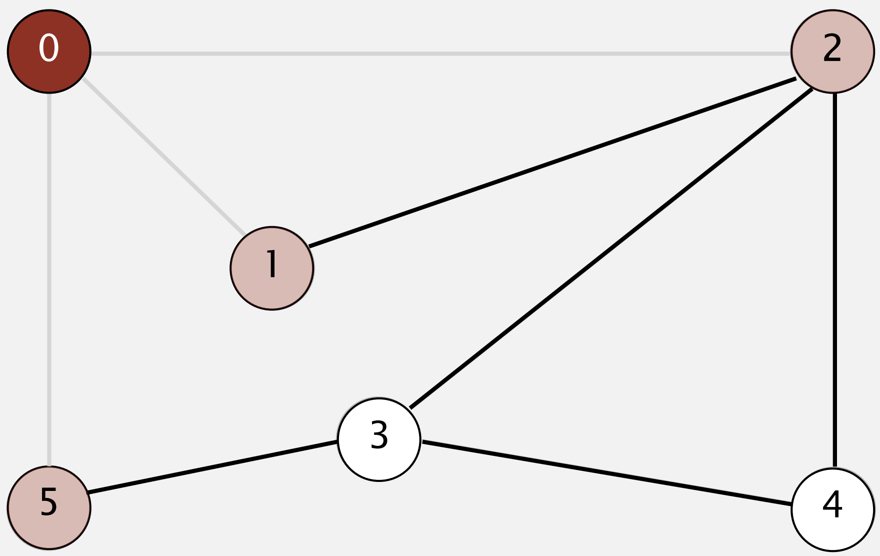
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	–	–
4	–	–
5	0	1

dequeue 0

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

5
1
2

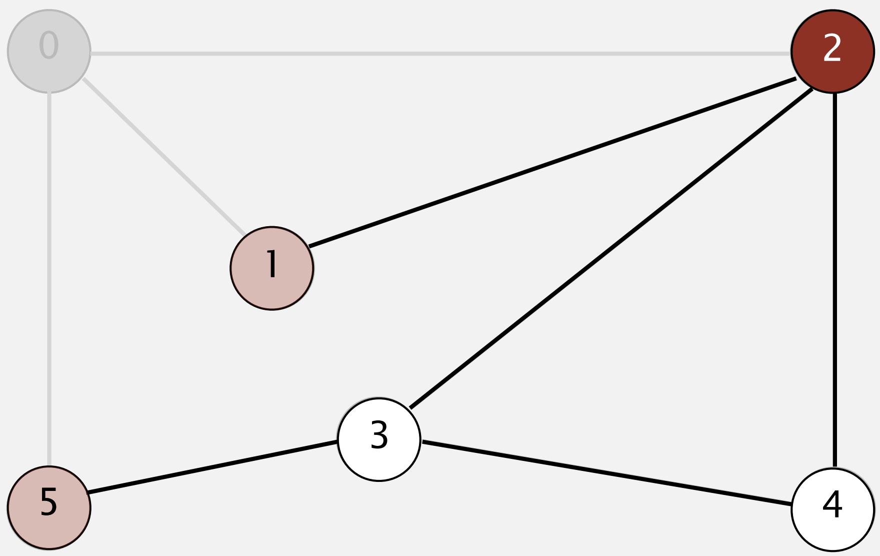
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	–	–
4	–	–
5	0	1

0 done

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
5
1
2

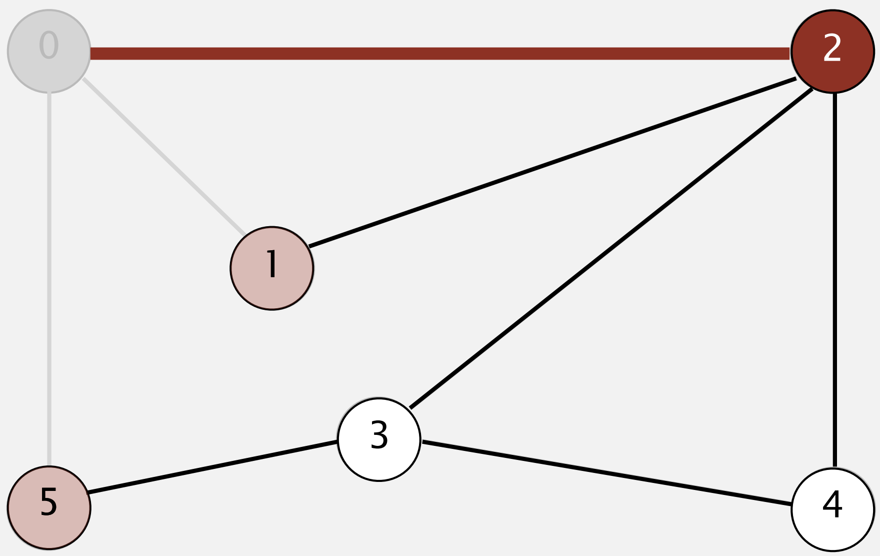
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	–	–
4	–	–
5	0	1

dequeue 2

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
5
1

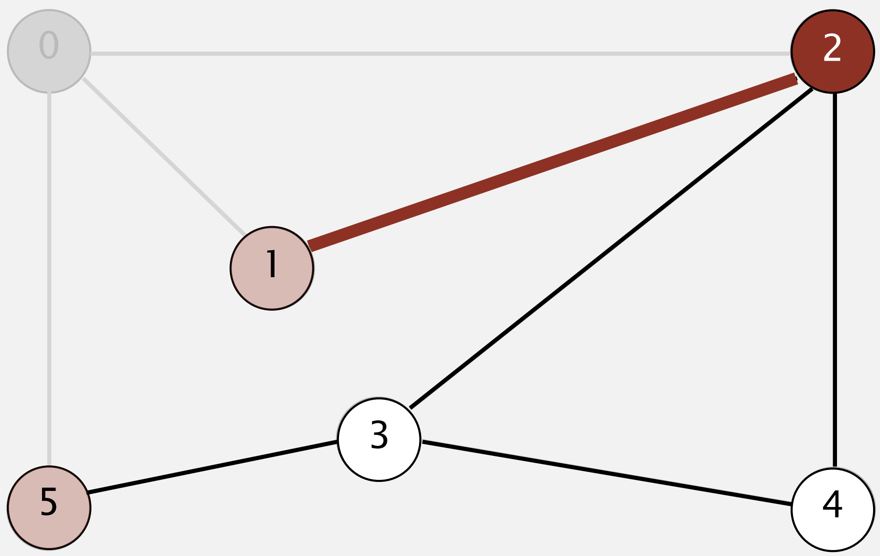
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	–	–
4	–	–
5	0	1

dequeue 2

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
5
1

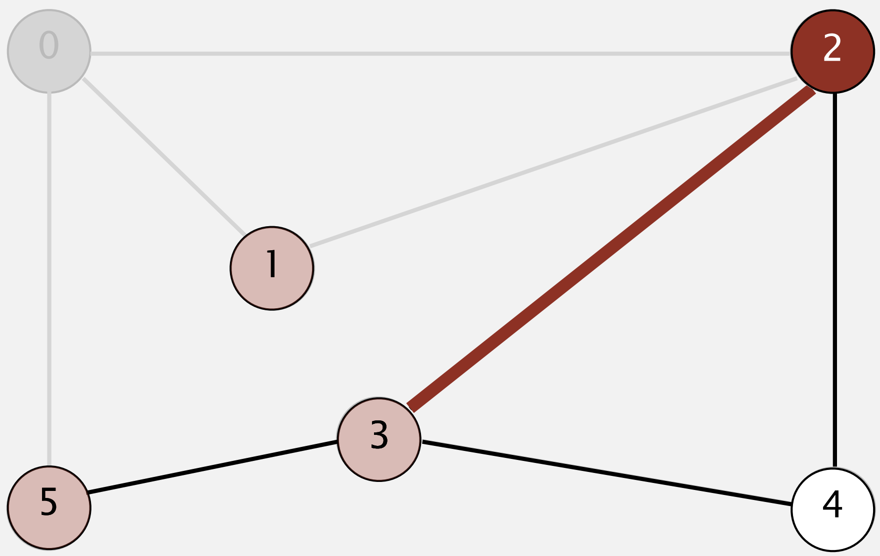
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	–	–
4	–	–
5	0	1

dequeue 2

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

5
1

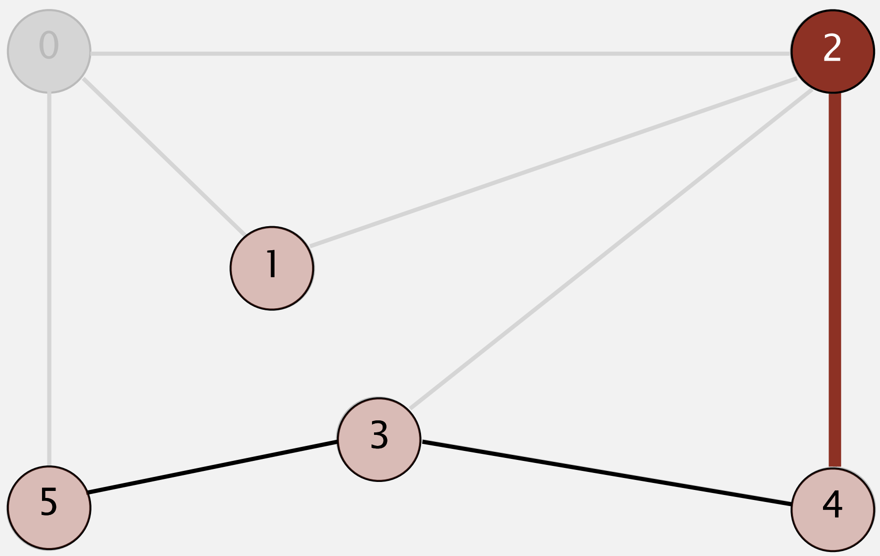
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	–	–
5	0	1

dequeue 2

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
3
5
1

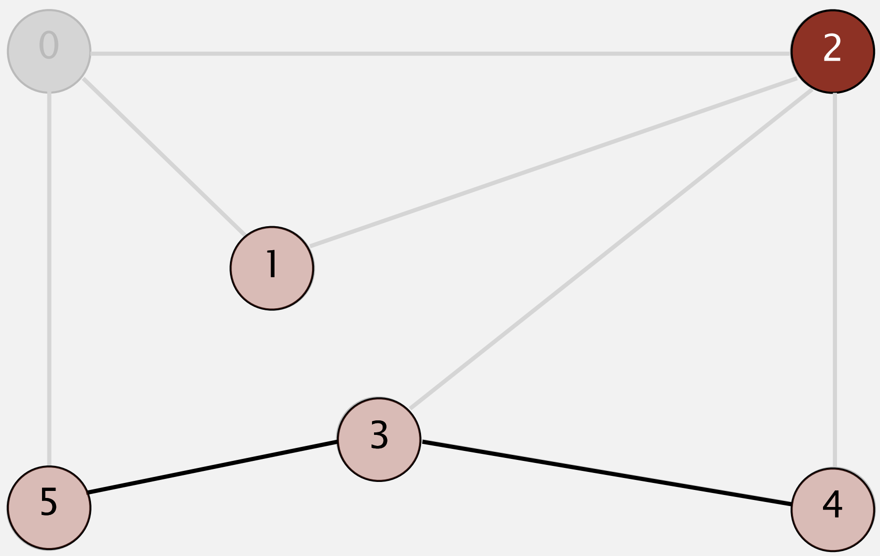
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 2

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
4
3
5
1

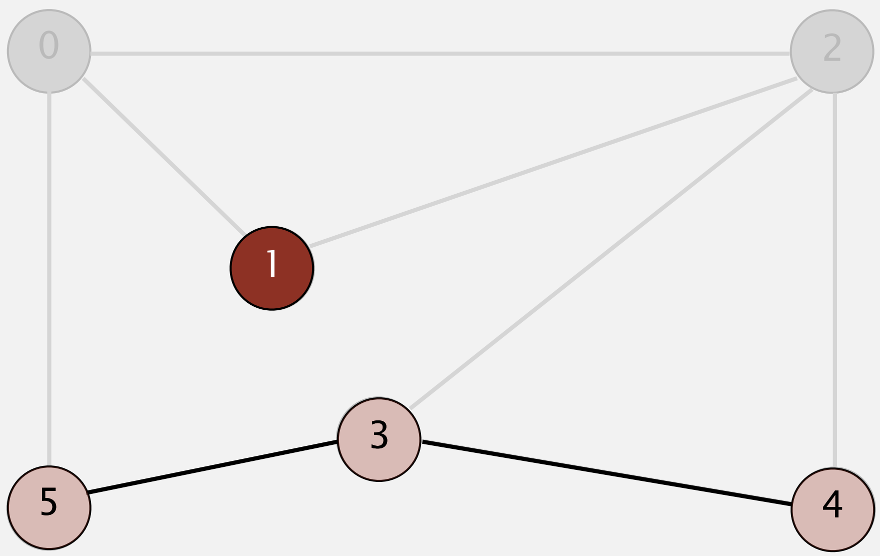
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

2 done

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

4
3
5
1

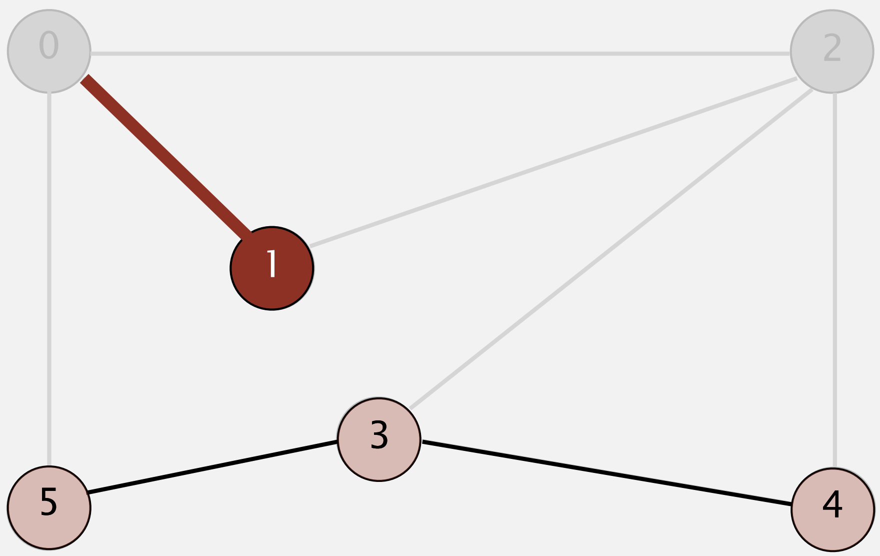
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 1

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
4
3
5

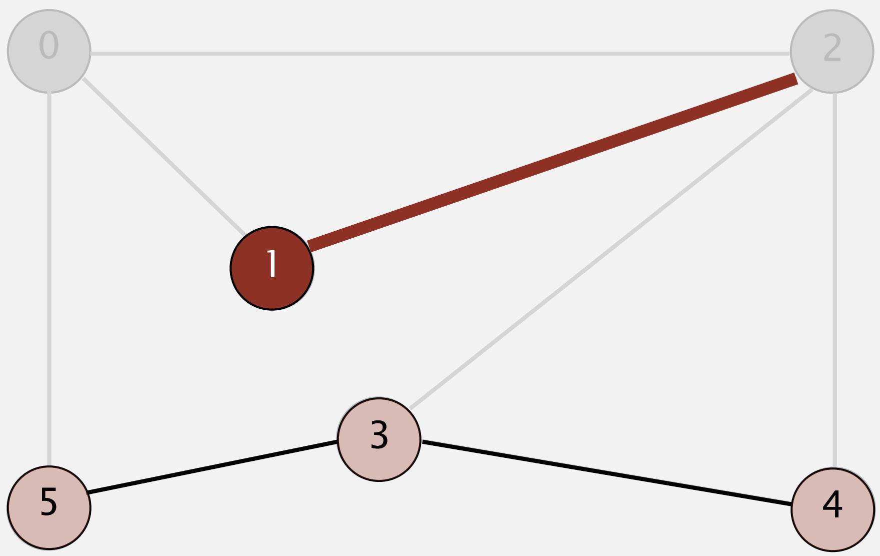
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 1

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
4
3
5

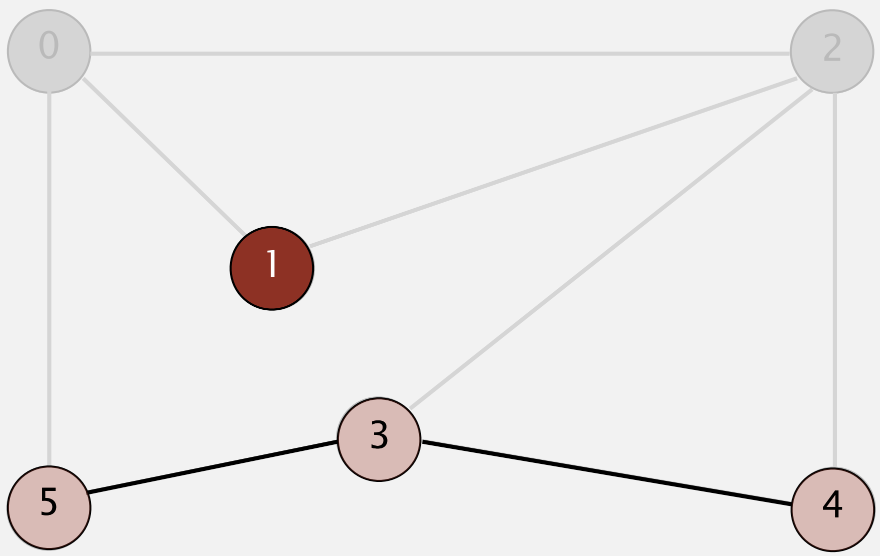
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 1

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
4
3
5

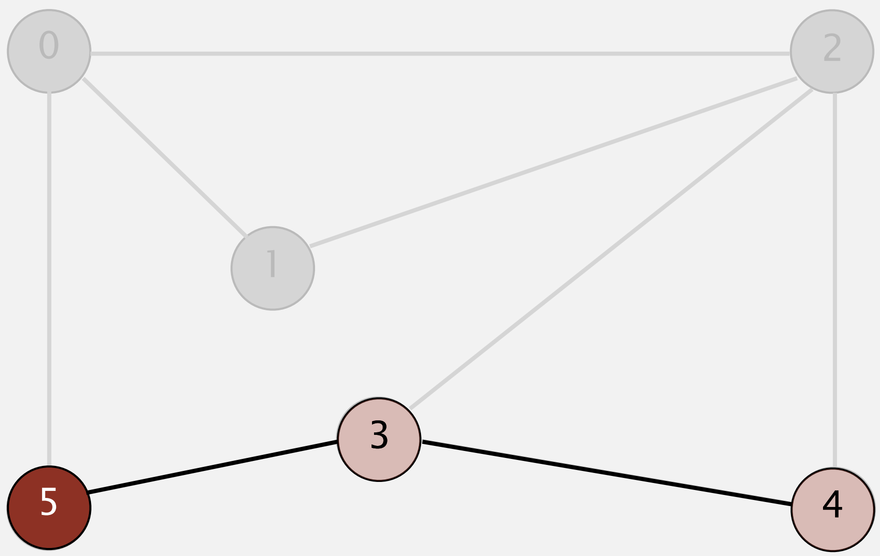
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

1 done

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

4
3
5

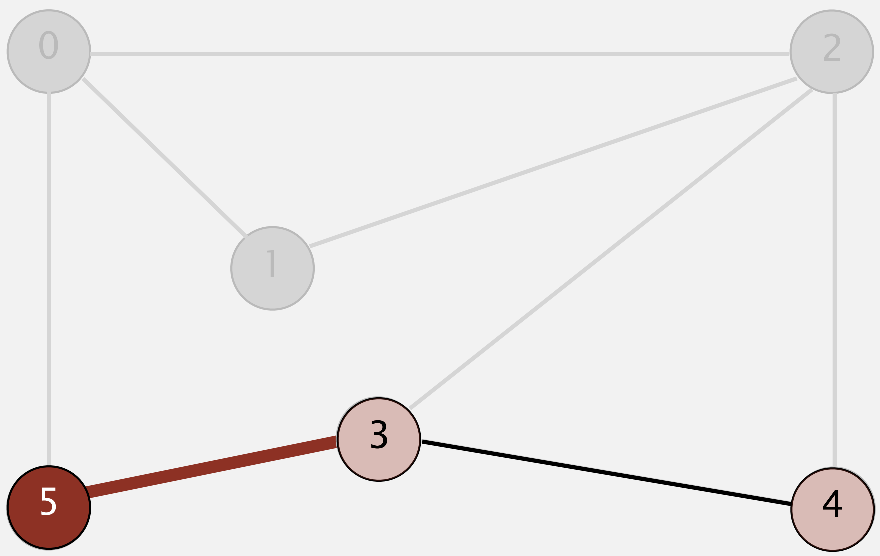
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 5

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

4
3

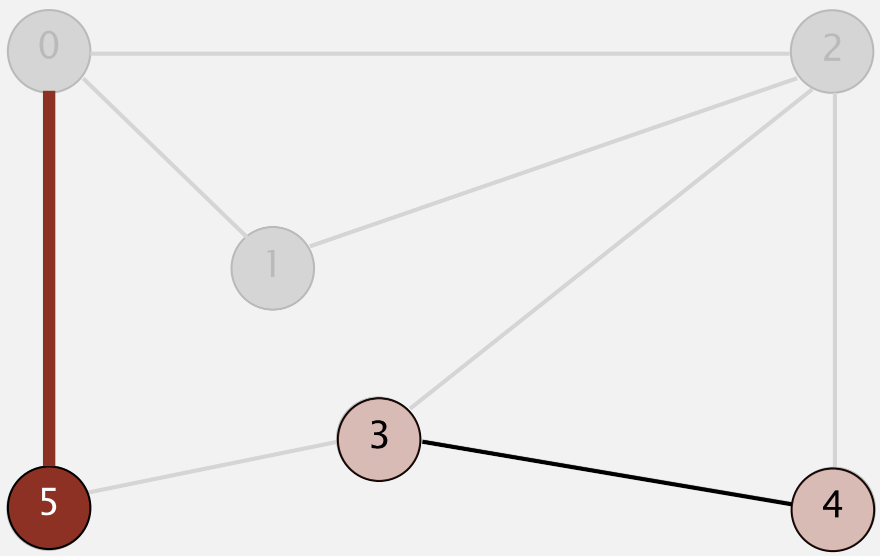
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 5

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
4
3

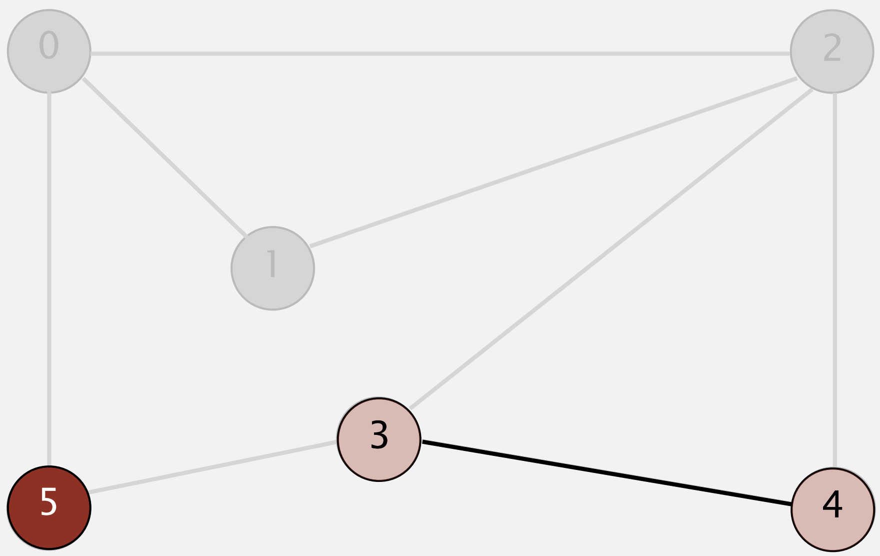
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 5

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

4
3

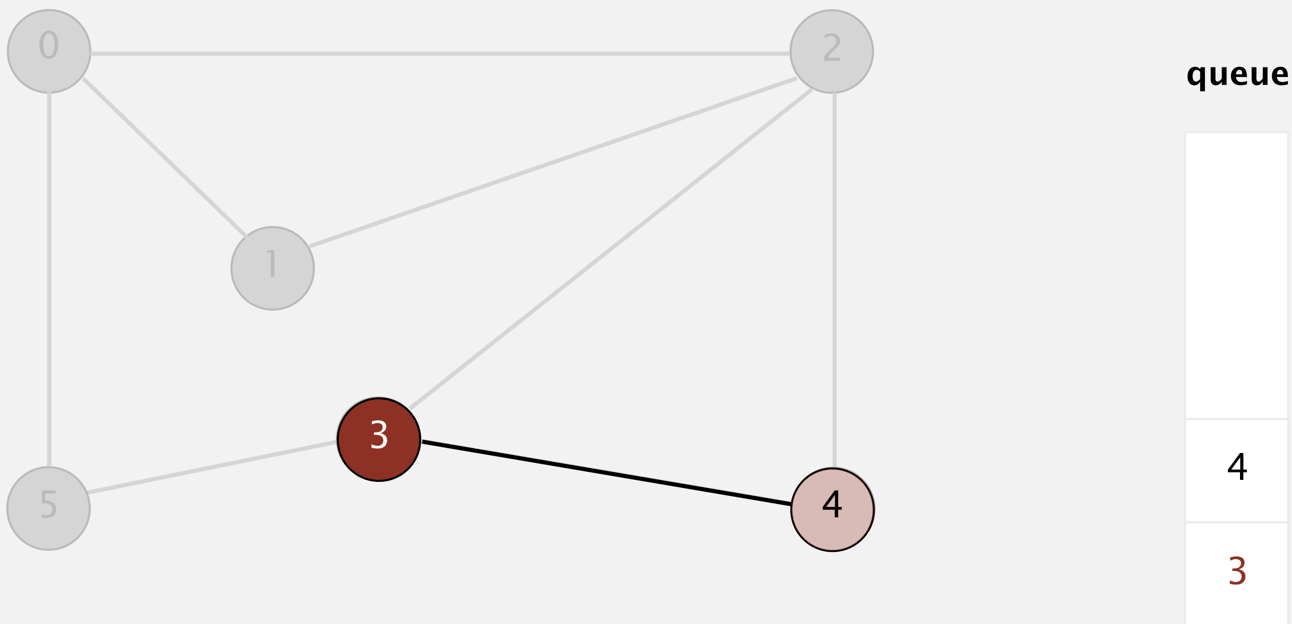
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

5 done

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



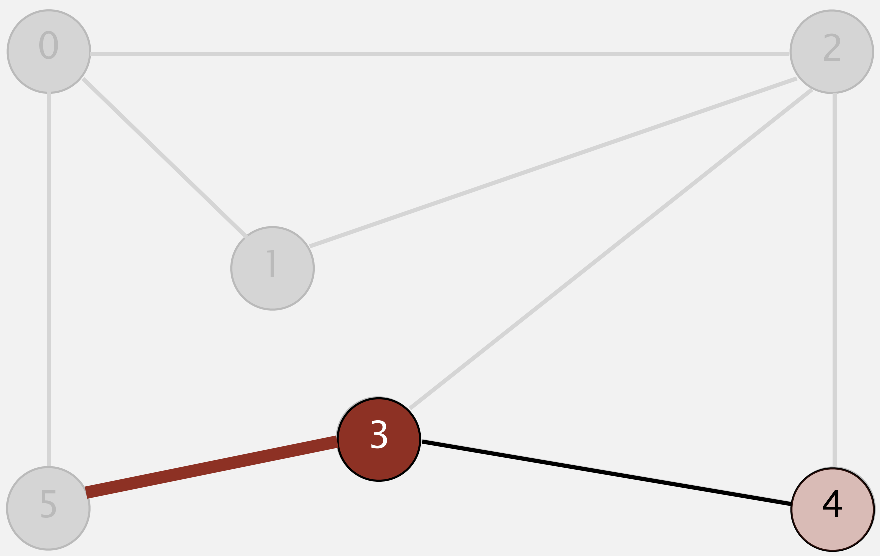
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

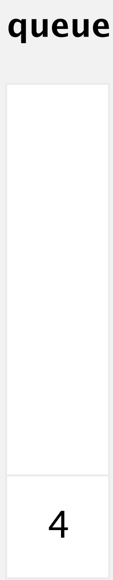
dequeue 3

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.





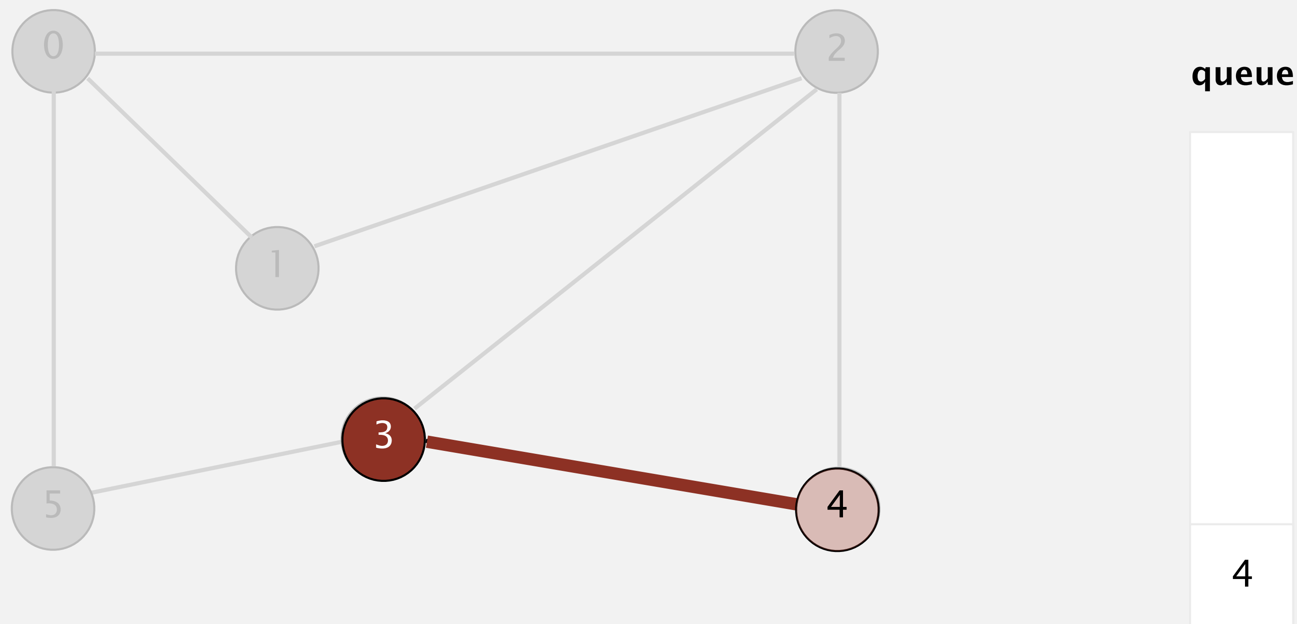
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 3

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



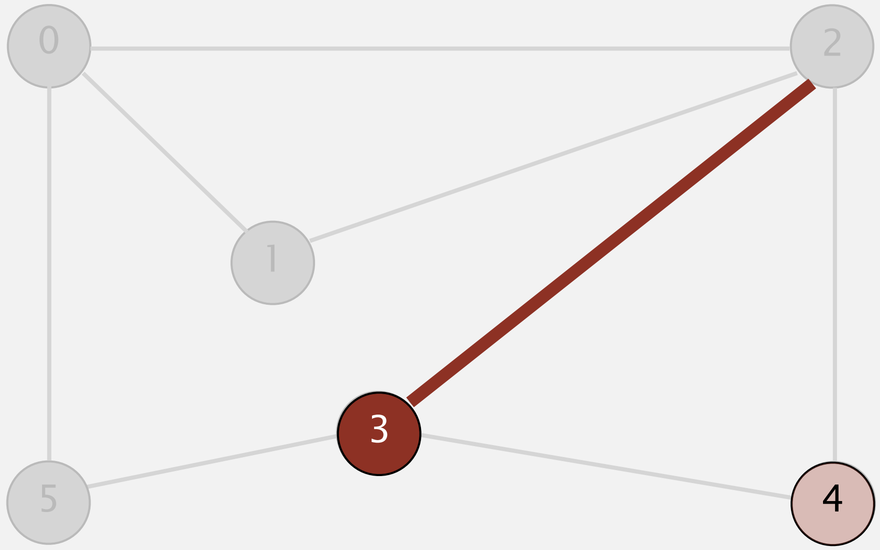
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 3

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

4

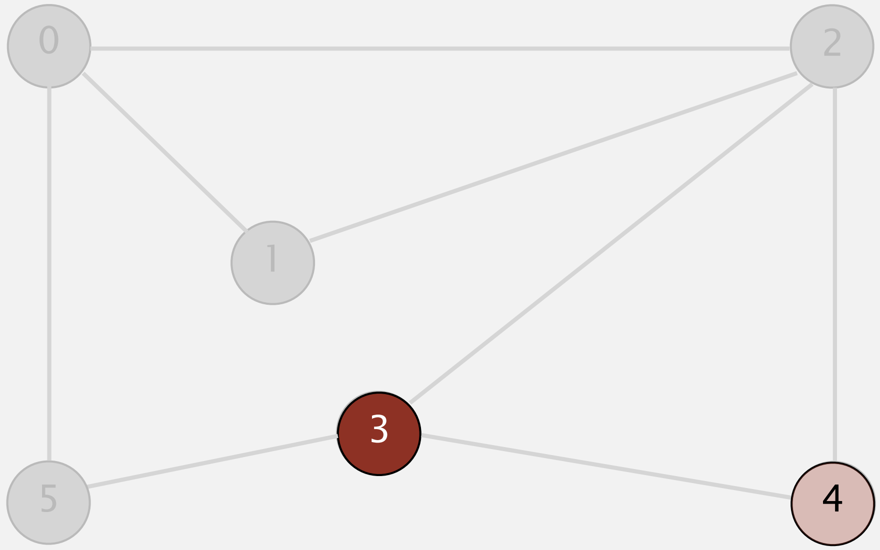
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 3

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue
4

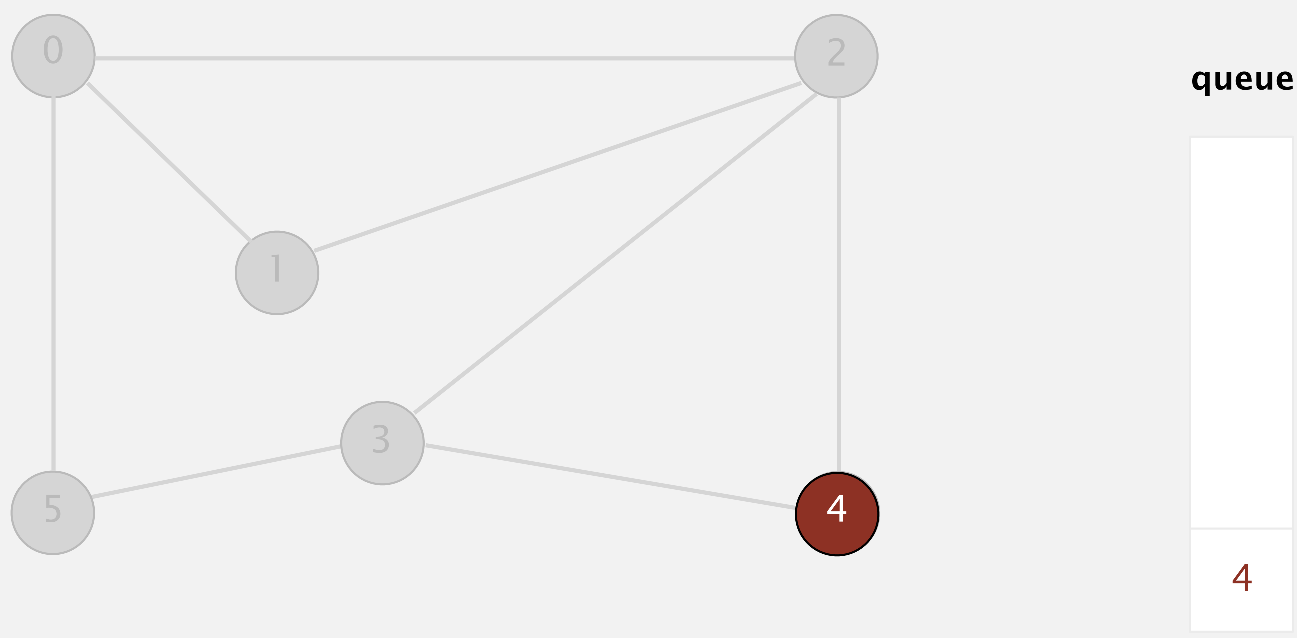
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

3 done

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



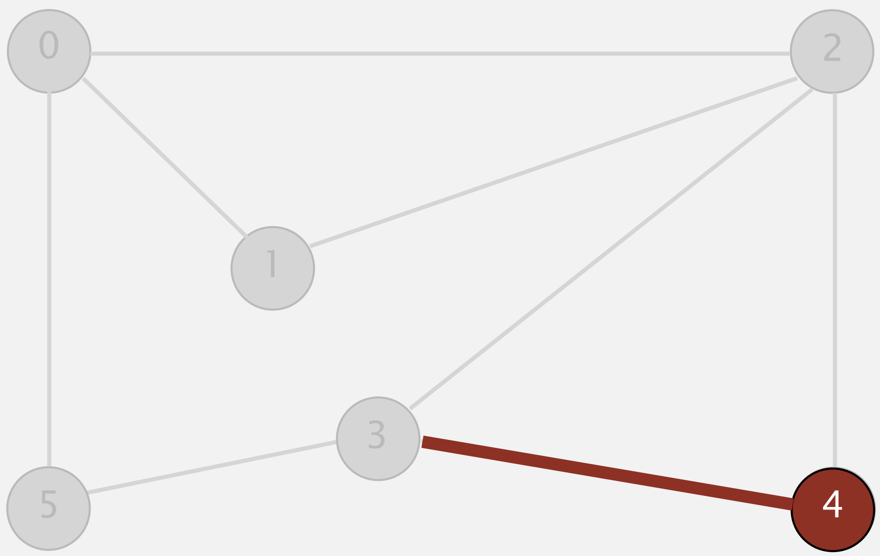
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 4

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

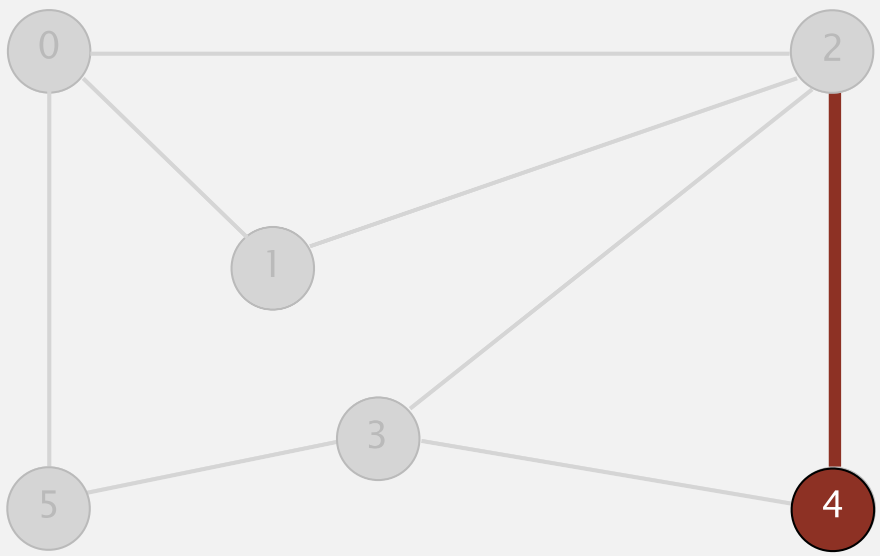
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 4

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

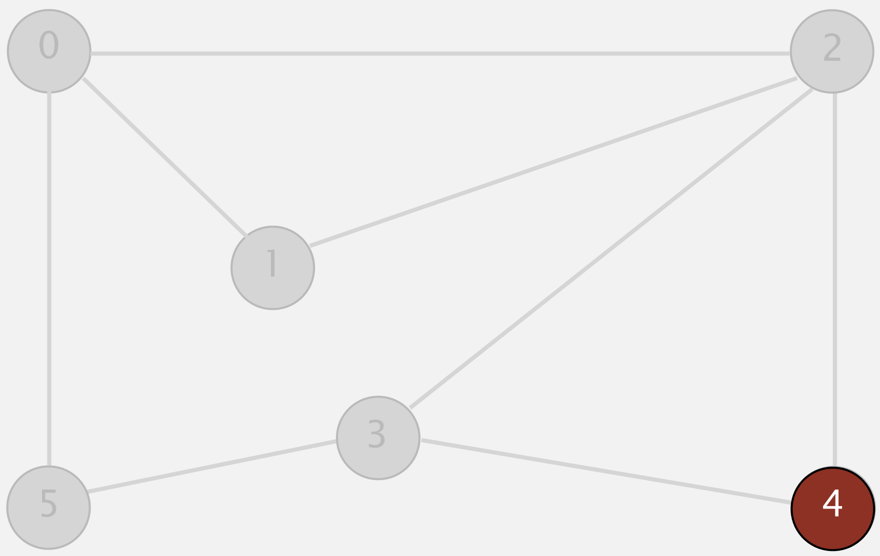
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

dequeue 4

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



queue

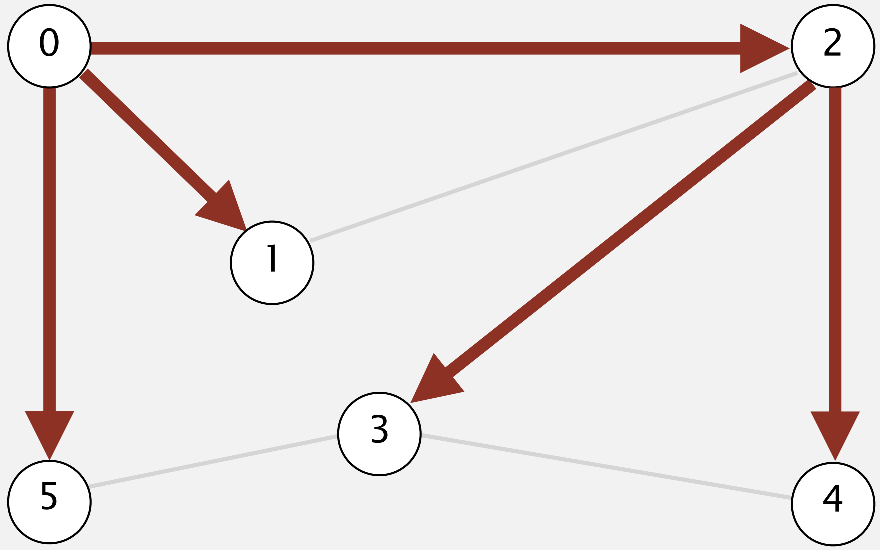
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

4 done

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



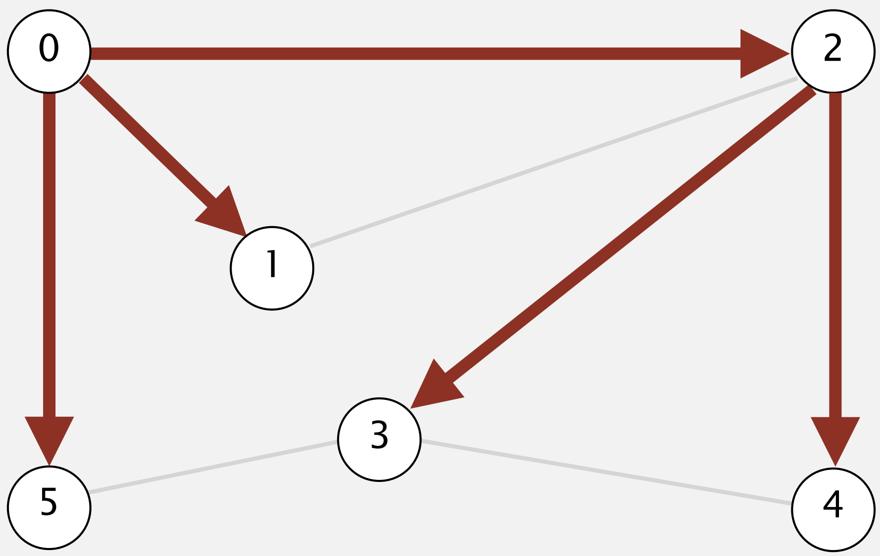
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

done

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



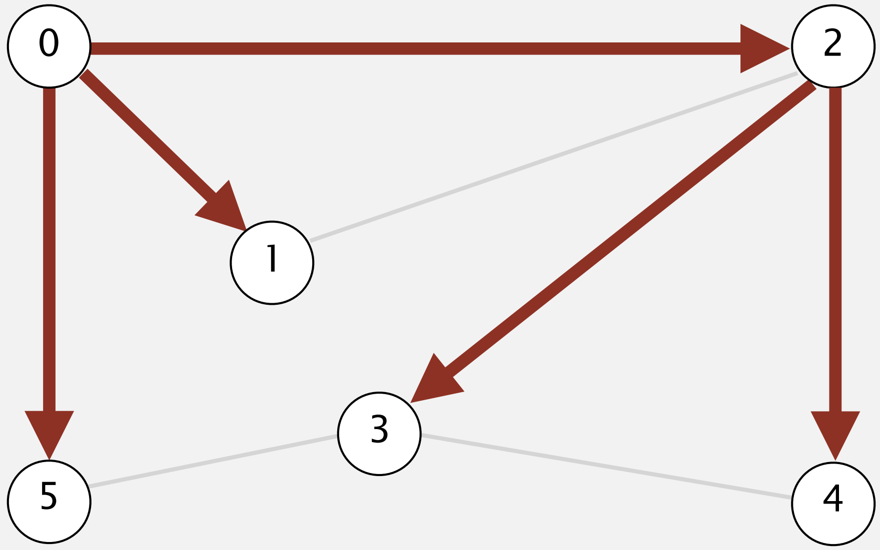
v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

done

Breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent to v and mark them.



v	edgeTo[]	distTo[]
0	–	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

Q. Draw another possible BFS tree of the same graph (also starting from 0)

A. Only one other BFS tree possible: replace 2→3 edge with 5→3 edge

Breadth-first search: Java implementation

```
public class BreadthFirstPaths
{
   private boolean[] marked;
   private int[] edgeTo;
   private int[] distTo;

   …

   private void bfs(Graph G, int s) {
      Queue<Integer> q = new Queue<Integer>();
      q.enqueue(s);
      marked[s] = true;
      distTo[s] = 0;

      while (!q.isEmpty()) {
         int v = q.dequeue();
         for (int w : G.adj(v)) {
            if (!marked[w]) {
               q.enqueue(w);
               marked[w] = true;
               edgeTo[w] = v;
               distTo[w] = distTo[v] + 1;
            }
         }
      }
   }
}
```

Skipped in class

initialize FIFO queue of vertices to explore

found new vertex w via edge v-w

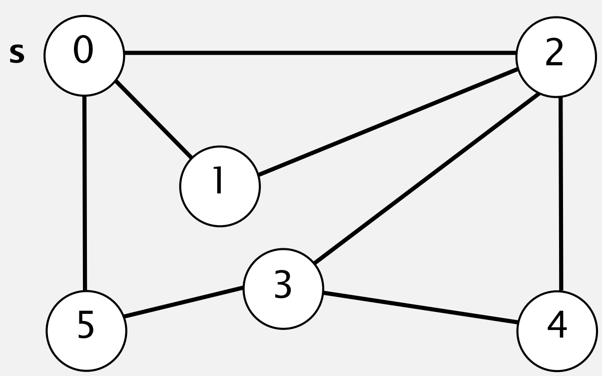
Breadth-first search properties

Q. In which order does BFS examine vertices?

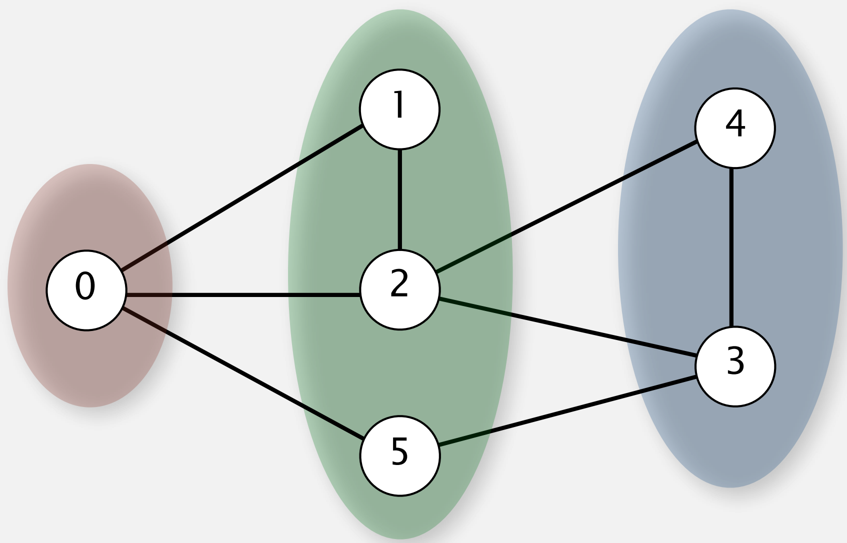
A. Increasing distance (number of edges) from s.

queue always consists of ≥ 0 vertices of distance k from s, followed by ≥ 0 vertices of distance k+1

Proposition. In any connected graph G, BFS computes shortest paths from s to all other vertices in time proportional to $E + V$.



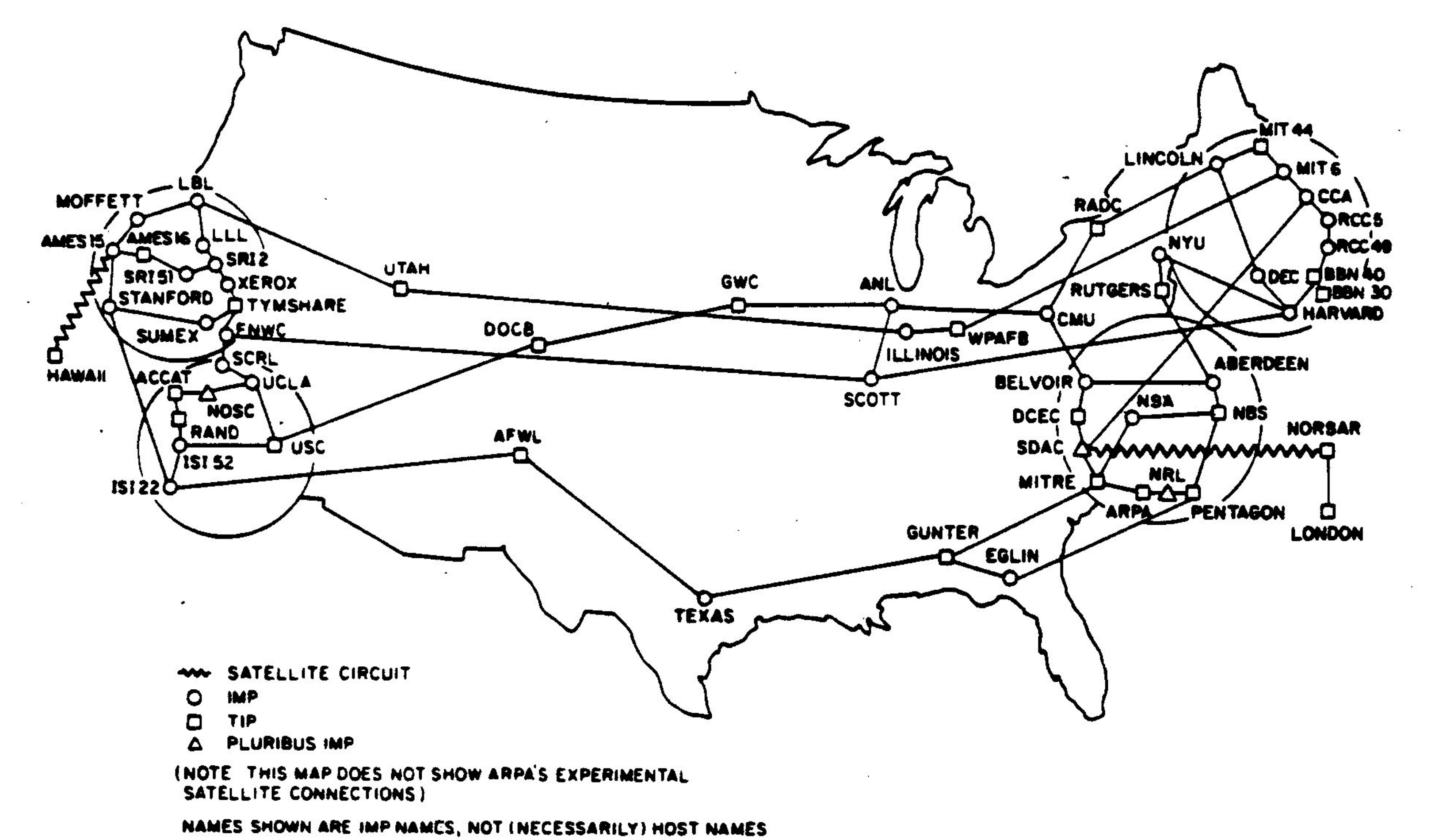
graph G



dist = 0 dist = 1 dist = 2

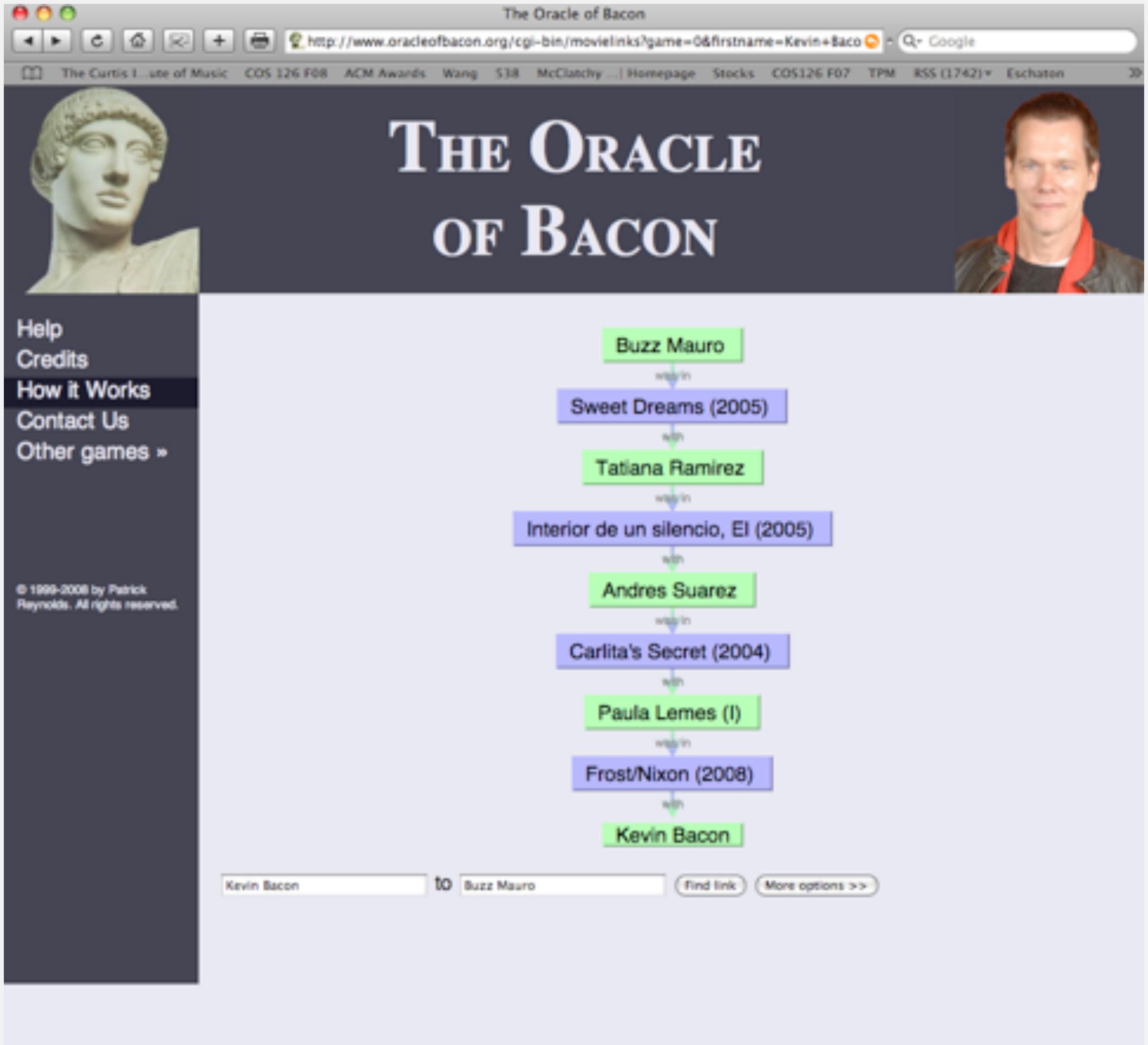
Breadth-first search application: routing

Fewest number of hops in a communication network.



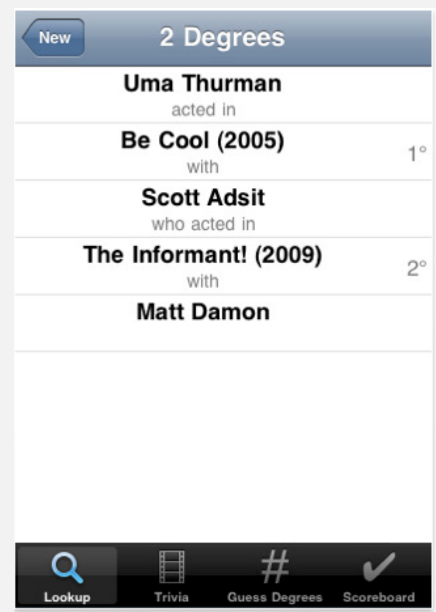
ARPANET, July 1977

Breadth-first search application: Kevin Bacon numbers



http://oracleofbacon.org

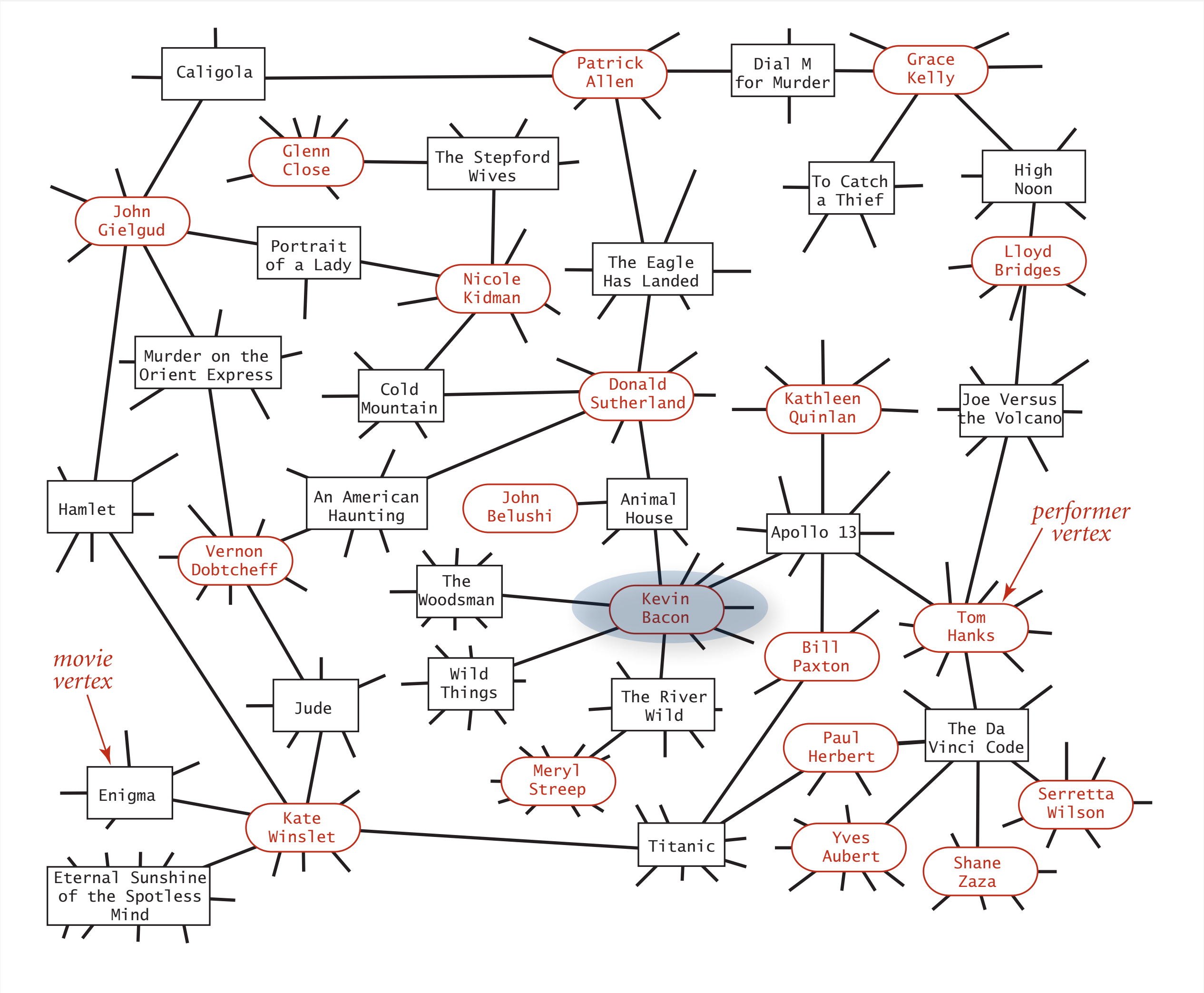
Endless Games board game



SixDegrees iPhone App

Kevin Bacon graph

- Include one vertex for each performer and one for each movie.
- Connect a movie to all performers that appear in that movie.
- Compute shortest path from s = Kevin Bacon.



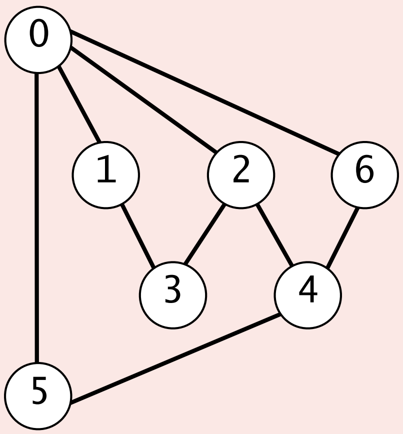
http://algs4.cs.princeton.edu

4.1 Undirected Graphs

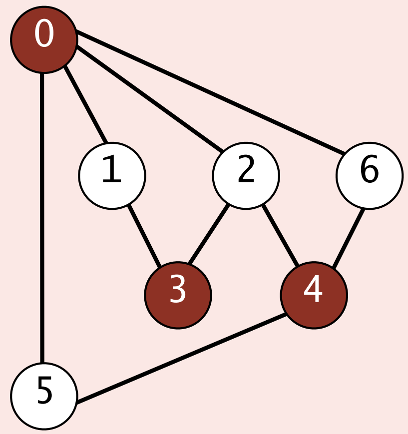
- introduction
- graph API
- depth-first search
- breadth-first search
- challenges

Graph-processing challenge 1

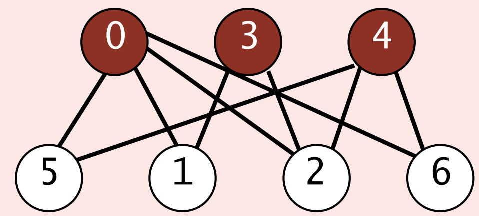
Problem. Is a graph bipartite?



0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6



0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6



Solution:

modify DFS so that each node is colored opposite of its parent

while iterating over adjacent nodes check color

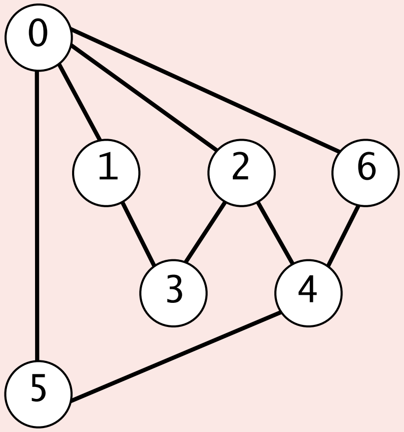
if same color as current node: not bipartite!

if graph not connected: check if each component is bipartite

Graph-processing challenge 2

Problem. Find a cycle in a graph (if one exists).

Simple DFS-based solution (see textbook).

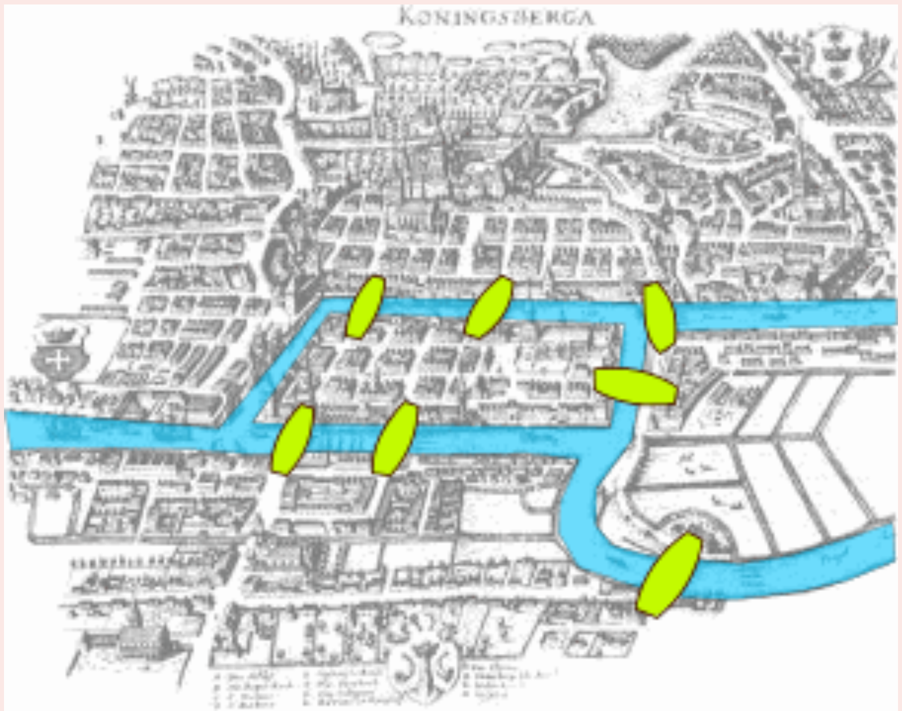


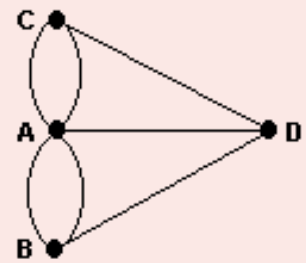
0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

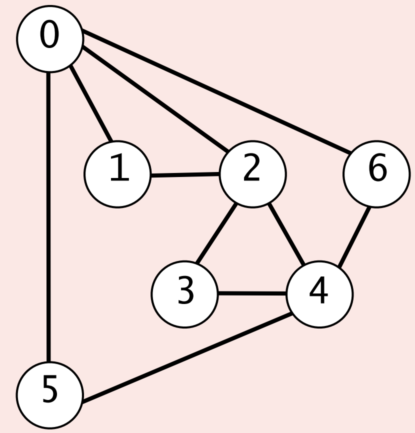
0-5-4-6-0

Graph-processing challenge 3

Problem. Find a cycle that uses every edge exactly once (if one exists).







0-1
0-2
0-5
0-6
1-2
2-3
2-4
3-4
4-5
4-6

0-1-2-3-4-2-0-6-4-5-0

Bridges of Koenigsberg problem. Famously solved by Euler in 1736.

Cycle exists if and only if graph connected & each vertex has even degree

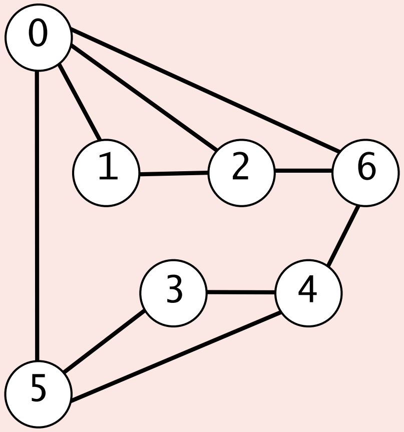
Finding Euler cycle (if it exists): another easy application of DFS.

Graph-processing challenge 4

Problem. Is there a cycle that contains every vertex exactly once?

"Hamiltonian circuit" problem.

Famously NP-complete.



0-1
0-2
0-5
0-6
1-2
2-6
3-4
3-5
4-5
4-6

0-5-3-4-6-2-1-0

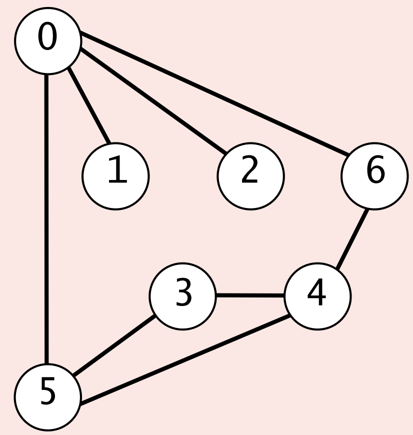
Graph-processing challenge 5

Problem. Are two graphs identical except for vertex names?

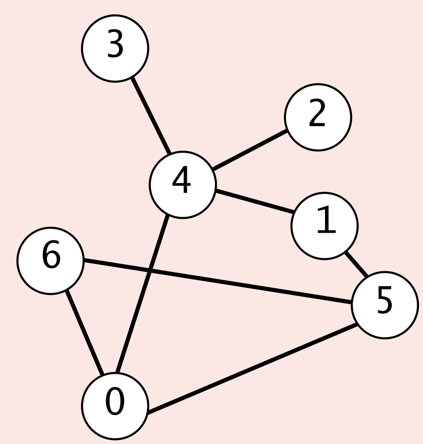
"Graph isomorphism" problem.

Complexity is famously unresolved.

Not known to be solvable in polynomial time
nor known to be NP-complete.



0-1
0-2
0-5
0-6
3-4
3-5
4-5
4-6



0-4
0-5
0-6
1-4
1-5
2-4
3-4
5-6

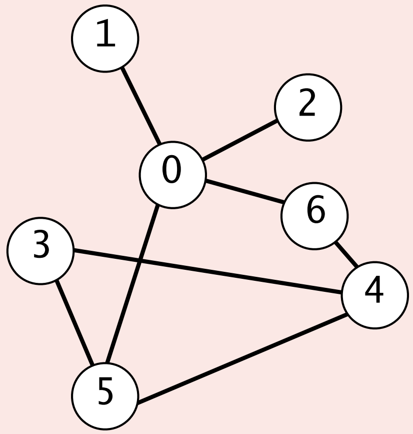
0↔4, 1↔3, 2↔2, 3↔6, 4↔5, 5↔0, 6↔1

Graph-processing challenge 6

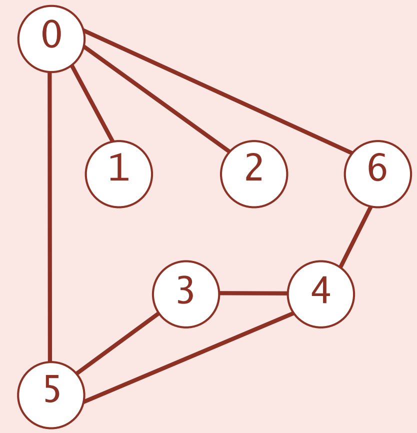
Problem. Can you draw a graph in the plane with no crossing edges?

try it yourself at http://planarity.net

Linear-time but complicated DFS-based algorithm
(by Bob Tarjan)



0-1
0-2
0-5
0-6
3-4
3-5
4-5
4-6



Graph traversal summary

BFS and DFS enables efficient solution of many (but not all) graph problems.

graph problem	BFS	DFS	time
s–t path	✔	✔	$E + V$
shortest s–t path	✔		$E + V$
cycle	✔	✔	$E + V$
Euler cycle		✔	$E + V$
Hamilton cycle			$2^{1.657\,V}$
bipartiteness (odd cycle)	✔	✔	$E + V$
connected components	✔	✔	$E + V$
biconnected components		✔	$E + V$
planarity		✔	$E + V$
graph isomorphism			$2^{c\sqrt{V \log V}}$

Exciting new theorem claimed in Nov 2015
Would improve this bound dramatically
Not yet verified and accepted by community

Algorithms

Robert Sedgewick | Kevin Wayne

http://algs4.cs.princeton.edu

4.1 Undirected Graphs

- introduction
- graph API
- depth-first search
- breadth-first search
- challenges
- **flipped lecture experiment**

Next 4 lectures will be flipped

No class Wednesday 3/23

Before Monday 3/28:

- Watch *directed graphs* and *minimum spanning trees* lectures
- Guna will lead flipped session (usual time and place on 3/28)

No class Wednesday 3/30

Before Monday 4/4:

- Watch *shortest paths* and *maximum flow* lectures
- Arvind will lead flipped session (usual time and place on 4/4)

Regular lectures will resume Wednesday 4/6