Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

http://algs4.cs.princeton.edu

2.3 QUICKSORT

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

Last updated on 2/24/16 8:21 PM

Two classic sorting algorithms: mergesort and quicksort

Critical components in the world's computational infrastructure.
- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

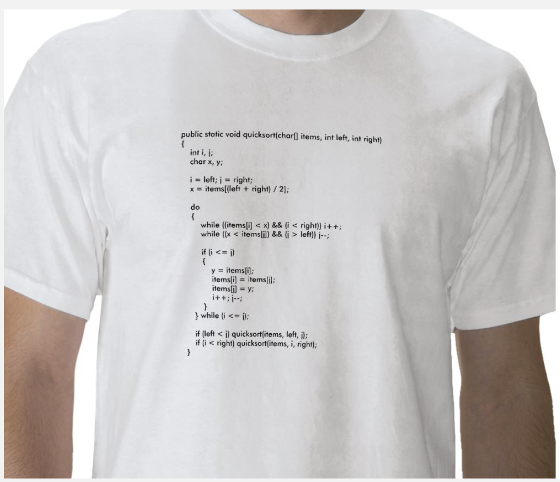
Mergesort. [last lecture]

JS …

Quicksort. [this lecture]

UNIX JS …

Quicksort t-shirt



Quicksort t-shirt

2.3 Quicksort

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

Quicksort overview

Step 1. Shuffle the array.
Step 2. Partition the array so that, for some `j`
- Entry `a[j]` is in its eventual sorted position.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Step 3. Sort each subarray recursively.

input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

partitioning item
not greater
not less

Quicksort overview

input

Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E

Quicksort overview

Step 1. Shuffle the array.

shuffle

Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E

Quicksort overview

Step 1. Shuffle the array.

shuffle

K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S

Quicksort overview

Step 2. Partition the array so that, for some `j`

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

partition

K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S

Quicksort overview

Step 2. Partition the array so that, for some `j`

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

partition

E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

<= K

>= K

Quicksort overview

Step 3. Sort each subarray recursively.

sort the left subarray

E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

Quicksort overview

Step 3. Sort each subarray recursively.

sort the left subarray

A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S

sorted

Quicksort overview

Step 3. Sort each subarray recursively.

sort the right subarray

A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S

sorted

Quicksort overview

Step 3. Sort each subarray recursively.

sort the right subarray

A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

sorted

sorted

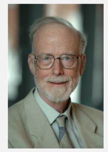
Quicksort overview

sorted array

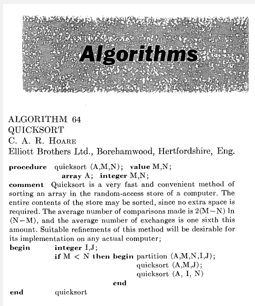
A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

Tony Hoare

- Invented quicksort to translate Russian into English.
  [but couldn't explain his algorithm or implement it!]
- Learned Algol 60 (and recursion).
- Implemented quicksort.



Tony Hoare
1980 Turing Award



Algorithms

ALGORITHM 64
QUICKSORT
C. A. R. Hoare
Elliott Brothers Ltd., Borehamwood, Hertfordshire, Eng.

```
procedure  quicksort (A,M,N);  value M,N;
              array A;  integer M,N;
comment  Quicksort is a very fast and convenient method of
sorting an array in the random-access store of a computer. The
entire contents of the store may be sorted, since no extra space is
required. The average number of comparisons made is 2(M−N) ln
(N−M), and the average number of exchanges is one sixth this
amount. Suitable refinements of this method will be desirable for
its implementation on any actual computer;
begin      integer I,J;
           if M < N then begin partition (A,M,N,I,J);
                               quicksort (A,M,J);
                               quicksort (A, I, N)
                          end
end        quicksort
```

Communications of the ACM (July 1961)

Tony Hoare

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  [but couldn't explain his algorithm or implement it!]
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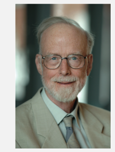


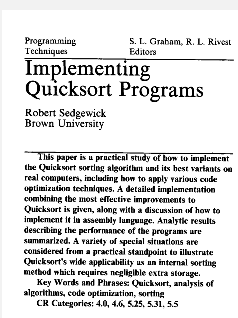
Tony Hoare
1980 Turing Award

> *" I call it my billion-dollar mistake. It was the invention of the null reference in 1965… This has led to innumerable errors, vulnerabilities, and system crashes, which have probably caused a billion dollars of pain and damage in the last forty years."*

Bob Sedgewick

- Refined and popularized quicksort.
- Analyzed many versions of quicksort.

Bob Sedgewick



Programming Techniques — S. L. Graham, R. L. Rivest, Editors

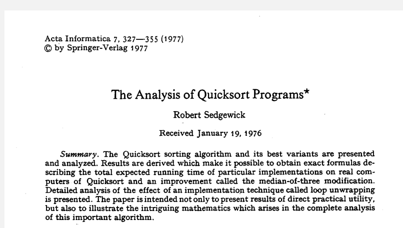
Implementing Quicksort Programs

Robert Sedgewick
Brown University

This paper is a practical study of how to implement the Quicksort sorting algorithm and its best variants on real computers, including how to apply various code optimization techniques. A detailed implementation combining the most effective improvements to Quicksort is given, along with a discussion of how to implement it in assembly language. Analytic results describing the performance of the programs are summarized. A variety of special situations are considered from a practical standpoint to illustrate Quicksort's wide applicability as an internal sorting method which requires negligible extra storage.

Key Words and Phrases: Quicksort, analysis of algorithms, code optimization, sorting

CR Categories: 4.0, 4.6, 5.25, 5.31, 5.5



Acta Informatica 7, 327—355 (1977)
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The Analysis of Quicksort Programs*

Robert Sedgewick

Received January 19, 1976

Summary. The Quicksort sorting algorithm and its best variants are presented and analyzed. Results are derived which make it possible to obtain exact formulas describing the total expected running time of particular implementations on real computers of Quicksort and an improvement called the median-of-three modification. Detailed analysis of the effect of an implementation technique called loop unwrapping is presented. The paper is intended not only to present results of direct practical utility, but also to illustrate the intriguing mathematics which arises in the complete analysis of this important algorithm.

Quicksort partitioning: first try

1. Pick `a[0]` as the partitioning element
2. Create an auxiliary array `aux`
3. Scan the array and copy each item less than `a[0]` to `aux`
4. Scan the array and copy each item not less than `a[0]` to `aux`
5. Copy `aux` back to `a`

Problems

- Requires space for auxiliary array
- Requires multiple scans of the array

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
lo	i														j

Quicksort partitioning demo

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K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
lo	i														j

stop i scan because a[i] >= a[lo]

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

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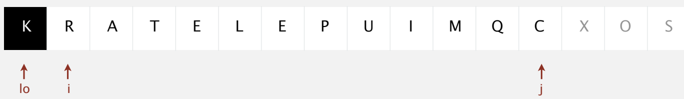
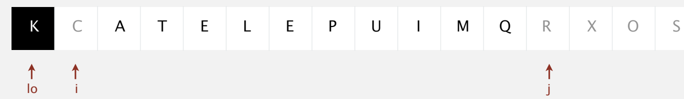


stop j scan and exchange a[i] with a[j]

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.



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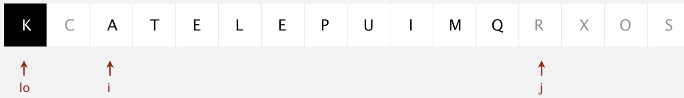

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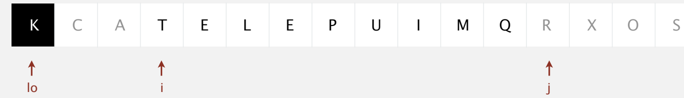

stop i scan because a[i] >= a[lo]

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
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K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
lo			i								j				

Quicksort partitioning demo

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lo			i							j					

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lo			i						j						

stop j scan and exchange a[i] with a[j]

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↑ lo				↑ i					↑ j						

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K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
↑ lo					↑ i				↑ j						

stop i scan because a[i] >= a[lo]

Quicksort partitioning demo

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K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
↑ lo					↑ i			↑ j							

Quicksort partitioning demo

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↑ lo					↑ i		↑ j								

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K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
lo					i	j									

stop j scan and exchange a[i] with a[j]

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K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
lo						i j									

stop i scan because a[i] >= a[lo]

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

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- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
lo					j	i									

stop j scan because a[j] <= a[lo]

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.

K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
lo					j	i									

pointers cross: exchange a[lo] with a[j]

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.

E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
lo					j										hi

partitioned!

Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.

E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
lo					j										hi

partitioned!

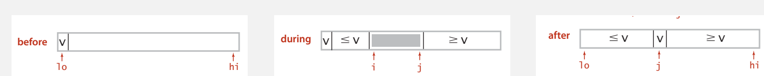
Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
   int i = lo, j = hi+1;
   while (true)
   {
      while (less(a[++i], a[lo]))          find item on left to swap
         if (i == hi) break;

      while (less(a[lo], a[--j]))          find item on right to swap
         if (j == lo) break;

      if (i >= j) break;                   check if pointers cross
      exch(a, i, j);                                         swap
   }

   exch(a, lo, j);                         swap with partitioning item
   return j;                   return index of item now known to be in place
}
```



Quicksort quiz 1

Are the array bounds checks in the previous slide necessary?

A. Yes

B. No

C. Both of the above

D. Neither of the above

E. *I don't know.*

Trick question! One of them is necessary and the other isn't.

Quicksort quiz 2

How many compares to partition an array of length N?

A. $\sim \frac{1}{4} N$

B. $\sim \frac{1}{2} N$

C. $\sim N$

D. $\sim N \lg N$

E. *I don't know.*

Quicksort: Java implementation

```
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {  /* see previous slide */  }

   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
   }
}
```

shuffle needed for performance guarantee (stay tuned)

Quicksort trace

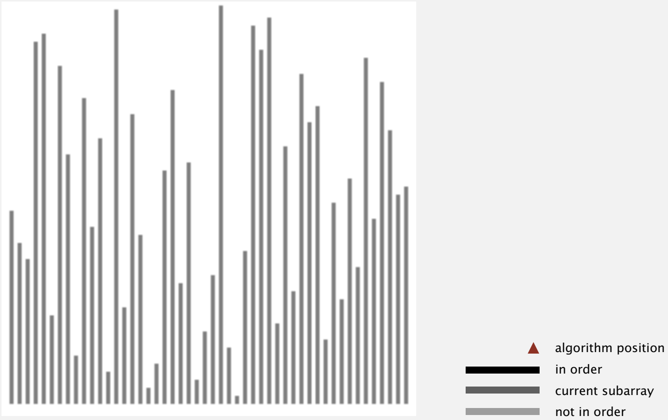
	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Quicksort trace (array contents after each partition)

Quicksort animation

50 random items



http://www.sorting-algorithms.com/quick-sort

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is trickier than it might seem.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop scans on keys equal to the partitioning item's key. ← stay tuned

Preserving randomness. Shuffling is needed for performance guarantee.
Equivalent alternative. Pick a random partitioning item in each subarray.

Quicksort: empirical analysis (1961)

Running time estimates:
- Algol 60 implementation.
- National-Elliott 405 computer.

Table 1

NUMBER OF ITEMS	MERGE SORT	QUICKSORT
500	2 min 8 sec	1 min 21 sec
1,000	4 min 48 sec	3 min 8 sec
1,500	8 min 15 sec*	5 min 6 sec
2,000	11 min 0 sec*	6 min 47 sec

* These figures were computed by formula, since they cannot be achieved on the 405 owing to limited store size.

sorting N 6-word items with 1-word keys

Elliott 405 magnetic disc (16K words)

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

a[ ]

lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.

lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Due to bad randomness, not bad input

Quicksort: analysis of *expected* running time

Proposition. The expected number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf. C_N satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = \underbrace{(N+1)}_{\text{partitioning}} + \left(\frac{C_0 + C_{N-1}}{N}\right) + \left(\frac{\overset{\text{left}}{C_1} + \overset{\text{right}}{C_{N-2}}}{N}\right) + \ldots + \left(\frac{C_{N-1} + C_0}{N}\right)$$

partitioning probability

- Multiply both sides by N and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \ldots + C_{N-1})$$

- Subtract from this equation the same equation for $N - 1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N+1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

Quicksort: analysis of expected running time

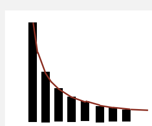
- Repeatedly apply previous equation:

$$\begin{aligned} \frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\ &= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\ &= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\ &= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \ldots + \frac{2}{N+1} \end{aligned}$$

substitute previous equation

- Approximate sum by an integral:

$$\begin{aligned} C_N &= 2(N+1)\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{N+1}\right) \\ &\sim 2(N+1)\int_3^{N+1} \frac{1}{x}\,dx \end{aligned}$$



- Finally, the desired result:

$$C_N \sim 2(N+1)\ln N \approx 1.39 N \lg N$$

Quicksort: worst case is exponentially unlikely

Probability (# compares > $0.1\,N^2$) < $1/2^N$ for large N.

Things more likely than quicksort being quadratic on a million-item array:

- Lightning bolt strikes computer during execution.
- Get trampled by a herd of zebra above the Arctic Circle, while being hit by a meteor.
- I become the next president of these United States.

The probability of needing even $2N \lg N$ compares (instead of $\sim 1.39\,N \lg N$) is negligible for large N.

Important caveats!

Bottom line. Assuming good randomness and no implementation bugs, this is as good as a worst-case $\sim 1.39\,N \lg N$ guarantee.

Quicksort: summary of performance characteristics

Quicksort is a randomized algorithm.

- Guaranteed to be correct.
- Running time depends on random shuffle.

Expected running time.

- Expected number of compares is $\sim 1.39\, N \lg N$.
- Independent of the input.

Comparison to mergesort.

- 39% more compares than mergesort.
- Faster than mergesort in practice because of less data movement.

Best case. Number of compares is $\sim N \lg N$.
Worst case. Number of compares is $\sim \frac{1}{2} N^2$.
[but more likely that lightning bolt strikes computer during execution]

Quicksort quiz 3

How much extra space does quicksort use?

A. $\Theta(1)$

B. $\Theta(\ln N)$

C. $\Theta(N)$

D. $\Theta(N \ln N)$

E. *I don't know.*

Quicksort properties

Proposition. Quicksort is an in-place sorting algorithm.
Pf.

- Partitioning: constant extra space.
- Depth of recursion: logarithmic extra space (with high probability).

can guarantee logarithmic depth by recurring
on smaller subarray before larger subarray
(but requires using an explicit stack)

Proposition. Quicksort is not stable.
Pf. [by counterexample]

i	j	0	1	2	3
		B_1	C_1	C_2	A_1
1	3	B_1	C_1	C_2	A_1
1	3	B_1	A_1	C_2	C_1
0	1	A_1	B_1	C_2	C_1

Quicksort: practical improvements

Insertion sort small subarrays.

- Like mergesort, quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

~ 12/7 N ln N compares (14% fewer)
~ 12/35 N ln N exchanges (3% more)

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;

   int median = medianOf3(a, lo, lo + (hi - lo)/2, hi);
   swap(a, lo, median);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```


2.3 Quicksort

Selection

Goal. Given an array of N items, find the k^{th} smallest item.
Ex. Min $(k = 0)$, max $(k = N - 1)$, median $(k = N/2)$.

Applications.

- Order statistics.
- Find the "top k."

Use theory as a guide.

- Easy $N \log N$ upper bound. How?
- Easy N upper bound for $k = 1, 2, 3$. How?
- Easy N lower bound. Why?

Which is true?

- $N \log N$ lower bound? ← is selection as hard as sorting?
- N upper bound? ← is there a linear-time algorithm?

Quick-select

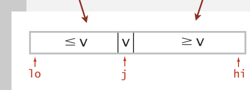
Partition array so that:

- Entry a[j] is in place.
- No larger entry to the left of j.
- No smaller entry to the right of j.

Repeat in one subarray, depending on j; finished when j equals k.

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
       int j = partition(a, lo, hi);
       if      (j < k) lo = j + 1;
       else if (j > k) hi = j - 1;
       else            return a[k];
    }
    return a[k];
}
```

if a[k] is here set hi to j-1
if a[k] is here set lo to j+1



Quick-select: mathematical analysis

Proposition. Quick-select takes expected linear time.

Pf.
Omitted, similar to the analysis of expected running time of quicksort.

There exists a deterministic algorithm with linear running time, but we don't use it because the constants are bad.

http://algs4.cs.princeton.edu

2.3 Quicksort

Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

key

War story (system sort in C)

Bug. A `qsort()` call that should have taken seconds was taking minutes.

At the time, almost all `qsort()` implementations based on those in:

- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.

Duplicate keys: stop on equal keys

Our partitioning subroutine stops both scans on equal keys.

scan until ≥ P — scan until ≤ P

P	G	E	P	A	Q	B	P	Y	C	O	U	P	Z	S	R

Q. Why not continue scans on equal keys?

scan until > P — scan until < P

P	G	E	P	A	Q	B	P	Y	C	O	U	P	Z	S	R

Quicksort quiz 4

What is the result of partitioning the following array (skip over equal keys)?

scan until > A — scan until < A

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

A.

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

B.

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

C.

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

D. *I don't know.*

Quicksort quiz 5

What is the result of partitioning the following array (stop on equal keys)?

scan until ≥ A — scan until ≤ A

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

A.

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

B.

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

C.

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

D. *I don't know.*

Partitioning an array with all equal keys

a[ ]

i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
		A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
1	15	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
1	15	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
2	14	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
2	14	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
3	13	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
3	13	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
4	12	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
4	12	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
5	11	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
5	11	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
6	10	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
6	10	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
7	9	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
7	9	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
	8	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
	8	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

Duplicate keys: partitioning strategies

Bad. Don't stop scans on equal keys.
[~ ½ N^2 compares when all keys equal]

B A A B A B B B C C C   A A A A A A A A A A A

Good. Stop scans on equal keys.
[~ $N \lg N$ compares when all keys equal]

B A A B A B C C B C B   A A A A A A A A A A A

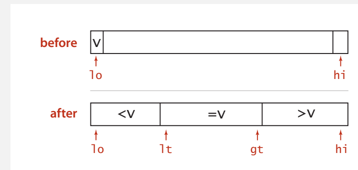
Better. Put all equal keys in place. How?
[~ N compares when all keys equal]

A A A B B B B B C C C   A A A A A A A A A A A

3-way partitioning

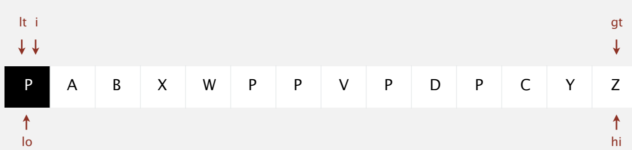
Goal. Partition array into three parts so that:

- Entries between `lt` and `gt` equal to the partition item.
- No larger entries to left of `lt`.
- No smaller entries to right of `gt`.

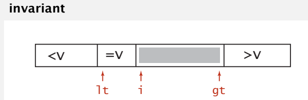


Dijkstra 3-way partitioning demo

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]`; increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]`; decrement `gt`
  - `(a[i] == v)`: increment `i`



invariant



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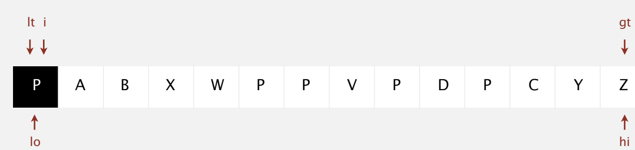

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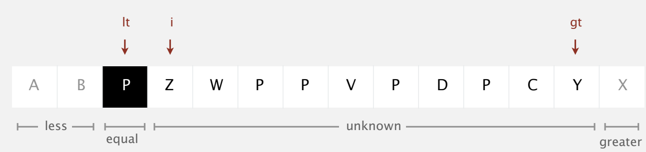

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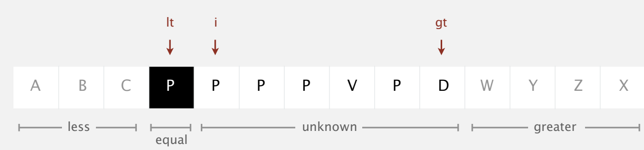

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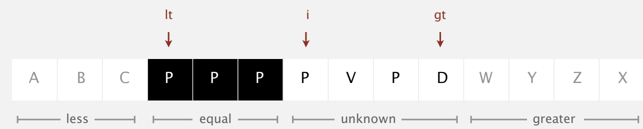

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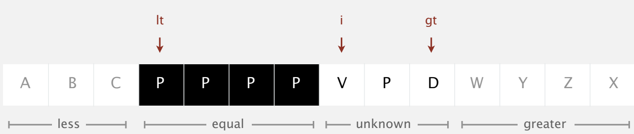
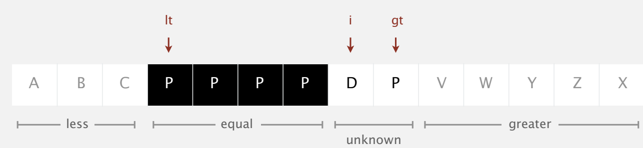


Dijkstra 3-way partitioning demo

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]`; increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]`; decrement `gt`
  - `(a[i] == v)`: increment `i`



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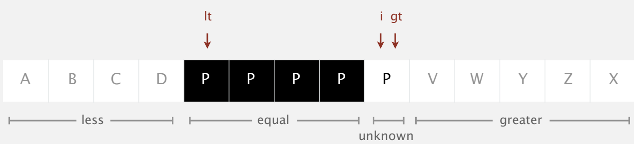

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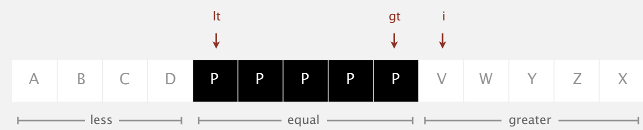

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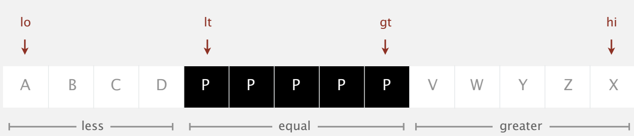

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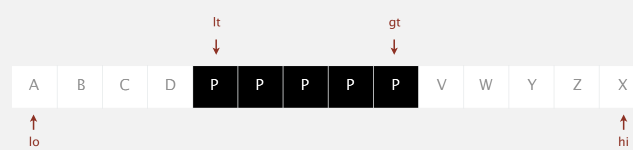
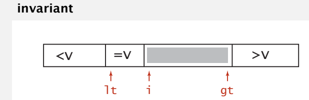
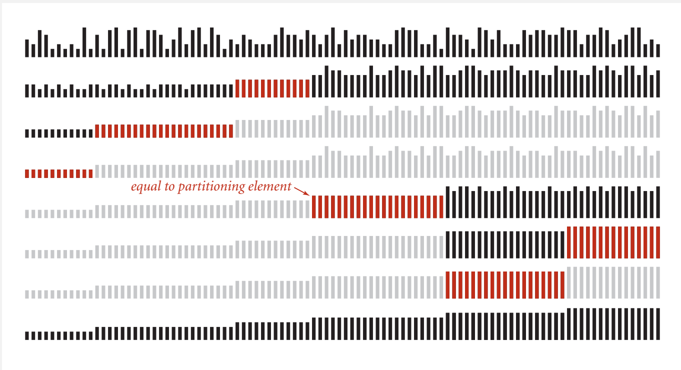


invariant



3-way quicksort: visual trace



2.3 Quicksort

Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

obvious applications

- Find the median.
- Identify statistical outliers.
- Binary search in a database.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

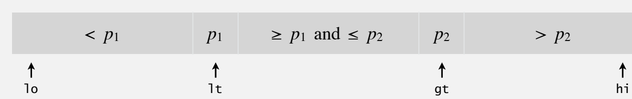
- Data compression.
- Computer graphics.
- Computational biology.
- Load balancing on a parallel computer.
- . . .

non-obvious applications

Dual-pivot quicksort

Use two partitioning items p_1 and p_2 and partition into three subarrays:

- Keys less than p_1.
- Keys between p_1 and p_2.
- Keys greater than p_2.



Recursively sort three subarrays.

degenerates to Dijkstra's 3-way partitioning

Note. Skip middle subarray if $p_1 = p_2$.

Dual-pivot quicksort

Use two partitioning items p_1 and p_2 and partition into three subarrays:

- Keys less than p_1.
- Keys between p_1 and p_2.
- Keys greater than p_2.

$< p_1$	p_1	$\geq p_1$ and $\leq p_2$	p_2	$> p_2$

↑ lo ↑ lt ↑ gt ↑ hi

Now widely used. Java 7, Python unstable sort, Android, …

System sort in Java 7

`Arrays.sort()`.

- Has one method for objects that are `Comparable`.
- Has an overloaded method for each primitive type.
- Has an overloaded method for use with a `Comparator`.
- Has overloaded methods for sorting subarrays.

Algorithms.

- Dual-pivot quicksort for primitive types.
- Timsort for reference types.

Q. Why use different algorithms for primitive and reference types?

Bottom line. Use the system sort!

Review: three types of averages in measuring efficiency of algorithms

Average-case. Average over all possible inputs.
Expected.* Average over all possible values of RNG.
Worst-case over all possible inputs.
Amortized. Average over a sequence of inputs.
(Must be stateful, such as a data structure.)

Example 1. The ______ running time of quicksort is O($N \lg N$). But if we omitted the shuffling step, only the ______ running time would be O($N \lg N$).

Example 2. The ______ running time of selection is O(N) with quick-select, but if we only care about the ______ running time, we'd first sort the array.

*Some people use average-case to refer to both.
If you do, it's important to always know which one you're talking about.

Quicksort quiz 6

The ______ running time of quicksort is O($N \lg N$). But if we omitted the shuffling step, only the ______ running time would be O($N \lg N$).

A. Average-case, expected

B. Expected, average-case

C. Amortized, expected

D. Expected, amortized

E. *I don't know.*

Quicksort quiz 7

The _______ running time of selection is O(N) with quick-select, but if we only care about the _______ running time, we'd first sort the array.

- **A.** Average-case, amortized
- **B.** Amortized, average-case
- **C.** Amortized, expected
- **D.** Expected, amortized
- **E.** *I don't know.*

Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	N exchanges
insertion	✔	✔	N	$\frac{1}{4} N^2$	$\frac{1}{2} N^2$	use for small N or partially ordered
merge		✔	$\frac{1}{2} N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
timsort		✔	N	$N \lg N$	$N \lg N$	improves mergesort when preexisting order
quick	✔		$N \lg N$	$2 N \ln N$ (expected)	$\frac{1}{2} N^2$	$N \log N$ probabilistic guarantee; fastest in practice
3-way quick	✔		N	$2 N \ln N$ (expected)	$\frac{1}{2} N^2$	improves quicksort when duplicate keys
?	✔	✔	N	$N \lg N$	$N \lg N$	holy sorting grail