Algorithms   Robert Sedgewick | Kevin Wayne

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5.3 Substring Search

- introduction
- brute force
- Knuth–Morris–Pratt
- Boyer–Moore
- Rabin–Karp

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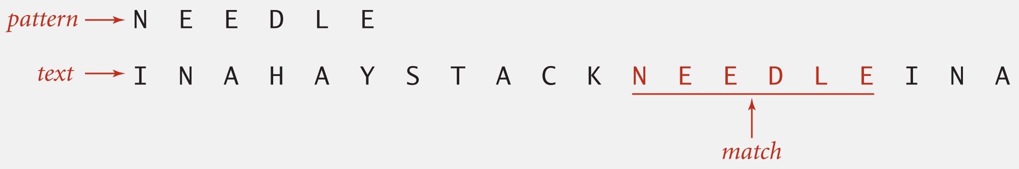
Substring search

Goal. Find pattern of length M in a text of length N.

typically N >> M


pattern → N E E D L E

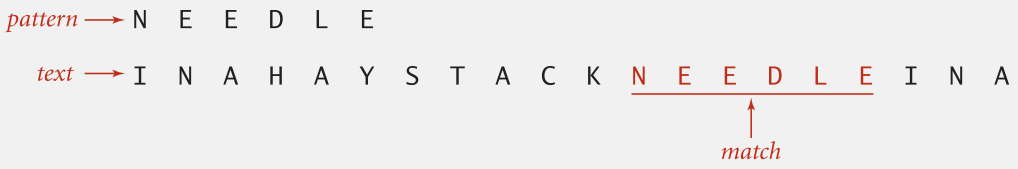
text → I N A H A Y S T A C K N E E D L E I N A

match

Substring search applications

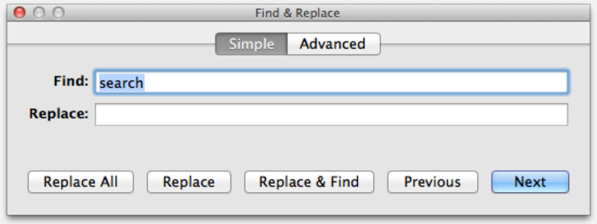
Goal. Find pattern of length M in a text of length N.

typically N >> M


pattern → N E E D L E

text → I N A H A Y S T A C K N E E D L E I N A

match



Find: search

Replace:

Substring search applications

Goal. Find pattern of length M in a text of length N.

typically N >> M

pattern → N E E D L E

text → I N A H A Y S T A C K N E E D L E I N A

match

Computer forensics. Search memory or disk for signatures, e.g., all URLs or RSA keys that the user has entered.

http://citp.princeton.edu/memory

Substring search applications

Goal. Find pattern of length M in a text of length N.

typically N >> M

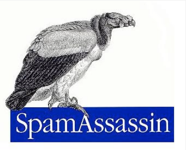
pattern → N E E D L E

text → I N A H A Y S T A C K N E E D L E I N A

match

Identify patterns indicative of spam.

- PROFITS
- L0SE WE1GHT
- herbal Viagra
- There is no catch.
- This is a one-time mailing.
- This message is sent in compliance with spam regulations.



Substring search applications

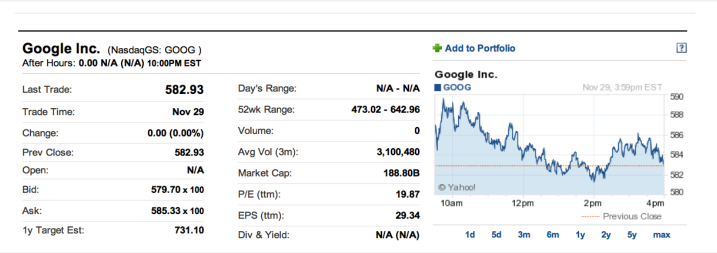
Electronic surveillance.

Need to monitor all internet traffic. (security)
No way! (privacy)
Well, we're mainly interested in "ATTACK AT DAWN"
OK. Build a machine that just looks for that.
"ATTACK AT DAWN" substring search machine
found

Substring search applications

Screen scraping. Extract relevant data from web page.

Ex. Find string delimited by `<b>` and `</b>` after first occurrence of pattern `Last Trade:`.



Google Inc. (NasdaqGS: GOOG)
After Hours: **0.00 N/A (N/A) 10:00PM EST**

Last Trade:	**582.93**	Day's Range:	**N/A - N/A**
Trade Time:	**Nov 29**	52wk Range:	**473.02 - 642.96**
Change:	**0.00 (0.00%)**	Volume:	**0**
Prev Close:	**582.93**	Avg Vol (3m):	**3,100,480**
Open:	**N/A**	Market Cap:	**188.80B**
Bid:	**579.70 x 100**	P/E (ttm):	**19.87**
Ask:	**585.33 x 100**	EPS (ttm):	**29.34**
1y Target Est:	**731.10**	Div & Yield:	**N/A (N/A)**

http://finance.yahoo.com/q?s=goog

```
...
<tr>
<td class= "yfnc_tablehead1"
width= "48%">
Last Trade:
</td>
<td class= "yfnc_tabledata1">
<big><b>582.93</b></big>
</td></tr>
<td class= "yfnc_tablehead1"
width= "48%">
Trade Time:
</td>
<td class= "yfnc_tabledata1">
...
```

Screen scraping: Java implementation

Java library. The `indexOf()` method in Java's `String` data type returns the index of the first occurrence of a given string, starting at a given offset.

```
public class StockQuote
{
   public static void main(String[] args)
   {
      String name = "http://finance.yahoo.com/q?s=";
      In in = new In(name + args[0]);
      String text = in.readAll();
      int start    = text.indexOf("Last Trade:", 0);
      int from     = text.indexOf("<b>",  start);
      int to       = text.indexOf("</b>", from);
      String price = text.substring(from + 3, to);
      StdOut.println(price);
   }
}
```

```
% java StockQuote goog
582.93
```

Caveat. Must update program if Yahoo format changes.

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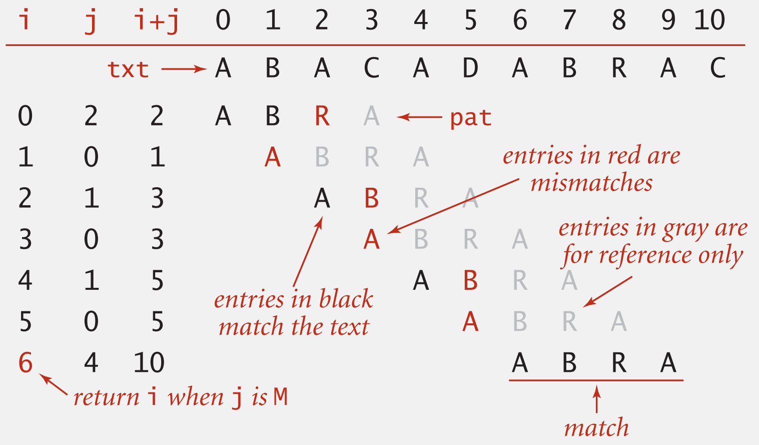
Brute-force substring search

Check for pattern starting at each text position.



i	j	i+j	0	1	2	3	4	5	6	7	8	9	10
		txt →	A	B	A	C	A	D	A	B	R	A	C
0	2	2	A	B	R	A ← pat							
1	0	1		A	B	R	A						
2	1	3			A	B	R	A					
3	0	3				A	B	R	A				
4	1	5					A	B	R	A			
5	0	5						A	B	R	A		
6	4	10							A	B	R	A	

entries in red are mismatches

entries in gray are for reference only

entries in black match the text

return i when j is M

match

Brute-force substring search: Java implementation

Check for pattern starting at each text position.

i	j	i+j	0	1	2	3	4	5	6	7	8	9	10
			A	B	A	C	A	D	A	B	R	A	C
4	3	7					A	D	A	C	R		
5	0	5						A	D	A	C	R	

```
public static int search(String pat, String txt)
{
   int M = pat.length();
   int N = txt.length();
   for (int i = 0; i <= N - M; i++)
   {
      int j;
      for (j = 0; j < M; j++)
         if (txt.charAt(i+j) != pat.charAt(j))
            break;
      if (j == M) return i;   ← index in text where pattern starts
   }
   return N;   ← not found
}
```

Substring search quiz 1

What is the worst-case running time of brute-force substring search as a function of the number of characters in the pattern M and text N?

A. $M + N$

B. M^2

C. $M N$

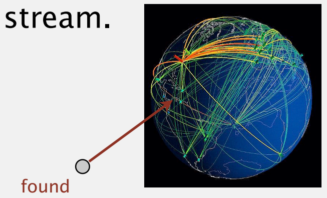
D. N^2

E. *I don't know.*

Backup

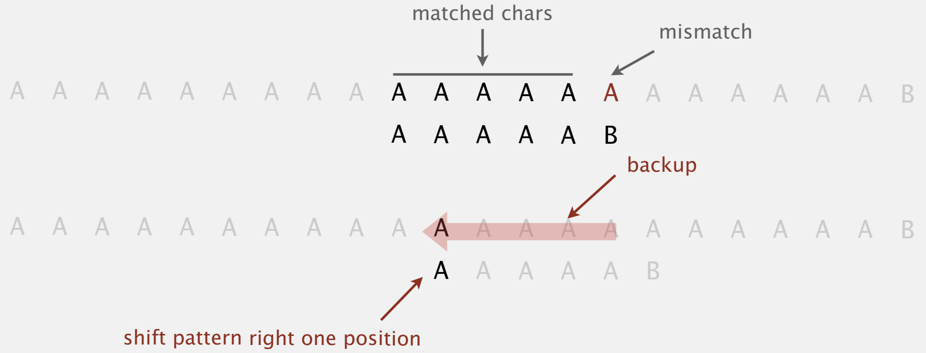
In many applications, we want to avoid backup in text stream.

- Treat input as stream of data.
- Abstract model: standard input.



"ATTACK AT DAWN" substring search machine

Brute-force algorithm needs backup for every mismatch.


matched chars
mismatch
A A A A A A A A A A A A A A A A A A A A A A B
A A A A A B
backup
A A A A A A A A A A A A A A A A A A A A A A B
A A A A A B
shift pattern right one position

Approach 1. Maintain buffer of last M characters.

Approach 2. Stay tuned.

Brute-force substring search: alternate implementation

Same sequence of character compares as previous implementation.

- `i` points to end of sequence of already-matched characters in text.
- `j` stores # of already-matched characters (end of sequence in pattern).

i	j	0	1	2	3	4	5	6	7	8	9	10
		A	B	A	C	A	D	A	B	R	A	C
7	3					A	D	A	C	R		
5	0						A	D	A	C	R	

```
public static int search(String pat, String txt)
{
   int i, N = txt.length();
   int j, M = pat.length();
   for (i = 0, j = 0; i < N && j < M; i++)
   {
      if (txt.charAt(i) == pat.charAt(j)) j++;
      else { i -= j; j = 0;  }
   }
   if (j == M) return i - M;
   else            return N;
}
```

explicit backup

Algorithmic challenges in substring search

Brute-force is not always good enough.

Theoretical challenge. Linear-time guarantee. ← fundamental algorithmic problem

Practical challenge. Avoid backup in text stream. ← often no space (or time) to save text

```
Now is the time for all people to come to the aid of their party. Now is the time for all good people to
come to the aid of their party. Now is the time for many good people to come to the aid of their party.
Now is the time for all good people to come to the aid of their party. Now is the time for a lot of good
people to come to the aid of their party. Now is the time for all of the good people to come to the aid of
their party. Now is the time for all good people to come to the aid of their party. Now is the time for
each good person to come to the aid of their party. Now is the time for all good people to come to the aid
of their party. Now is the time for all good Republicans to come to the aid of their party. Now is the
time for all good people to come to the aid of their party. Now is the time for many or all good people to
come to the aid of their party. Now is the time for all good people to come to the aid of their party. Now
is the time for all good Democrats to come to the aid of their party. Now is the time for all people to
come to the aid of their party. Now is the time for all good people to come to the aid of their party. Now
is the time for many good people to come to the aid of their party. Now is the time for all good people to
come to the aid of their party. Now is the time for a lot of good people to come to the aid of their
party. Now is the time for all of the good people to come to the aid of their party. Now is the time for
all good people to come to the aid of their attack at dawn party. Now is the time for each person to come
to the aid of their party. Now is the time for all good people to come to the aid of their party. Now is
the time for all good Republicans to come to the aid of their party. Now is the time for all good people
to come to the aid of their party. Now is the time for many or all good people to come to the aid of their
party. Now is the time for all good people to come to the aid of their party. Now is the time for all good
Democrats to come to the aid of their party.
```

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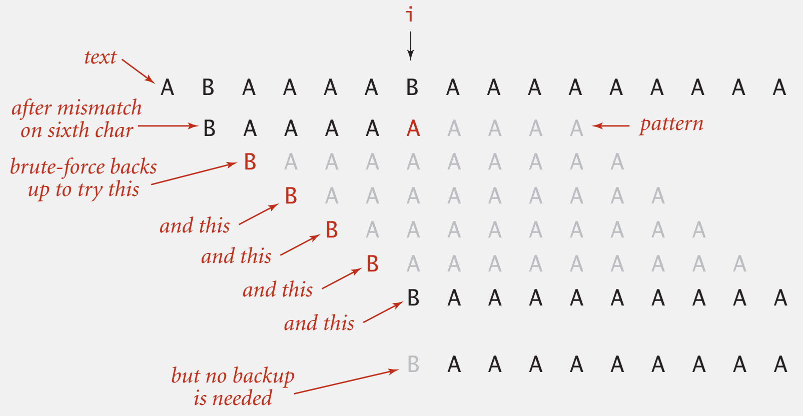
Knuth–Morris–Pratt substring search

Intuition. Suppose we are searching in text for pattern BAAAAAAAAA.

- Suppose we match 5 chars in pattern, with mismatch on 6th char.
- We know previous 6 chars in text are BAAAAB.
- Don't need to back up text pointer!

assuming { A, B } alphabet



i

text A B A A A A B A A A A A A A A A

after mismatch on sixth char B A A A A A A A A A pattern

brute-force backs up to try this B A A A A A A A A A

and this B A A A A A A A A A

and this B A A A A A A A A A

and this B A A A A A A A A A

and this B A A A A A A A A A

but no backup is needed B A A A A A A A A A

Knuth–Morris–Pratt algorithm. Clever method to always avoid backup!

Deterministic finite state automaton (DFA)

DFA is abstract string-searching machine.

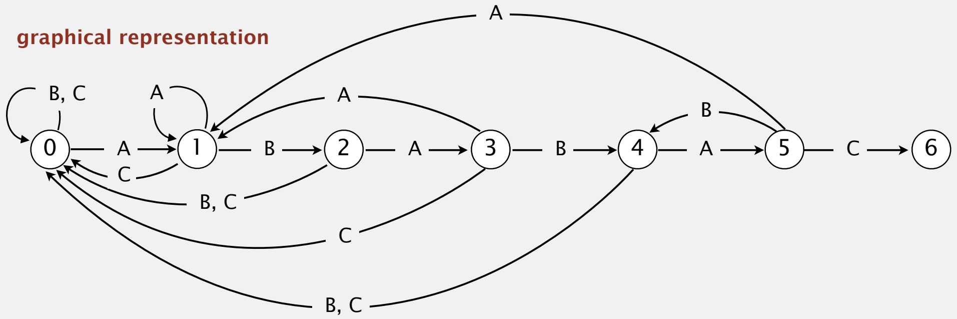
- Finite number of states (including start and halt).
- Exactly one state transition for each char in alphabet.
- Accept if sequence of state transitions leads to halt state.

internal representation

	j	0	1	2	3	4	5
	pat.charAt(j)	A	B	A	B	A	C
dfa[][j]	A	1	1	3	1	5	1
	B	0	2	0	4	0	4
	C	0	0	0	0	0	6

If in state j reading char c:
if j is 6 halt and accept
else move to state dfa[c][j]

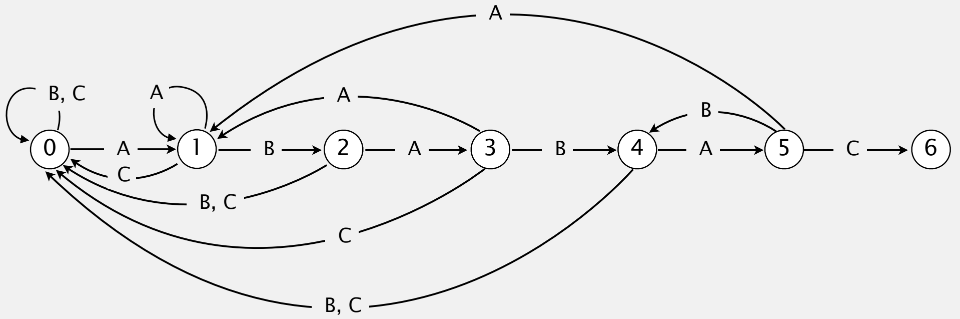
graphical representation



Knuth–Morris–Pratt demo: DFA simulation

A A B A C A A B A B A C A A

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
	A	1	1	3	1	5	1
dfa[][j]	B	0	2	0	4	0	4
	C	0	0	0	0	0	6



Interpretation of Knuth-Morris-Pratt DFA

Q. What is interpretation of DFA state after reading in `txt[i]`?

A. State = number of characters in pattern that have been matched.

length of longest prefix of `pat[]` that is a suffix of `txt[0..i]`

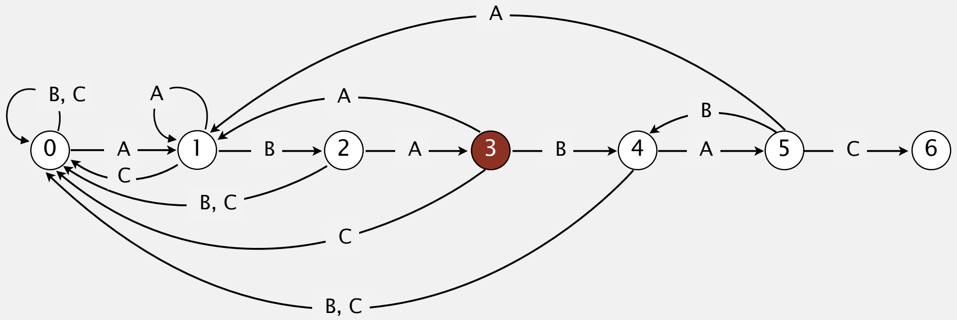
Ex. DFA is in state 3 after reading in `txt[0..6]`.

							i		
	0	1	2	3	4	5	6	7	8
txt →	B	C	B	A	A	B	A	C	A

suffix of `txt[0..6]`

	0	1	2	3	4	5
pat →	A	B	A	B	A	C

prefix of `pat[]`



Substring search quiz 2

Which state is the DFA in after processing the following input?

A A B B A B A B C A B A A B A A C A A A B A B A B A A C A A B A A B A B A B

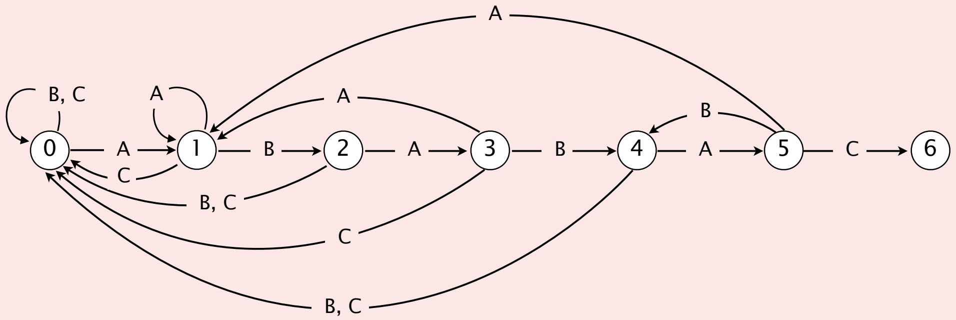
A. 0

B. 1

C. 3

D. 4

E. *I don't know.*



Knuth-Morris-Pratt substring search: Java implementation

Key differences from brute-force implementation.

- Need to precompute `dfa[][]` from pattern.
- Text pointer `i` never decrements.

```
public int search(String txt)
{
   int i, j, N = txt.length();
   for (i = 0, j = 0; i < N && j < M; i++)
      j = dfa[txt.charAt(i)][j];
   if (j == M) return i - M;
   else        return N;
}
```

no backup

Running time.

- Simulate DFA on text: at most N character accesses.
- Build DFA: how to do efficiently? [warning: tricky algorithm ahead]

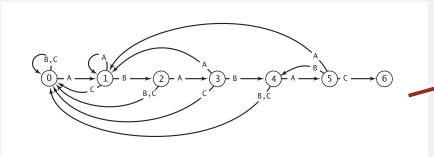
Knuth-Morris-Pratt substring search: Java implementation

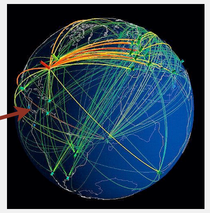
Key differences from brute-force implementation.

- Need to precompute `dfa[][]` from pattern.
- Text pointer `i` never decrements.
- Could use input stream.

```
public int search(In in)
{
   int i, j;
   for (i = 0, j = 0; !in.isEmpty() && j < M; i++)
      j = dfa[in.readChar()][j];
   if (j == M) return i - M;
   else        return NOT_FOUND;
}
```

no backup

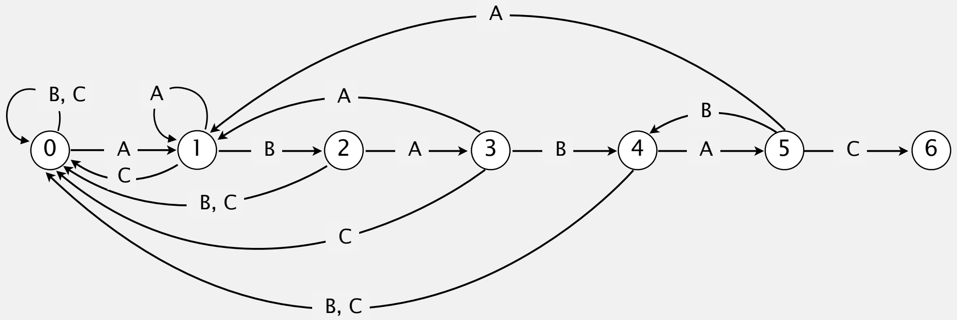




Knuth-Morris-Pratt demo: DFA construction

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
	A	1	1	3	1	5	1
dfa[][j]	B	0	2	0	4	0	4
	C	0	0	0	0	0	6

Constructing the DFA for KMP substring search for A B A B A C



How to build DFA from pattern?

Include one state for each character in pattern (plus accept state).

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
	A						
dfa[][j]	B						
	C						

How to build DFA from pattern?

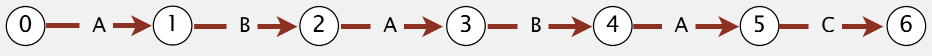
Match transition. If in state j and next char c == pat.charAt(j), go to j+1.

first j characters of pattern have already been matched

next char matches

now first j +1 characters of pattern have been matched

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
	A	1		3		5	
dfa[][j]	B		2		4		
	C						6



How to build DFA from pattern?

Mismatch transition. If in state j and next char c != pat.charAt(j), then the last j-1 characters of input are pat[1..j-1], followed by c.

To compute dfa[c][j]: Simulate pat[1..j-1] on DFA and take transition c.

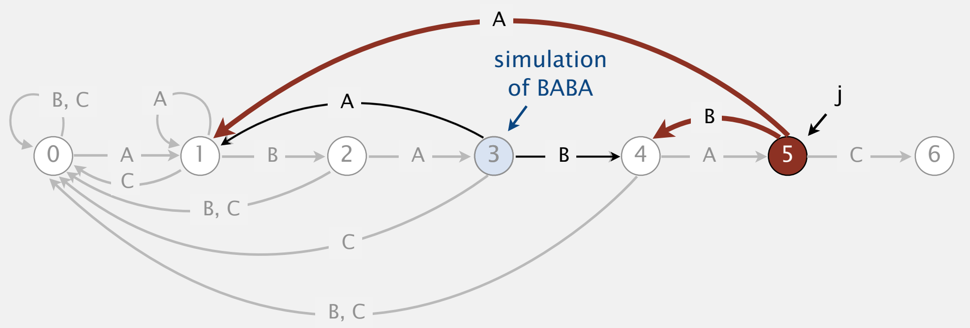
still under construction (!)

Running time. Seems to require j steps.

Ex. dfa['A'][5] = 1 dfa['B'][5] = 4

simulate BABAA simulate BABAB

j	0	1	2	3	4	5
pat.charAt(j)	A	B	A	B	A	C



How to build DFA from pattern?

Mismatch transition. If in state `j` and next char `c != pat.charAt(j)`, then the last `j-1` characters of input are `pat[1..j-1]`, followed by `c`.

state x

To compute `dfa[c][j]`: Simulate `pat[1..j-1]` on DFA and take transition `c`.

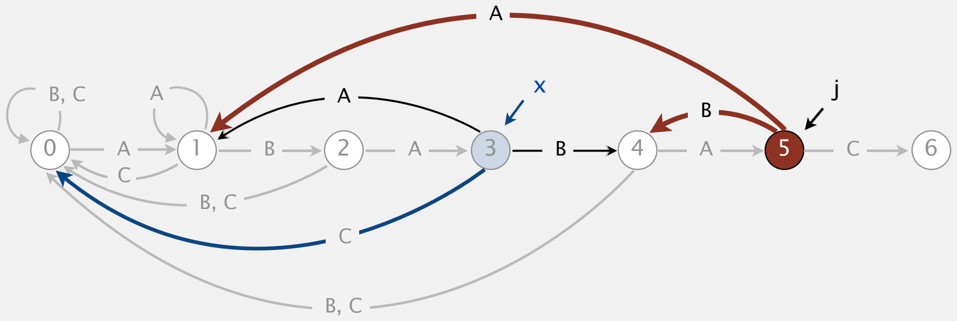
Running time. Takes only constant time if we maintain state x.

Ex. `dfa['A'][5] = 1` — from state x, take transition 'A' = dfa['A'][x]

`dfa['B'][5] = 4` — from state x, take transition 'B' = dfa['B'][x]

`x' = 0` — from state x, take transition 'C' = dfa['C'][x]

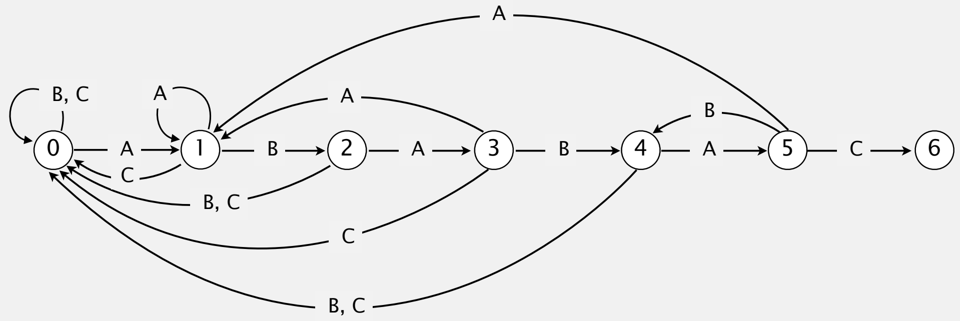
0	1	2	3	4	5
A	B	A	B	A	C



Knuth-Morris-Pratt demo: DFA construction in linear time

		0	1	2	3	4	5
pat.charAt(j)		A	B	A	B	A	C
dfa[][j]	A	1	1	3	1	5	1
	B	0	2	0	4	0	4
	C	0	0	0	0	0	6

Constructing the DFA for KMP substring search for A B A B A C



Constructing the DFA for KMP substring search: Java implementation

For each state `j`:

- Copy `dfa[][x]` to `dfa[][j]` for mismatch case.
- Set `dfa[pat.charAt(j)][j]` to `j+1` for match case.
- Update `x`.

```
public KMP(String pat)
{
   this.pat = pat;
   M = pat.length();
   dfa = new int[R][M];
   dfa[pat.charAt(0)][0] = 1;
   for (int x = 0, j = 1; j < M; j++)
   {
      for (int c = 0; c < R; c++)
         dfa[c][j] = dfa[c][x];        ← copy mismatch cases
      dfa[pat.charAt(j)][j] = j+1;     ← set match case
      x = dfa[pat.charAt(j)][x];       ← update restart state
   }
}
```

Running time. M character accesses (but space/time proportional to $R\,M$).

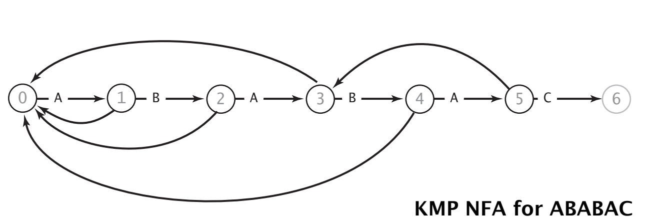
KMP substring search analysis

Proposition. KMP substring search accesses no more than $M + N$ chars to search for a pattern of length M in a text of length N.

Pf. Each pattern character accessed once when constructing the DFA; each text character accessed once (in the worst case) when simulating the DFA.

Proposition. KMP constructs `dfa[][]` in time and space proportional to $R\,M$.

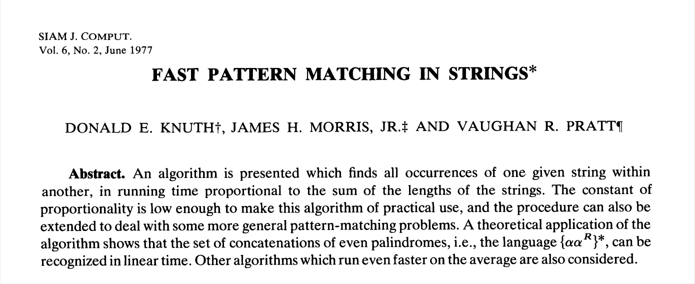
Larger alphabets. Improved version of KMP constructs `nfa[]` in time and space proportional to M.



KMP NFA for ABABAC

Knuth–Morris–Pratt: brief history

- Independently discovered by two theoreticians and a hacker.
  - Knuth: inspired by esoteric theorem, discovered linear algorithm
  - Pratt: made running time independent of alphabet size
  - Morris: built a text editor for the CDC 6400 computer
- Theory meets practice.



SIAM J. Comput.
Vol. 6, No. 2, June 1977

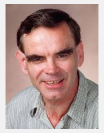
FAST PATTERN MATCHING IN STRINGS*

DONALD E. KNUTH†, JAMES H. MORRIS, JR.‡ AND VAUGHAN R. PRATT¶

Abstract. An algorithm is presented which finds all occurrences of one given string within another, in running time proportional to the sum of the lengths of the strings. The constant of proportionality is low enough to make this algorithm of practical use, and the procedure can also be extended to deal with some more general pattern-matching problems. A theoretical application of the algorithm shows that the set of concatenations of even palindromes, i.e., the language $\{\alpha\alpha^R\}^*$, can be recognized in linear time. Other algorithms which run even faster on the average are also considered.

Don Knuth

Jim Morris



Vaughan Pratt

Cyclic Rotation

A string s is a cyclic rotation of t if s and t have the same length and s is a suffix of t followed by a prefix of t.

yes	yes	no
ROTATEDSTRING	ABABABBABBABA	ROTATEDSTRING
STRINGROTATED	BABBABBABAABA	GNIRTSDETATOR

Problem. Given two binary strings s and t, design a linear-time algorithm to determine if s is a cyclic rotation of t.

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5.3 Substring Search

- introduction
- brute force
- Knuth–Morris–Pratt
- Boyer–Moore
- Rabin–Karp

Robert Boyer

J. Strother Moore

Boyer–Moore: mismatched character heuristic

Intuition.

- Scan characters in pattern from right to left.
- Can skip as many as M text chars when finding one not in the pattern.

i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
	text →	F	I	N	D	I	N	A	H	A	Y	S	T	A	C	K	N	E	E	D	L	E	I	N	A
0	5	N	E	E	D	L	E ← pattern																		
5	5						N	E	E	D	L	E													
11	4												N	E	E	D	L	E							
15	0																N	E	E	D	L	E			

return i = 15

no S in pattern

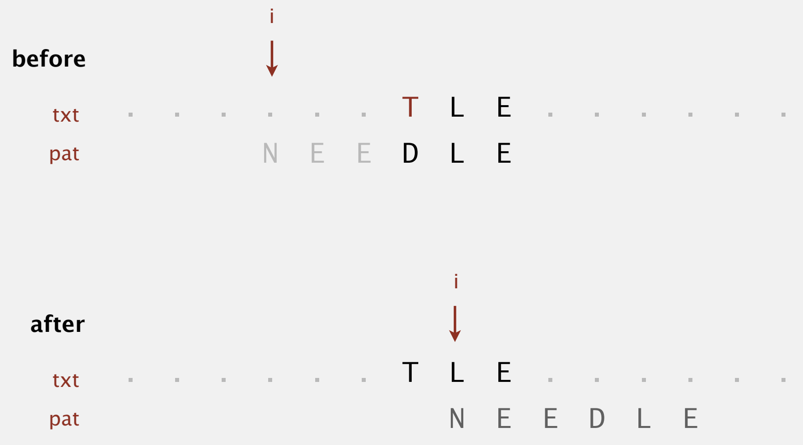
align N in text with rightmost N in pattern

align N in text with rightmost N in pattern

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

Case 1. Mismatch character not in pattern.



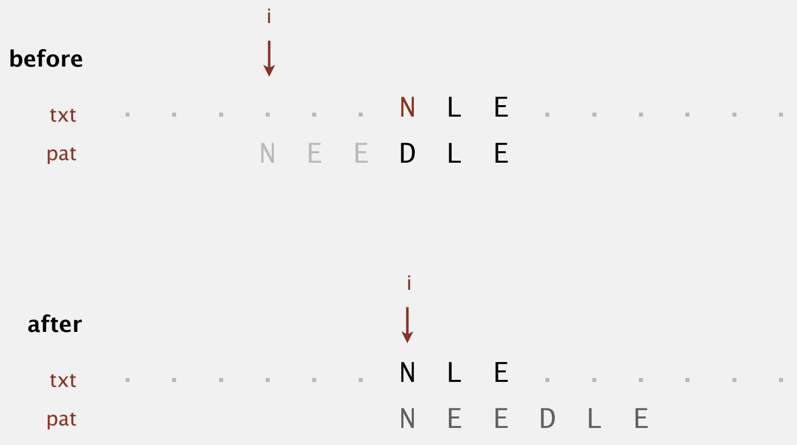
```
                i
before          ↓
txt   . . . . . . T L E . . . . . .
pat         N E E D L E


                    i
after               ↓
txt   . . . . . . T L E . . . . . .
pat                 N E E D L E
```

mismatch character 'T' not in pattern: increment i one character beyond 'T'

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

Case 2a. Mismatch character in pattern.



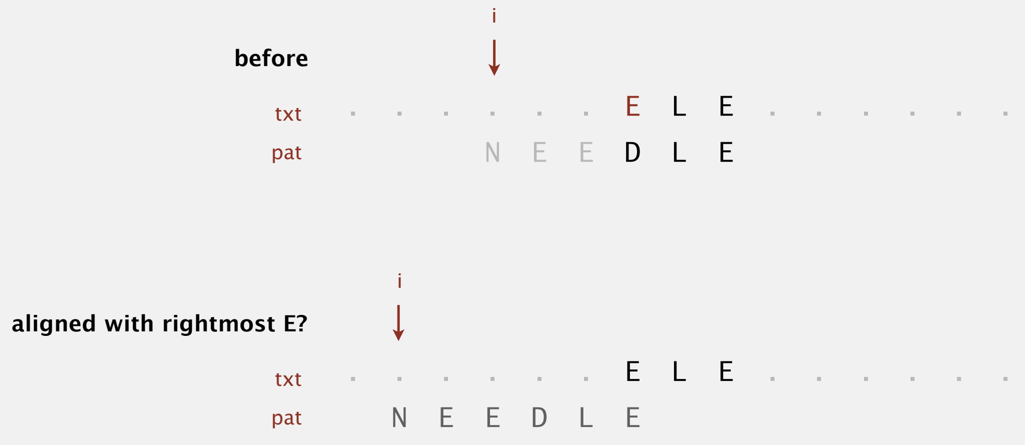
```
                i
before          ↓
txt   . . . . . . N L E . . . . . .
pat         N E E D L E


                  i
after             ↓
txt   . . . . . . N L E . . . . . .
pat               N E E D L E
```

mismatch character 'N' in pattern: align text 'N' with rightmost pattern 'N'

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

Case 2b. Mismatch character in pattern (but heuristic no help).



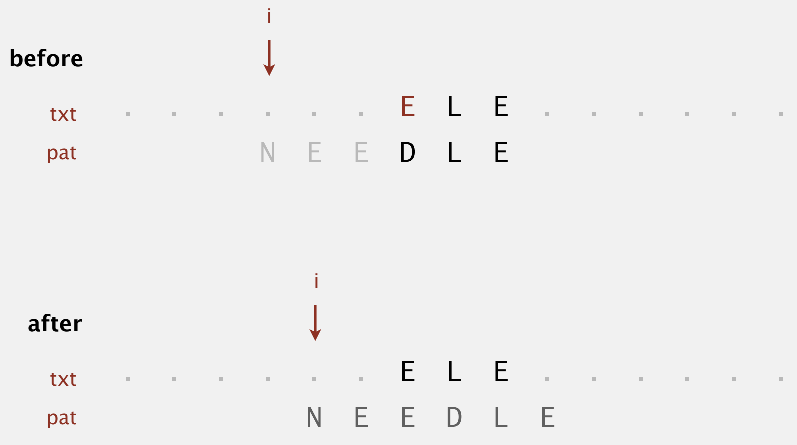
```
                            i
before                      ↓
txt                   . . . . . . E L E . . . . . .
pat                         N E E D L E


                          i
aligned with rightmost E? ↓
txt                   . . . . . . E L E . . . . . .
pat                     N E E D L E
```

mismatch character 'E' in pattern: align text 'E' with rightmost pattern 'E' ?

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

Case 2b. Mismatch character in pattern (but heuristic no help).



```
                i
before          ↓
txt   . . . . . . E L E . . . . . .
pat         N E E D L E


                  i
after             ↓
txt   . . . . . . E L E . . . . . .
pat           N E E D L E
```

mismatch character 'E' in pattern: increment i by 1

Boyer-Moore: mismatched character heuristic

Q. How much to skip?

A. Precompute index of rightmost occurrence of character c in pattern.
(-1 if character not in pattern)

```
right = new int[R];
for (int c = 0; c < R; c++)
   right[c] = -1;
for (int j = 0; j < M; j++)
   right[pat.charAt(j)] = j;
```

c		N 0	E 1	E 2	D 3	L 4	E 5	right[c]
A	-1	-1	-1	-1	-1	-1	-1	-1
B	-1	-1	-1	-1	-1	-1	-1	-1
C	-1	-1	-1	-1	-1	-1	-1	-1
D	-1	-1	-1	-1	3	3	3	3
E	-1	-1	1	2	2	2	5	5
...								-1
L	-1	-1	-1	-1	-1	4	4	4
M	-1	-1	-1	-1	-1	-1	-1	-1
N	-1	0	0	0	0	0	0	0
...								-1

Boyer-Moore skip table computation

Boyer-Moore: Java implementation

```
public int search(String txt)
{
   int N = txt.length();
   int M = pat.length();
   int skip;
   for (int i = 0; i <= N-M; i += skip)
   {
      skip = 0;
      for (int j = M-1; j >= 0; j--)
      {
         if (pat.charAt(j) != txt.charAt(i+j))
         {
            skip = Math.max(1, j - right[txt.charAt(i+j)]);
            break;
         }
      }
      if (skip == 0) return i;
   }
   return N;
}
```

compute skip value

in case other term is zero or negative

match

Boyer-Moore: analysis

Property. Substring search with the Boyer–Moore mismatched character heuristic takes about $\sim N / M$ character compares to search for a pattern of length M in a text of length N.

sublinear!

Worst-case. Can be as bad as $\sim M N$.

```
i skip      0  1  2  3  4  5  6  7  8  9
     txt ⟶  B  B  B  B  B  B  B  B  B  B
0  0        A  B  B  B  B ⟵ pat
1  1           A  B  B  B  B
2  1              A  B  B  B  B
3  1                 A  B  B  B  B
4  1                    A  B  B  B  B
5  1                       A  B  B  B  B
```

Boyer–Moore variant. Can improve worst case to $\sim 3\,N$ character compares by adding a KMP-like rule to guard against repetitive patterns.

http://algs4.cs.princeton.edu

5.3 Substring Search

- introduction
- brute force
- Knuth-Morris-Pratt
- Boyer-Moore
- Rabin-Karp

Michael Rabin
Dick Karp

Rabin-Karp fingerprint search

Basic idea = modular hashing.

- Compute a hash of `pat[0..M-1]`.
- For each `i`, compute a hash of `txt[i..M+i-1]`.
- If pattern hash = text substring hash, check for a match.

```
        pat.charAt(i)
 i      0  1  2  3  4
        2  6  5  3  5   % 997 = 613

                        txt.charAt(i)
 i      0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15
        3  1  4  1  5  9  2  6  5  3  5  8  9  7  9  3
 0      3  1  4  1  5   % 997 = 508
 1         1  4  1  5  9   % 997 = 201
 2            4  1  5  9  2   % 997 = 715
 3               1  5  9  2  6   % 997 = 971
 4                  5  9  2  6  5   % 997 = 442
 5                     9  2  6  5  3   % 997 = 929       match
 6  ← return i = 6        2  6  5  3  5   % 997 = 613
```

modular hashing with R = 10 and hash(s) = s (mod 997)

Modular arithmetic

Math trick. To keep numbers small, take intermediate results modulo Q.

Ex.

$$\begin{aligned}(10000 + 535) * 1000 &\quad (\text{mod } 997)\\ = (30 + 535) * 3 &\quad (\text{mod } 997)\\ = 1695 &\quad (\text{mod } 997)\\ = 698 &\quad (\text{mod } 997)\end{aligned}$$

1000 mod 997 = 3

10000 mod 997 = 30

$$(a + b) \bmod Q \;=\; ((a \bmod Q) + (b \bmod Q)) \bmod Q$$

$$(a * b) \bmod Q \;=\; ((a \bmod Q) * (b \bmod Q)) \bmod Q$$

two useful modular arithmetic identities

Efficiently computing the hash function

Modular hash function. Using the notation t_i for `txt.charAt(i)`, we wish to compute

$$x_i \;=\; t_i\, R^{M-1} + t_{i+1}\, R^{M-2} + \ldots + t_{i+M-1}\, R^0 \;\; (\text{mod } Q)$$

Intuition. M-digit, base-R integer, modulo Q.

Horner's method. Linear-time method to evaluate degree-M polynomial.

```
      pat.charAt()
 i    0  1  2  3  4
      2  6  5  3  5
 0    2  % 997 = 2                  R            Q
 1    2  6  % 997 = (2*10 + 6) % 997 = 26
 2    2  6  5  % 997 = (26*10 + 5) % 997 = 265
 3    2  6  5  3  % 997 = (265*10 + 3) % 997 = 659
 4    2  6  5  3  5  % 997 = (659*10 + 5) % 997 = 613
```

```
// Compute hash for M-digit key
private long hash(String key, int M)
{
   long h = 0;
   for (int j = 0; j < M; j++)
      h = (h * R + key.charAt(j)) % Q;
   return h;
}
```

26535 = 2*10000 + 6*1000 + 5*100 + 3*10 + 5

= ((((2) *10 + 6) * 10 + 5) * 10 + 3) * 10 + 5

Efficiently computing the hash function

Challenge. How to efficiently compute x_{i+1} given that we know x_i.

$$x_i \;=\; t_i\, R^{M-1} + t_{i+1}\, R^{M-2} + \ldots + t_{i+M-1}\, R^0$$

$$x_{i+1} \;=\; t_{i+1}\, R^{M-1} + t_{i+2}\, R^{M-2} + \ldots + t_{i+M}\, R^0$$

Key property. Can update "rolling" hash function in constant time!

$$x_{i+1} \;=\; (\underset{\substack{\uparrow\\ \text{current}\\ \text{value}}}{x_i} \;-\; \underset{\substack{\uparrow\\ \text{subtract}\\ \text{leading digit}}}{t_i\, R^{M-1}})\;\; \underset{\substack{\uparrow\\ \text{multiply}\\ \text{by radix}}}{R} \;+\; \underset{\substack{\uparrow\\ \text{add new}\\ \text{trailing digit}}}{t_{i+M}}$$

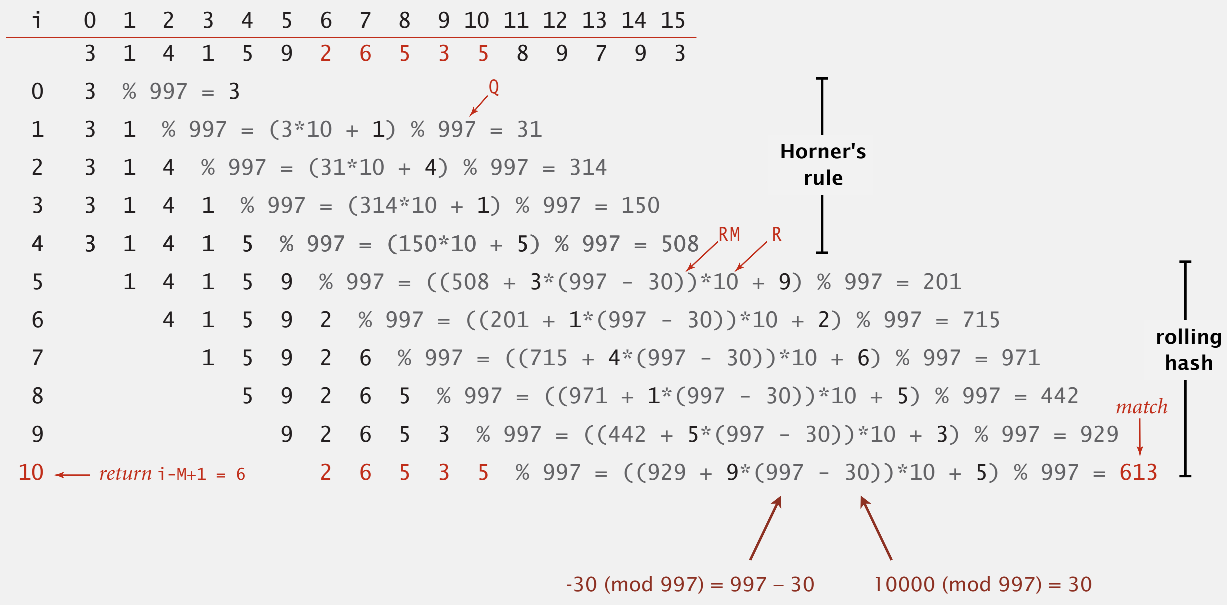
(can precompute R^{M-1})

```
      i       ...  2  3  4  5  6  7  ...
current value  1   4  1  5  9  2  6  5   > text
    new value      4  1  5  9  2  6  5

                   4  1  5  9  2   current value
                -  4  0  0  0  0
                      1  5  9  2   subtract leading digit
                         *  1  0   multiply by radix
                   1  5  9  2  0
                            +  6   add new trailing digit
                   1  5  9  2  6   new value
```

Rabin-Karp substring search example

First R entries: Use Horner's rule.

Remaining entries: Use rolling hash (and % to avoid overflow).



```
 i    0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15
      3  1  4  1  5  9  2  6  5  3  5  8  9  7  9  3
 0    3  % 997 = 3
 1    3  1  % 997 = (3*10 + 1) % 997 = 31
 2    3  1  4  % 997 = (31*10 + 4) % 997 = 314
 3    3  1  4  1  % 997 = (314*10 + 1) % 997 = 150
 4    3  1  4  1  5  % 997 = (150*10 + 5) % 997 = 508
 5       1  4  1  5  9  % 997 = ((508 + 3*(997 - 30))*10 + 9) % 997 = 201
 6          4  1  5  9  2  % 997 = ((201 + 1*(997 - 30))*10 + 2) % 997 = 715
 7             1  5  9  2  6  % 997 = ((715 + 4*(997 - 30))*10 + 6) % 997 = 971
 8                5  9  2  6  5  % 997 = ((971 + 1*(997 - 30))*10 + 5) % 997 = 442
 9                   9  2  6  5  3  % 997 = ((442 + 5*(997 - 30))*10 + 3) % 997 = 929
10 ← return i-M+1 = 6   2  6  5  3  5  % 997 = ((929 + 9*(997 - 30))*10 + 5) % 997 = 613
```

Rabin-Karp: Java implementation

```
public class RabinKarp
{
   private long patHash;    // pattern hash value
   private int M;           // pattern length
   private long Q;          // modulus
   private int R;           // radix
   private long RM1;        // R^(M-1) % Q

   public RabinKarp(String pat) {
      M = pat.length();
      R = 256;
      Q = longRandomPrime();

      RM1 = 1;
      for (int i = 1; i <= M-1; i++)
         RM1 = (R * RM1) % Q;
      patHash = hash(pat, M);
   }

   private long hash(String key, int M)
   {  /* as before */  }

   public int search(String txt)
   {  /* see next slide */  }
}
```

a large prime (but avoid overflow)

precompute R^{M-1} (mod Q)

Rabin-Karp: Java implementation (continued)

Monte Carlo version. Return match if hash match.

```
public int search(String txt)
{
   int N = txt.length();
   int txtHash = hash(txt, M);
   if (patHash == txtHash) return 0;
   for (int i = M; i < N; i++)
   {
      txtHash = (txtHash + Q - RM*txt.charAt(i-M) % Q) % Q;
      txtHash = (txtHash*R + txt.charAt(i)) % Q;
      if (patHash == txtHash) return i - M + 1;
   }
   return N;
}
```

check for hash collision using rolling hash function

Las Vegas version. Modify code to check for substring match if hash match; continue search if false collision.

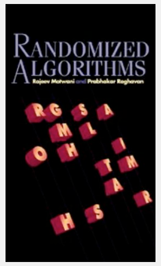
Rabin-Karp analysis

Theory. If Q is a sufficiently large random prime (about $M N^2$), then the probability of a false collision is about $1 / N$.

Practice. Choose Q to be a large prime (but not so large to cause overflow). Under reasonable assumptions, probability of a collision is about $1 / Q$.

Monte Carlo version.

- Always runs in linear time.
- Extremely likely to return correct answer (but not always!).



Las Vegas version.

- Always returns correct answer.
- Extremely likely to run in linear time (but worst case is $M N$).

Rabin–Karp fingerprint search

Advantages.

- Extends to two-dimensional patterns.
- Extends to finding multiple patterns.

Disadvantages.

- Arithmetic ops slower than char compares.
- Las Vegas version requires backup.
- Poor worst-case guarantee.

Q. How would you extend Rabin–Karp to efficiently search for any one of P possible patterns in a text of length N?

Substring search cost summary

Cost of searching for an M-character pattern in an N-character text.

algorithm	version	operation count		backup in input?	correct?	extra space
		guarantee	typical			
brute force	—	$M N$	$1.1\ N$	*yes*	*yes*	1
Knuth-Morris-Pratt	*full DFA* *(Algorithm 5.6)*	$2N$	$1.1\ N$	*no*	*yes*	MR
	mismatch *transitions only*	$3N$	$1.1\ N$	*no*	*yes*	M
Boyer-Moore	*full algorithm*	$3N$	N / M	*yes*	*yes*	R
	mismatched char *heuristic only* *(Algorithm 5.7)*	$M N$	N / M	*yes*	*yes*	R
Rabin-Karp†	*Monte Carlo* *(Algorithm 5.8)*	$7\ N$	$7\ N$	*no*	*yes* †	1
	Las Vegas	$7\ N$ †	$7\ N$	*yes*	*yes*	1

† probabilisitic guarantee, with uniform hash function