http://algs4.cs.princeton.edu

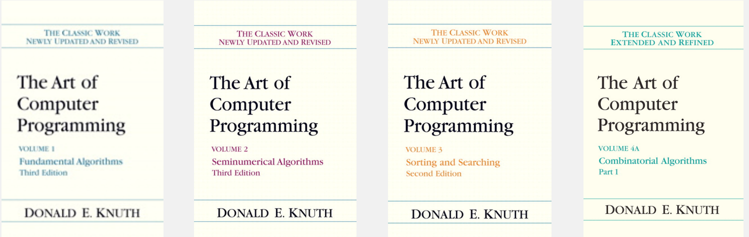
3.4 HASH TABLES

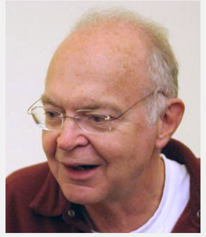
- *hash functions*
- *separate chaining*
- *linear probing*
- *context*

Last updated on Mar 4, 2015, 9:56 AM

Premature optimization

> " *Programmers waste enormous amounts of time thinking about, or worrying about, the speed of noncritical parts of their programs, and these attempts at efficiency actually have a strong negative impact when debugging and maintenance are considered.*
>
> *We should forget about small efficiencies, say about 97% of the time: premature optimization is the root of all evil.*
>
> *Yet we should not pass up our opportunities in that critical 3%.* "





Symbol table implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N	N	N		`equals()`
binary search (ordered array)	$\log N$	N	N	$\log N$	N	N	✔	`compareTo()`
BST	N	N	N	$\log N$	$\log N$	$\sqrt{N}$	✔	`compareTo()`
red-black BST	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`

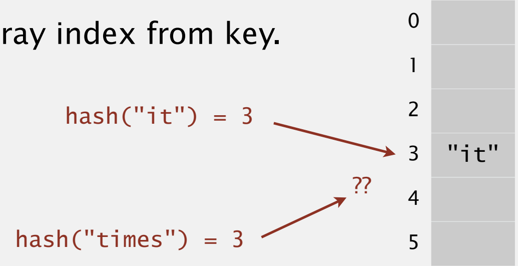
Q. Can we do better?

A. Yes, but with different access to the data.

Hashing: basic plan

Save items in a key-indexed table (index is a function of the key).

Hash function. Method for computing array index from key.

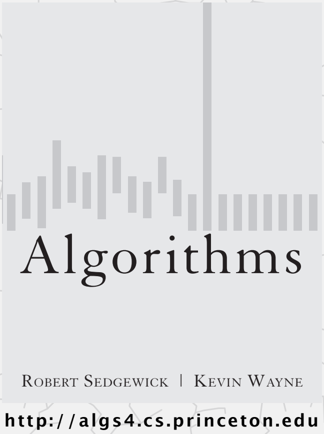


Issues.

- Computing the hash function.
- Equality test: Method for checking whether two keys are equal.
- Collision resolution: Algorithm and data structure to handle two keys that hash to the same array index.

Classic space-time tradeoff.

- No space limitation: trivial hash function with key as index.
- No time limitation: trivial collision resolution with sequential search.
- Space and time limitations: hashing (the real world).



3.4 HASH TABLES

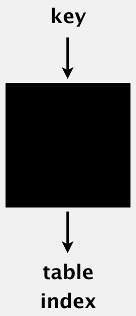
- *hash functions*
- *separate chaining*
- *linear probing*
- *context*

Computing the hash function

Idealistic goal. Scramble the keys uniformly to produce a table index.

- Efficiently computable.
- Each table index equally likely for each key.

thoroughly researched problem,
still problematic in practical applications



Ex. Social Security numbers.

- Bad: first three digits.
- Better: last three digits.

573 = California, 574 = Alaska
(assigned in chronological order within geographic region)

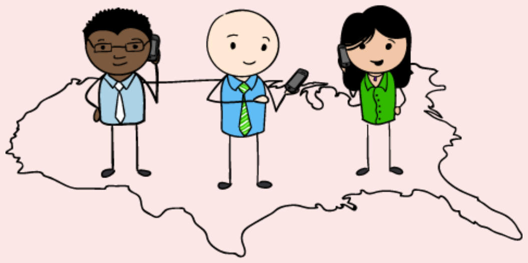
Practical challenge. Need different approach for each key type.

Hash tables: quiz 1

Which of the following would be a good hash function for U.S. phone numbers to integers between 0 and 999?

A. First three digits.
B. Second three digits.
C. Last three digits.
D. Either B or C.
E. *I don't know.*

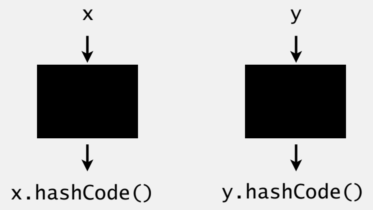
(609) 867-5309



Java's hash code conventions

All Java classes inherit a method `hashCode()`, which returns a 32-bit `int`.

Requirement. If `x.equals(y)`, then `(x.hashCode() == y.hashCode())`.
Highly desirable. If `!x.equals(y)`, then `(x.hashCode() != y.hashCode())`.



Default implementation. Memory address of `x`.
Legal (but poor) implementation. Always return `17`.
Customized implementations. `Integer`, `Double`, `String`, `File`, `URL`, `Date`, …
User-defined types. Users are on their own.

Implementing hash code: integers, booleans, and doubles

Java library implementations

```
public final class Integer
{
   private final int value;
   ...

   public int hashCode()
   {  return value;  }
}
```

```
public final class Boolean
{
   private final boolean value;
   ...

   public int hashCode()
   {
      if (value) return 1231;
      else       return 1237;
   }
}
```

```
public final class Double
{
   private final double value;
   ...

   public int hashCode()
   {
      long bits = doubleToLongBits(value);
      return (int) (bits ^ (bits >>> 32));
   }
}
```

convert to IEEE 64-bit representation;
xor most significant 32-bits
with least significant 32-bits

Warning: -0.0 and +0.0 have different hash codes

Implementing hash code: strings

Treat string of length L as L-digit, base-31 number:

$$h = s[0] \cdot 31^{L-1} + \ldots + s[L-3] \cdot 31^2 + s[L-2] \cdot 31^1 + s[L-1] \cdot 31^0$$

```
public final class String
{
   private final char[] s;
   :
   public int hashCode()
   {
      int hash = 0;
      for (int i = 0; i < length(); i++)
         hash = s[i] + (31 * hash);
      return hash;
   }
}
```

Java library implementation

char	Unicode
…	…
'a'	97
'b'	98
'c'	99
…	…

Horner's method: only L multiplies/adds to hash string of length L.

```
String s = "call";
s.hashCode();
```

$3045982 = 99 \cdot 31^3 + 97 \cdot 31^2 + 108 \cdot 31^1 + 108 \cdot 31^0$
$= 108 + 31 \cdot (108 + 31 \cdot (97 + 31 \cdot (99)))$

Implementing hash code: strings

Performance optimization.

- Cache the hash value in an instance variable.
- Return cached value.

```
public final class String
{
   private int hash = 0;
   private final char[] s;
   ...

   public int hashCode()
   {
      int h = hash;
      if (h != 0) return h;
      for (int i = 0; i < length(); i++)
         h = s[i] + (31 * h);
      hash = h;
      return h;
   }
}
```

cache of hash code

return cached value

store cache of hash code

Q. What if `hashCode()` of string is 0?

`hashCode()` of `"pollinating sandboxes"` is 0

Implementing hash code: user-defined types

```
public final class Transaction implements Comparable<Transaction>
{
   private final String  who;
   private final Date    when;
   private final double  amount;

   public Transaction(String who, Date when, double amount)
   {  /* as before */  }

   ...

   public boolean equals(Object y)
   {  /* as before */  }

   public int hashCode()
   {
      int hash = 17;
      hash = 31*hash + who.hashCode();
      hash = 31*hash + when.hashCode();
      hash = 31*hash + ((Double) amount).hashCode();
      return hash;
   }
}
```

nonzero constant

for reference types,
use hashCode()

for primitive types,
use hashCode()
of wrapper type

typically a small prime

Hash code design

"Standard" recipe for user-defined types.

- Combine each significant field using the $31x + y$ rule.
- If field is a primitive type, use wrapper type `hashCode()`.
- If field is `null`, use 0.
- If field is a reference type, use `hashCode()`. ← applies rule recursively
- If field is an array, apply to each entry. ← or use `Arrays.deepHashCode()`

In practice. Recipe above works reasonably well; used in Java libraries.

In theory. Keys are bitstring; "universal" family of hash functions exist.

awkward in Java since only one (deterministic) `hashCode()`

Basic rule. Need to use the whole key to compute hash code; consult an expert for state-of-the-art hash codes.

Hash tables: quiz 1

Which of the following is an effective way to map a hashable key to an integer between `0` and `M-1` ?

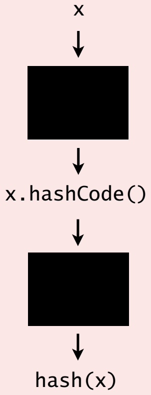
A.
```
private int hash(Key key)
{  return key.hashCode() % M;  }
```

B.
```
private int hash(Key key)
{  return Math.abs(key.hashCode()) % M;  }
```

C. Both A and B.

D. Neither A nor B.

E. *I don't know.*



Modular hashing

Hash code. An `int` between -2^{31} and $2^{31} - 1$.

Hash function. An `int` between `0` and `M - 1` (for use as array index).

typically a prime or power of 2

```
private int hash(Key key)
{  return key.hashCode() % M;  }
```

bug

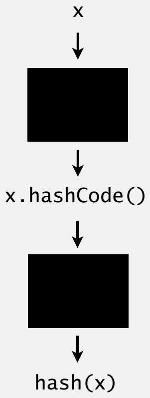
```
private int hash(Key key)
{  return Math.abs(key.hashCode()) % M;  }
```

1-in-a-billion bug

`hashCode()` of `"polygenelubricants"` is -2^{31}

```
private int hash(Key key)
{  return (key.hashCode() & 0x7fffffff) % M;  }
```

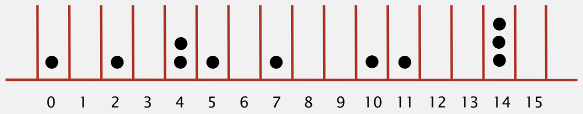
correct



Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to an integer between 0 and $M - 1$.

Bins and balls. Throw balls uniformly at random into M bins.



Birthday problem. Expect two balls in the same bin after $\sim \sqrt{\pi M / 2}$ tosses.

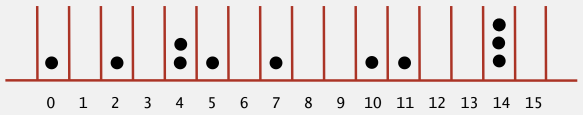
Coupon collector. Expect every bin has ≥ 1 ball after $\sim M \ln M$ tosses.

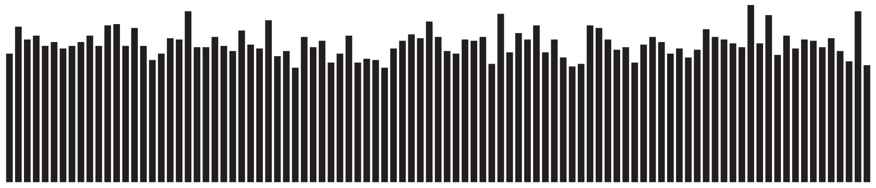
Load balancing. After M tosses, expect most loaded bin has $\sim \ln M / \ln \ln M$ balls.

Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to an integer between 0 and $M - 1$.

Bins and balls. Throw balls uniformly at random into M bins.





Hash value frequencies for words in Tale of Two Cities (M = 97)

Java's `String` data uniformly distribute the keys of Tale of Two Cities

---

http://algs4.cs.princeton.edu

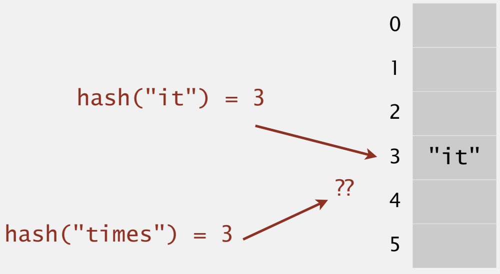
3.4 HASH TABLES

- hash functions
- *separate chaining*
- linear probing
- context

Collisions

Collision. Two distinct keys hashing to same index.

- Birthday problem ⇒ can't avoid collisions. ← unless you have a ridiculous (quadratic) amount of memory
- Coupon collector ⇒ not too much wasted space.
- Load balancing ⇒ no index gets too many collisions.



Challenge. Deal with collisions efficiently.

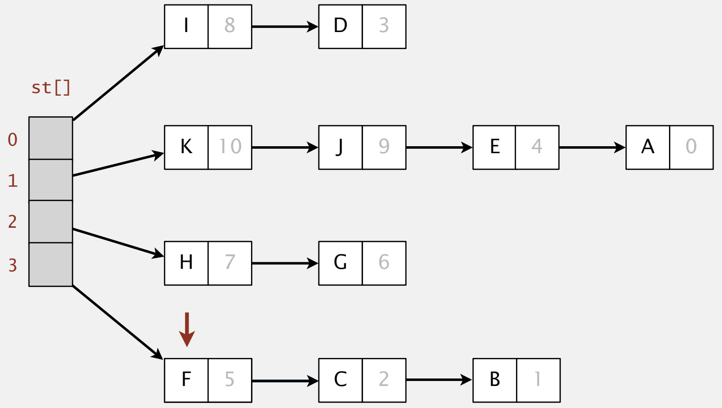
Separate-chaining symbol table

Use an array of $M < N$ linked lists. [H. P. Luhn, IBM 1953]

- Hash: map key to integer i between 0 and $M - 1$.
- Insert: put at front of i^{th} chain (if not already in chain).
- Search: sequential search in i^{th} chain.

put(L, 11)
hash(L) = 3

separate-chaining hash table (M = 4)

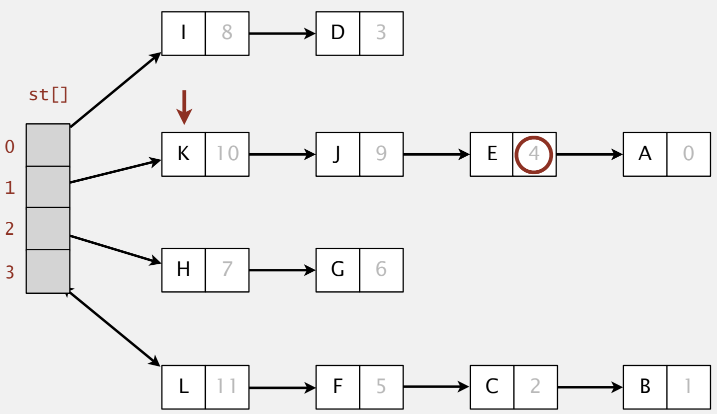


Separate-chaining symbol table

Use an array of $M < N$ linked lists. [H. P. Luhn, IBM 1953]

- Hash: map key to integer i between 0 and $M - 1$.
- Insert: put at front of i^{th} chain (if not already in chain).
- Search: sequential search in i^{th} chain.

separate-chaining hash table (M = 4)



get(E)
hash(E) = 1

Separate-chaining symbol table: Java implementation

```
public class SeparateChainingHashST<Key, Value>
{
   private int M = 97;                // number of chains
   private Node[] st = new Node[M];   // array of chains

   private static class Node
   {
      private Object key;
      private Object val;
      private Node next;
      ...
   }

   private int hash(Key key)
   {  return (key.hashCode() & 0x7fffffff) % M;  }

   public Value get(Key key) {
      int i = hash(key);
      for (Node x = st[i]; x != null; x = x.next)
         if (key.equals(x.key)) return (Value) x.val;
      return null;
   }

}
```

array doubling and halving code omitted

no generic array creation
(declare key and value of type Object)

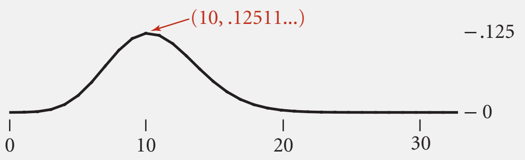
Separate-chaining symbol table: Java implementation

```
public class SeparateChainingHashST<Key, Value>
{
   private int M = 97;                // number of chains
   private Node[] st = new Node[M];   // array of chains

   private static class Node
   {
      private Object key;
      private Object val;
      private Node next;
      ...
   }

   private int hash(Key key)
   {  return (key.hashCode() & 0x7fffffff) % M;  }

   public void put(Key key, Value val) {
      int i = hash(key);
      for (Node x = st[i]; x != null; x = x.next)
         if (key.equals(x.key)) { x.val = val; return; }
      st[i] = new Node(key, val, st[i]);
   }

}
```

Analysis of separate chaining

Proposition. Under uniform hashing assumption, prob. that the number of keys in a list is within a constant factor of N / M is extremely close to 1.

Pf sketch. Distribution of list size obeys a binomial distribution.



Binomial distribution ($N = 10^4$, $M = 10^3$, $\alpha = 10$)

equals() and hashCode()

Consequence. Number of probes for search/insert is proportional to N / M.

- M too large $\Rightarrow$ too many empty chains.
- M too small $\Rightarrow$ chains too long.
- Typical choice: $M \sim \frac{1}{4} N$ $\Rightarrow$ constant-time ops.

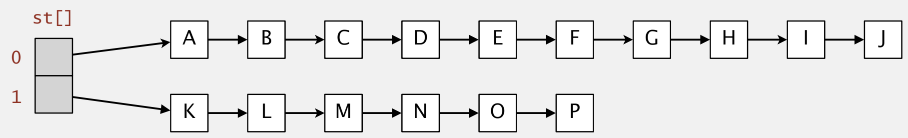
M times faster than sequential search

Resizing in a separate-chaining hash table

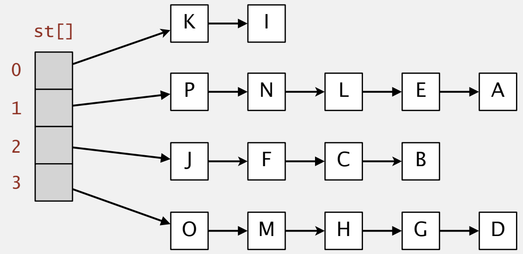
Goal. Average length of list N / M = constant.

- Double size of array M when $N / M \geq 8$;
  halve size of array M when $N / M \leq 2$.
- Note: need to rehash all keys when resizing. ← `x.hashCode()` does not change; but `hash(x)` can change

before resizing (N/M = 8)



after resizing (N/M = 4)

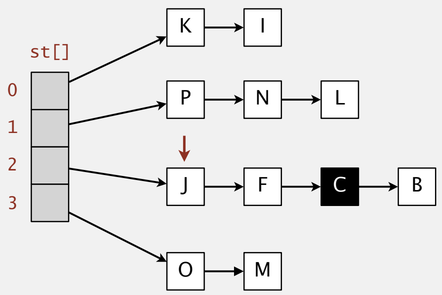


Deletion in a separate-chaining hash table

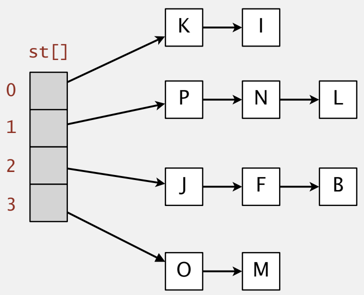
Q. How to delete a key (and its associated value)?

A. Easy: need to consider only chain containing key.

before deleting C



after deleting C



Symbol table implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N	N	N		`equals()`
binary search (ordered array)	$\log N$	N	N	$\log N$	N	N	✔	`compareTo()`
BST	N	N	N	$\log N$	$\log N$	$\sqrt{N}$	✔	`compareTo()`
red-black BST	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`
separate chaining	N	N	N	1 *	1 *	1 *		`equals()` `hashCode()`

* under uniform hashing assumption

http://algs4.cs.princeton.edu

3.4 HASH TABLES

- hash functions
- separate chaining
- *linear probing*
- context

Collision resolution: open addressing

Open addressing. [Amdahl-Boehme-Rocherster-Samuel, IBM 1953]

- Maintain keys and values in two parallel arrays.
- When a new key collides, find next empty slot, and put it there.

linear-probing hash table (M = 16, N =10)

put(K, 14)
hash(K) = 7

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C		H	L		E				R	X
								K								
								14								
vals[]	11	10			9	5		6	12		13				4	8

Linear-probing hash table summary

Hash. Map key to integer i between 0 and $M - 1$.

Insert. Put at table index i if free; if not try $i + 1$, $i + 2$, etc.

Search. Search table index i; if occupied but no match, try $i + 1$, $i + 2$, etc.

Note. Array size M must be greater than number of key-value pairs N.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C	S	H	L		E				R	X

M = 16

Linear-probing symbol table: Java implementation

```
public class LinearProbingHashST<Key, Value>
{
   private int M = 30001;
   private Value[] vals = (Value[]) new Object[M];
   private Key[]   keys = (Key[])   new Object[M];

   private int hash(Key key)            { /* as before  */  }

   private void put(Key key, Value val) { /* next slide */  }

   public Value get(Key key)
   {
      for (int i = hash(key); keys[i] != null; i = (i+1) % M)
         if (key.equals(keys[i]))
             return vals[i];
      return null;
   }

}
```

array doubling and halving code omitted

sequential search in chain i

Linear-probing symbol table: Java implementation

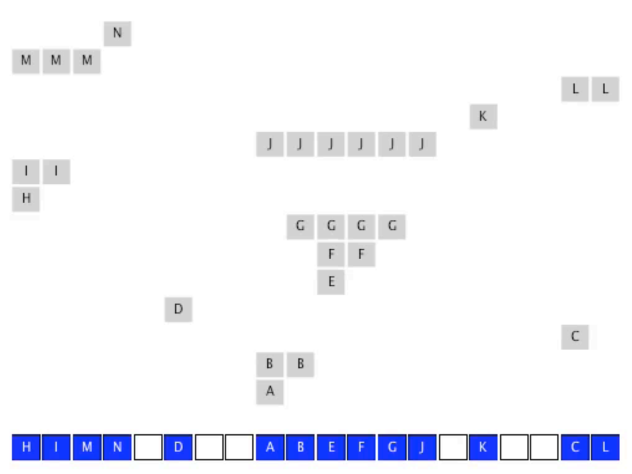
```
public class LinearProbingHashST<Key, Value>
{
   private int M = 30001;
   private Value[] vals = (Value[]) new Object[M];
   private Key[]   keys = (Key[])   new Object[M];

   private int hash(Key key)            { /* as before  */  }

   private Value get(Key key)           { /* prev slide */  }

   public void put(Key key, Value val)
   {
      int i;
      for (i = hash(key); keys[i] != null; i = (i+1) % M)
         if (keys[i].equals(key))
             break;
      keys[i] = key;
      vals[i] = val;
   }

}
```

sequential search in chain i

Clustering

Cluster. A contiguous block of items.

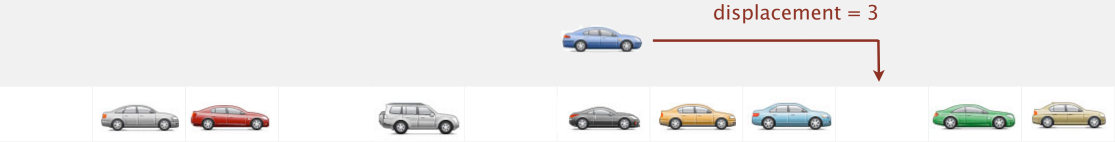
Observation. New keys likely to hash into middle of big clusters.



Knuth's parking problem

Model. Cars arrive at one-way street with M parking spaces. Each desires a random space i : if space i is taken, try $i + 1$, $i + 2$, etc.

Q. What is mean displacement of a car?



Half-full. With $M / 2$ cars, mean displacement is ~ $5 / 2$.

Full. With M cars, mean displacement is ~ $\sqrt{\pi M / 8}$.

Key insight. Cannot afford to let linear-probing hash table get too full.

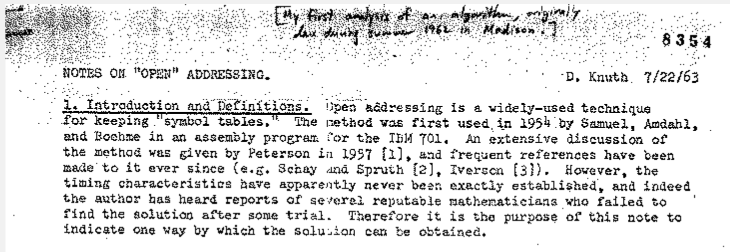
Analysis of linear probing

Proposition. Under uniform hashing assumption, the average # of probes in a linear probing hash table of size M that contains $N = \alpha M$ keys is:

$$\sim \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right) \qquad \sim \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

search hit **search miss / insert**

Pf.

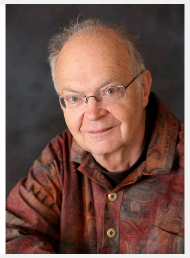


[My first analysis of an algorithm, originally done during Summer 1962 in Madison.]

8354

NOTES ON "OPEN" ADDRESSING. D. Knuth 7/22/63

1. Introduction and Definitions. Open addressing is a widely-used technique for keeping "symbol tables." The method was first used in 1954 by Samuel, Amdahl, and Boehme in an assembly program for the IBM 701. An extensive discussion of the method was given by Peterson in 1957 [1], and frequent references have been made to it ever since (e.g. Schay and Spruth [2], Iverson [3]). However, the timing characteristics have apparently never been exactly established, and indeed the author has heard reports of several reputable mathematicians who failed to find the solution after some trial. Therefore it is the purpose of this note to indicate one way by which the solution can be obtained.



Parameters.

- M too large ⇒ too many empty array entries.
- M too small ⇒ search time blows up.
- Typical choice: $\alpha = N / M \sim ½$. ←

probes for search hit is about 3/2
probes for search miss is about 5/2

Resizing in a linear-probing hash table

Goal. Average length of list $N / M \leq ½$.

- Double size of array M when $N / M \geq ½$.
- Halve size of array M when $N / M \leq ⅛$.
- Need to rehash all keys when resizing.

before resizing

	0	1	2	3	4	5	6	7
keys[]		E	S			R	A	
vals[]		1	0			3	2	

after resizing

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]					A		S				E				R	
vals[]					2		0				1				3	

Deletion in a linear-probing hash table

Q. How to delete a key (and its associated value)?

A. Requires some care: can't just delete array entries.

before deleting S

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
`keys[]`	P	M			A	C	S	H	L		E				R	X
`vals[]`	10	9			8	4	0	5	11		12				3	7

after deleting S ?

doesn't work, e.g., if hash(H) = 4

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
`keys[]`	P	M			A	C		H	L		E				R	X
`vals[]`	10	9			8	4		5	11		12				3	7

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N	N	N		`equals()`
binary search (ordered array)	$\log N$	N	N	$\log N$	N	N	✔	`compareTo()`
BST	N	N	N	$\log N$	$\log N$	$\sqrt{N}$	✔	`compareTo()`
red-black BST	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`
separate chaining	N	N	N	1 *	1 *	1 *		`equals()` `hashCode()`
linear probing	N	N	N	1 *	1 *	1 *		`equals()` `hashCode()`

* under uniform hashing assumption

3-SUM (REVISITED)

3-SUM. Given N distinct integers, find three such that $a + b + c = 0$.

Goal. N^2 expected time case, N extra space.

http://algs4.cs.princeton.edu

3.4 Hash Tables

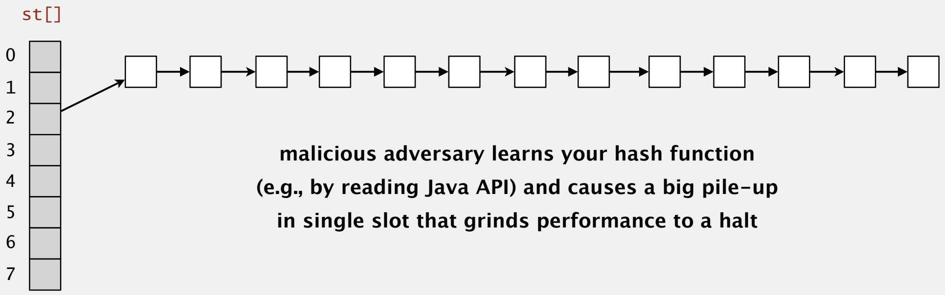
- hash functions
- separate chaining
- linear probing
- *context*

War story: algorithmic complexity attacks

Q. Is the uniform hashing assumption important in practice?

A. Obvious situations: aircraft control, nuclear reactor, pacemaker, HFT, …

A. Surprising situations: denial-of-service attacks.



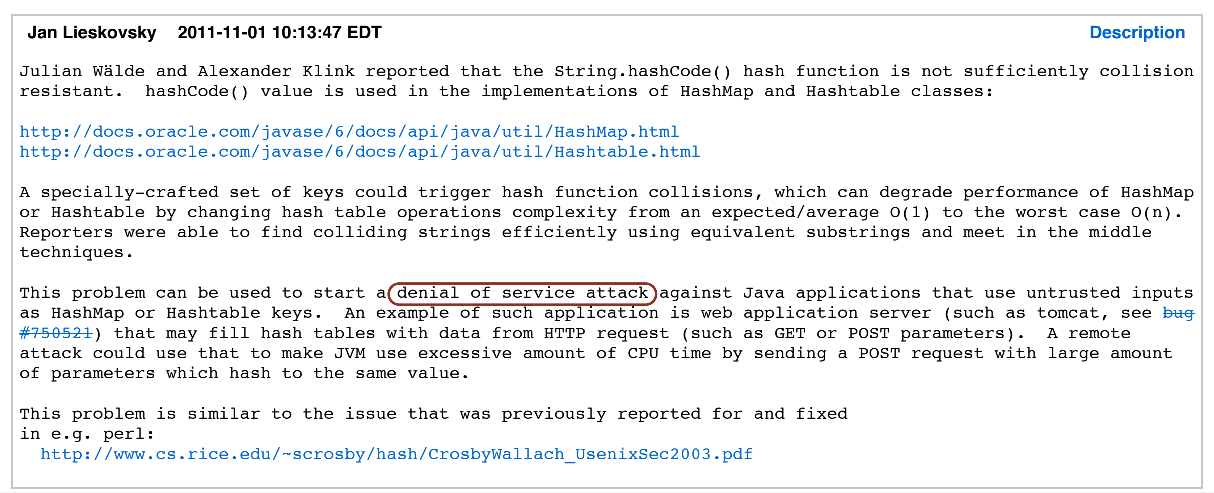
malicious adversary learns your hash function (e.g., by reading Java API) and causes a big pile-up in single slot that grinds performance to a halt

Real-world exploits. [Crosby-Wallach 2003]

- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem.
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

War story: algorithmic complexity attacks

A Java bug report.



Jan Lieskovsky **2011-11-01 10:13:47 EDT** **Description**

```
Julian Wälde and Alexander Klink reported that the String.hashCode() hash function is not sufficiently collision
resistant.  hashCode() value is used in the implementations of HashMap and Hashtable classes:

http://docs.oracle.com/javase/6/docs/api/java/util/HashMap.html
http://docs.oracle.com/javase/6/docs/api/java/util/Hashtable.html

A specially-crafted set of keys could trigger hash function collisions, which can degrade performance of HashMap
or Hashtable by changing hash table operations complexity from an expected/average O(1) to the worst case O(n).
Reporters were able to find colliding strings efficiently using equivalent substrings and meet in the middle
techniques.

This problem can be used to start a denial of service attack against Java applications that use untrusted inputs
as HashMap or Hashtable keys.  An example of such application is web application server (such as tomcat, see bug
#750521) that may fill hash tables with data from HTTP request (such as GET or POST parameters).  A remote
attack could use that to make JVM use excessive amount of CPU time by sending a POST request with large amount
of parameters which hash to the same value.

This problem is similar to the issue that was previously reported for and fixed
in e.g. perl:
  http://www.cs.rice.edu/~scrosby/hash/CrosbyWallach_UsenixSec2003.pdf
```

Algorithmic complexity attack on Java

Goal. Find family of strings with the same `hashCode()`.

Solution. The base-31 hash code is part of Java's `String` API.

key	hashCode()
"Aa"	2112
"BB"	2112

key	hashCode()
"AaAaAaAa"	-540425984
"AaAaAaBB"	-540425984
"AaAaBBAa"	-540425984
"AaAaBBBB"	-540425984
"AaBBAaAa"	-540425984
"AaBBAaBB"	-540425984
"AaBBBBAa"	-540425984
"AaBBBBBB"	-540425984

key	hashCode()
"BBAaAaAa"	-540425984
"BBAaAaBB"	-540425984
"BBAaBBAa"	-540425984
"BBAaBBBB"	-540425984
"BBBBAaAa"	-540425984
"BBBBAaBB"	-540425984
"BBBBBBAa"	-540425984
"BBBBBBBB"	-540425984

2^N strings of length 2N that hash to same value!

Diversion: one-way hash functions

One-way hash function. "Hard" to find a key that will hash to a desired value (or two keys that hash to same value).

Ex. $\underbrace{\text{MD4, MD5, SHA-0, SHA-1}}_{\text{known to be insecure}}$, SHA-2, WHIRLPOOL, RIPEMD-160, ….

```
String password = args[0];
MessageDigest sha1 = MessageDigest.getInstance("SHA1");
byte[] bytes = sha1.digest(password);

/* prints bytes as hex string */
```

Applications. Crypto, message digests, passwords, Bitcoin, ….

Caveat. Too expensive for use in ST implementations.

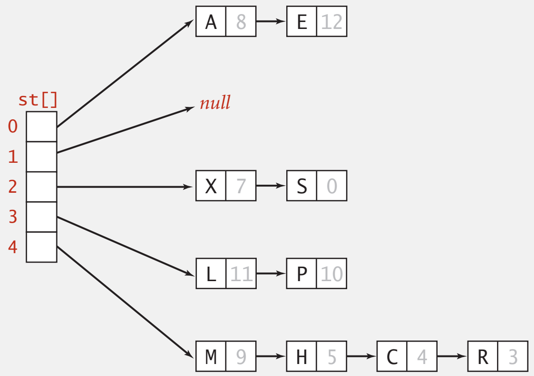
Separate chaining vs. linear probing

Separate chaining.

- Performance degrades gracefully.
- Clustering less sensitive to poorly-designed hash function.

Linear probing.

- Less wasted space.
- Better cache performance.



	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C	S	H	L		E				R	X
vals[]	10	9			8	4	0	5	11		12				3	7

Hashing: variations on the theme

Many improved versions have been studied.

Two-probe hashing. [separate-chaining variant]

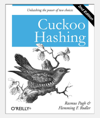
- Hash to two positions, insert key in shorter of the two chains.
- Reduces expected length of the longest chain to $\sim \lg \ln N$.

Double hashing. [linear-probing variant]

- Use linear probing, but skip a variable amount, not just 1 each time.
- Effectively eliminates clustering.
- Can allow table to become nearly full.
- More difficult to implement delete.

Cuckoo hashing. [linear-probing variant]

- Hash key to two positions; insert key into either position; if occupied, reinsert displaced key into its alternative position (and recur).
- Constant worst-case time for search.



Hash tables vs. balanced search trees

Hash tables.

- Simpler to code.
- No effective alternative for unordered keys.
- Faster for simple keys (a few arithmetic ops versus $\log N$ compares).
- Better system support in Java for `String` (e.g., cached hash code).

Balanced search trees.

- Stronger performance guarantee.
- Support for ordered ST operations.
- Easier to implement `compareTo()` correctly than `equals()` and `hashCode()`.

Java system includes both.

- Red-black BSTs: `java.util.TreeMap`, `java.util.TreeSet`.
- Hash tables: `java.util.HashMap`, `java.util.IdentityHashMap`.

linear probing

separate chaining