Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

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6.5 REDUCTIONS

- *introduction*
- *designing algorithms*
- *establishing lower bounds*
- *classifying problems*
- *intractability*

Overview: introduction to advanced topics

Main topics. [final two lectures]

- Reduction: relationship between two problems.
- Algorithm design: paradigms for solving problems.

Shifting gears.

- From individual problems to problem-solving models.
- From linear/quadratic to polynomial/exponential scale.
- From implementation details to conceptual frameworks.

Goals.

- Place algorithms and techniques we've studied in a larger context.
- Introduce you to important and essential ideas.
- Inspire you to learn more about algorithms!

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6.5 REDUCTIONS

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- *intractability*

Bird's-eye view

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	*min, max, median,* *Burrows-Wheeler transform, ...*
linearithmic	$N \log N$	*sorting, element distinctness,* *closest pair, Euclidean MST, ...*
quadratic	N^2	?
⋮	⋮	⋮
exponential	c^N	?

Frustrating news. Huge number of problems have defied classification.

Bird's-eye view

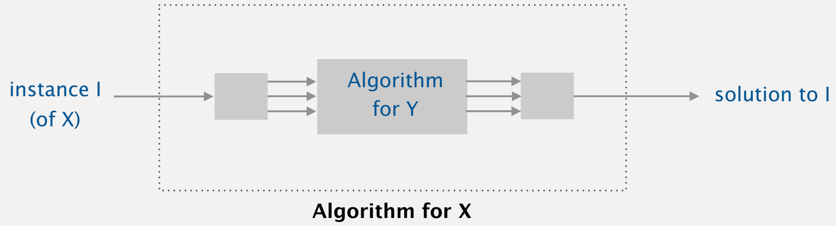
Desiderata. Classify problems according to computational requirements.

Desiderata'. Suppose we could (could not) solve problem X efficiently. What else could (could not) we solve efficiently?

> *" Give me a lever long enough and a fulcrum on which to place it, and I shall move the world. "* — *Archimedes*

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



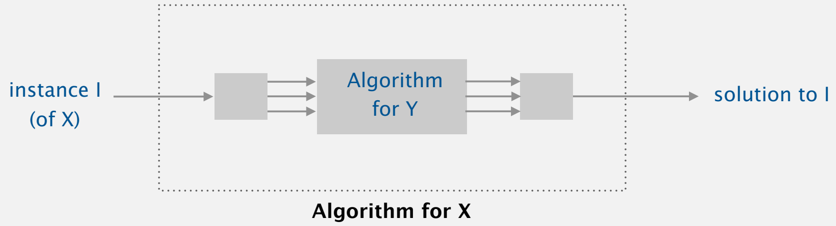
Cost of solving X = total cost of solving Y + cost of reduction.

perhaps many calls to Y on problems of different sizes (though, typically only one call)

preprocessing and postprocessing (typically less than cost of solving Y)

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 1. [finding the median reduces to sorting]

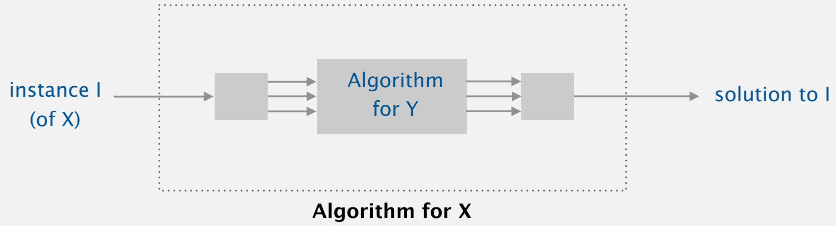
To find the median of N items:

- Sort N items.
- Return item in the middle.

Cost of solving finding the median. $\overset{\text{cost of sorting}}{N \log N} + \overset{\text{cost of reduction}}{1}$.

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 2. [element distinctness reduces to sorting]

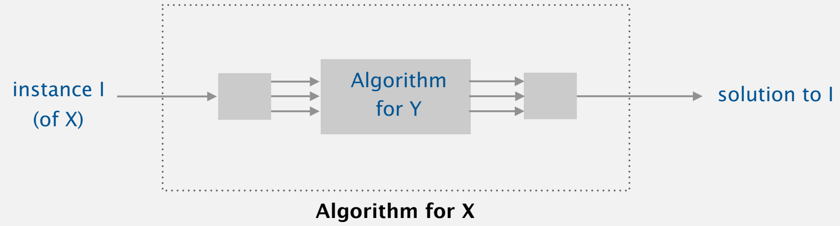
To solve element distinctness on N items:

- Sort N items.
- Check adjacent pairs for equality.

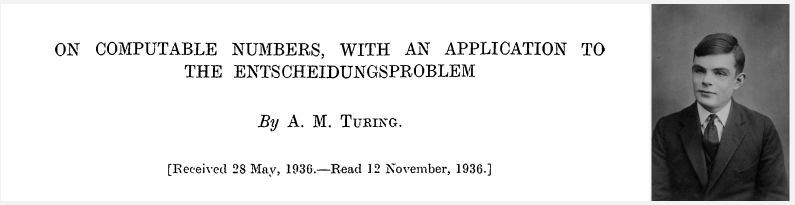
Cost of solving element distinctness. $\overset{\text{cost of sorting}}{N \log N} + \overset{\text{cost of reduction}}{N}$.

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Novice error. Confusing X *reduces to* Y with Y *reduces to* X.



ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO
THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

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6.5 Reductions

- *introduction*
- *designing algorithms*
- *establishing lower bounds*
- *classifying problems*
- *intractability*

Reduction: design algorithms

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.

Design algorithm. Given algorithm for Y, can also solve X.

More familiar reductions.

- CPM reduces to topological sort.
- Arbitrage reduces to negative cycles.
- Bipartite matching reduces to maxflow.
- Seam carving reduces to shortest paths in a DAG.
- Burrows-Wheeler transform reduces to suffix sort.

…

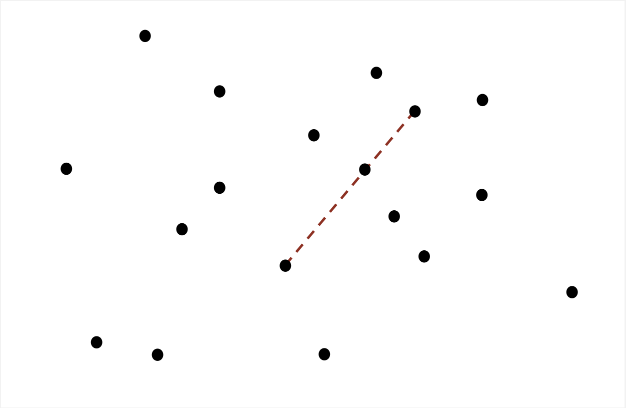
Mentality. Since I know how to solve Y, can I use that algorithm to solve X?

↑

programmer's version: I have code for Y. Can I use it for X?

3-collinear

3-COLLINEAR. Given N distinct points in the plane, are there 3 (or more) that all lie on the same line?



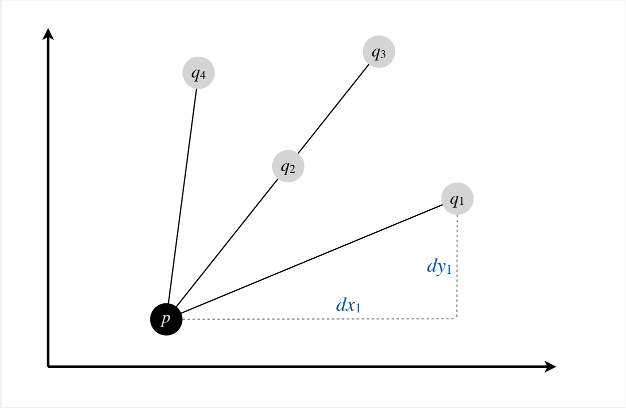
3-collinear

Brute force N^3. For all triples of points (p, q, r) check if they are collinear.

3-collinear reduces to sorting

Sorting-based algorithm. For each point p,

- Compute the slope that each other point q makes with p.
- Sort the remaining $N - 1$ points by slope.
- Collinear points are adjacent.



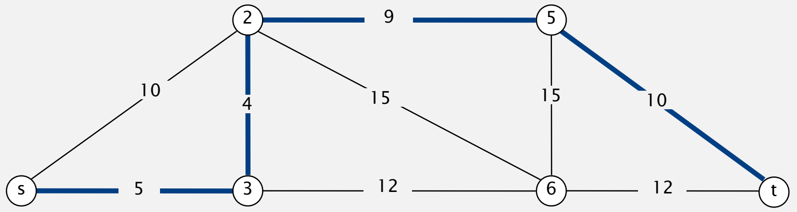
cost of sorting (N times)

cost of reduction

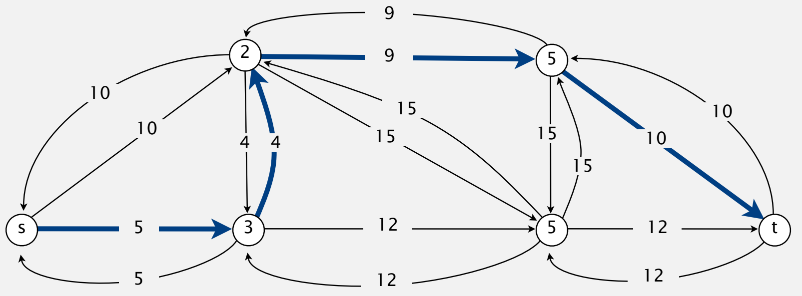
Cost of solving 3-collinear. $N^2 \log N + N^2$.

Shortest paths on edge-weighted graphs and digraphs

Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.

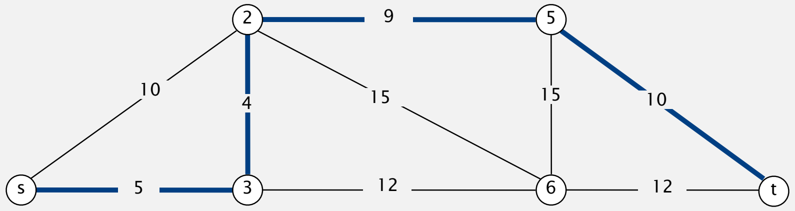


Pf. Replace each undirected edge by two directed edges.



Shortest paths on edge-weighted graphs and digraphs

Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.



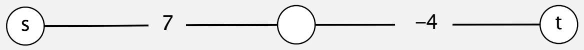
cost of shortest paths in digraph

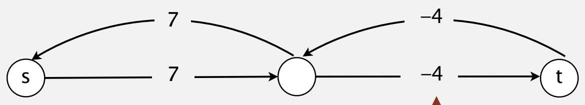
cost of reduction

Cost of undirected shortest paths. $E \log V + (E + V)$.

Shortest paths with negative weights

Caveat. Reduction is invalid for edge-weighted graphs with negative weights (even if no negative cycles).



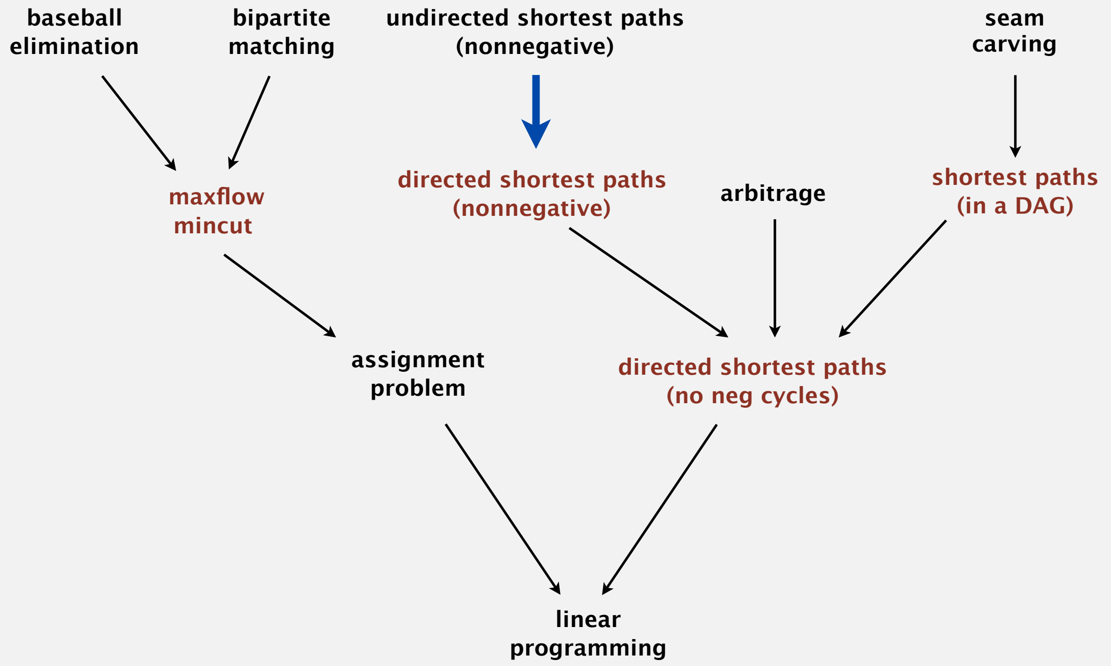


reduction creates negative cycles

Remark. Can still solve shortest-paths problem in undirected graphs (if no negative cycles), but need more sophisticated techniques.

reduces to weighted non-bipartite matching (!)

Some reductions in combinatorial optimization



baseball elimination

bipartite matching

undirected shortest paths (nonnegative)

seam carving

maxflow mincut

directed shortest paths (nonnegative)

arbitrage

shortest paths (in a DAG)

assignment problem

directed shortest paths (no neg cycles)

linear programming

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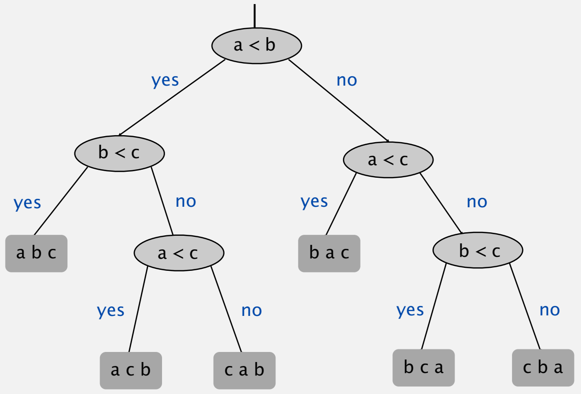
6.5 REDUCTIONS

- introduction
- designing algorithms
- *establishing lower bounds*
- classifying problems
- intractability

Bird's-eye view

Goal. Prove that a problem requires a certain number of steps.

Ex. In decision tree model, any compare-based sorting algorithm requires $\Omega(N \log N)$ compares in the worst case.



argument must apply to all conceivable algorithms

Bad news. Very difficult to establish lower bounds from scratch.

Good news. Spread $\Omega(N \log N)$ lower bound to Y by reducing sorting to Y.

assuming cost of reduction is not too high

Linear-time reductions

Def. Problem X linear-time reduces to problem Y if X can be solved with:

- Linear number of standard computational steps.
- Constant number of calls to Y.

Ex. Almost all of the reductions we've seen so far. [Exceptions?]

Establish lower bound:

- If X takes $\Omega(N \log N)$ steps, then so does Y.
- If X takes $\Omega(N^2)$ steps, then so does Y.

Mentality.

- If I could easily solve Y, then I could easily solve X.
- I can't easily solve X.
- Therefore, I can't easily solve Y.

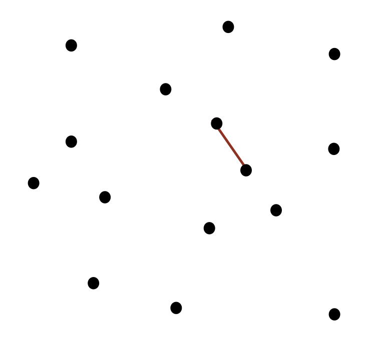
Element distinctness linear-time reduces to 2d closest pair

Element distinctness. Given N elements, are any two equal?

2d closest pair. Given N points in the plane, find the closest pair.

590584
-23439854
1251432
-2861534
3988818
-43434213
333255
13546464
89885444
-43434213
11998833

element distinctness



2d closest pair

Element distinctness linear-time reduces to 2d closest pair

Element distinctness. Given N elements, are any two equal?

2d closest pair. Given N points in the plane, find the closest pair.

Proposition. Element distinctness linear-time reduces to 2d closest pair.

Pf.

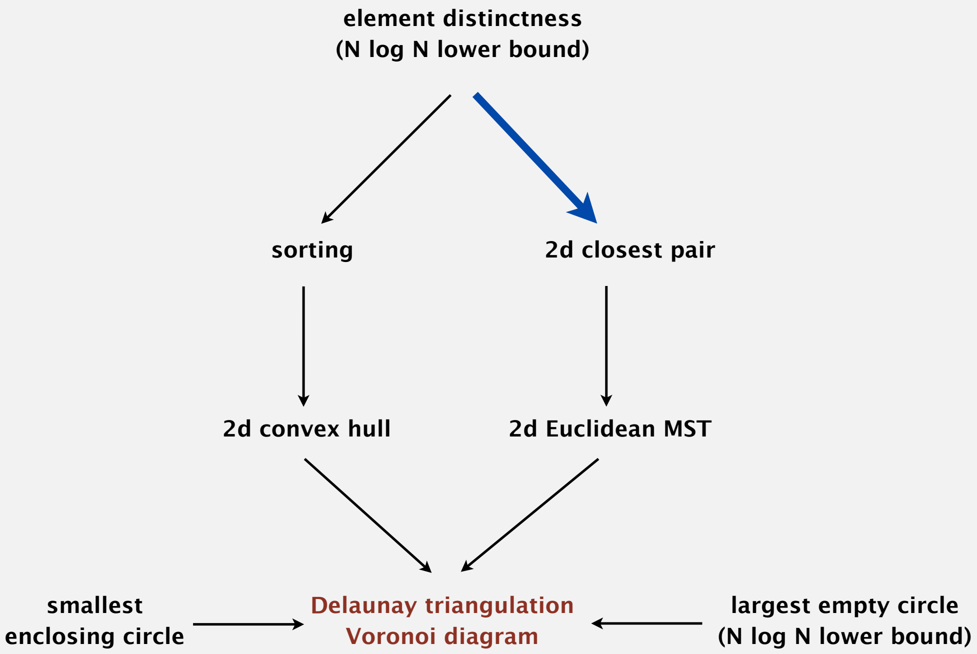
- Element distinctness instance: $x_1, x_2, \ldots, x_N$.
- 2d closest pair instance: $(x_1, x_1), (x_2, x_2), \ldots, (x_N, x_N)$.
- The N elements are distinct iff distance of closest pair > 0.

allows quadratic tests of the form: $x_i < x_j$ or $(x_i - x_k)^2 - (x_j - x_k)^2 < 0$

Element distinctness lower bound. In quadratic decision tree model, any algorithm that solves element distinctness takes $\Omega(N \log N)$ steps.

Implication. In quadratic decision tree model, any algorithm for closest pair takes $\Omega(N \log N)$ steps.

Some linear-time reductions in computational geometry



element distinctness
(N log N lower bound)

sorting

2d closest pair

2d convex hull

2d Euclidean MST

smallest
enclosing circle

Delaunay triangulation
Voronoi diagram

largest empty circle
(N log N lower bound)

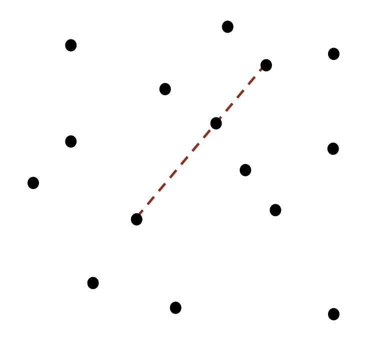
Lower bound for 3-COLLINEAR

3-SUM. Given N distinct integers, are there three that sum to 0?

3-COLLINEAR. Given N distinct points in the plane, are there 3 (or more) that all lie on the same line?

590584
-23439854
1251432
-2861534
3988818
-4190745
333255
13546464
89885444
-43434213
11998833

3-sum



3-collinear

Lower bound for 3-COLLINEAR

3-SUM. Given N distinct integers, are there three that sum to 0 ?

3-COLLINEAR. Given N distinct points in the plane, are there 3 (or more) that all lie on the same line?

Proposition. *3-SUM* linear-time reduces to *3-COLLINEAR*.

Pf. [next two slides]

lower-bound mentality:
if I can't solve 3-SUM in $N^{1.99}$ time,
I can't solve 3-COLLINEAR
in $N^{1.99}$ time either

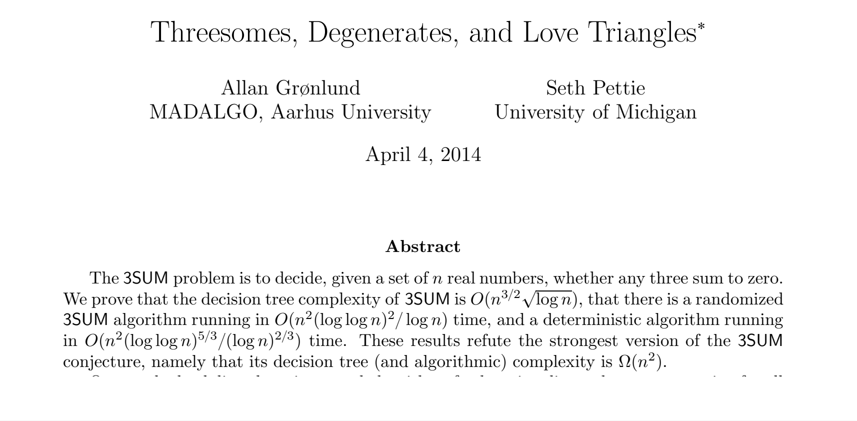
Conjecture. Any algorithm for *3-SUM* requires $\Omega(N^{2-\varepsilon})$ steps.

Implication. No sub-quadratic algorithm for *3-COLLINEAR* likely.

our $N^2 \log N$ algorithm was pretty good

Complexity of 3-SUM

April 2014. Some recent evidence that the complexity might be $N^{3/2}$.



Threesomes, Degenerates, and Love Triangles*

Allan Grønlund
MADALGO, Aarhus University

Seth Pettie
University of Michigan

April 4, 2014

Abstract

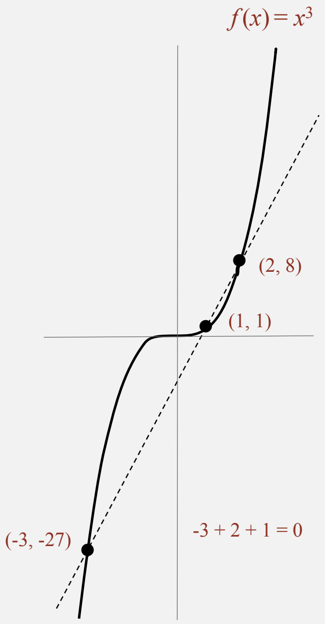
The 3SUM problem is to decide, given a set of n real numbers, whether any three sum to zero. We prove that the decision tree complexity of 3SUM is $O(n^{3/2}\sqrt{\log n})$, that there is a randomized 3SUM algorithm running in $O(n^2(\log\log n)^2/\log n)$ time, and a deterministic algorithm running in $O(n^2(\log\log n)^{5/3}/(\log n)^{2/3})$ time. These results refute the strongest version of the 3SUM conjecture, namely that its decision tree (and algorithmic) complexity is $\Omega(n^2)$.

3-SUM linear-time reduces to 3-COLLINEAR

Proposition. *3-SUM* linear-time reduces to *3-COLLINEAR*.

- *3-SUM* instance: $x_1, x_2, \ldots, x_N$.
- *3-COLLINEAR* instance: $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$.

Lemma. If a, b, and c are distinct, then $a + b + c = 0$ if and only if (a, a^3), (b, b^3), and (c, c^3) are collinear.



3-SUM linear-time reduces to 3-COLLINEAR

Proposition. *3-SUM* linear-time reduces to *3-COLLINEAR*.

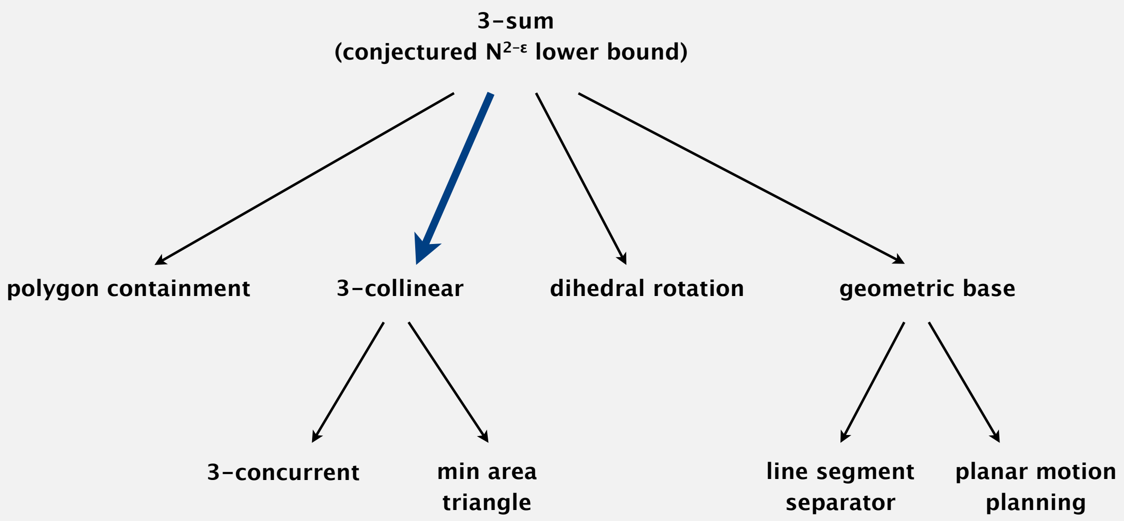
- *3-SUM* instance: $x_1, x_2, \ldots, x_N$.
- *3-COLLINEAR* instance: $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$.

Lemma. If a, b, and c are distinct, then $a + b + c = 0$ if and only if (a, a^3), (b, b^3), and (c, c^3) are collinear.

Pf. Three distinct points (a, a^3), (b, b^3), and (c, c^3) are collinear iff:

$$
\begin{aligned}
0 &= \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} \\
&= a(b^3 - c^3) - b(a^3 - c^3) + c(a^3 - b^3) \\
&= (a - b)(b - c)(c - a)(a + b + c)
\end{aligned}
$$

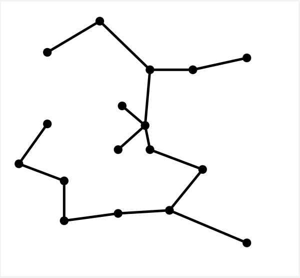
More geometric reductions and lower bounds



Establishing lower bounds: summary

Establishing lower bounds through reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself no linear-time Euclidean MST algorithm exists?
A1. [hard way] Long futile search for a linear-time algorithm.
A2. [easy way] Linear-time reduction from element distinctness.


2d Euclidean MST

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6.5 REDUCTIONS

Classifying problems: summary

Desiderata. Problem with algorithm that matches lower bound.
Ex. Sorting and element distinctness have complexity $N \log N$.

Desiderata'. Prove that two problems X and Y have the same complexity.

First, show that problem X linear-time reduces to Y.
- Second, show that Y linear-time reduces to X.
- Conclude that X and Y have the same complexity.
  (even if we don't know what it is)

assuming both take at least linear time

Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.

Brute force. N^2 bit operations.

								1	1	0	1	0	1	0	1
							×	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.

Brute force. N^2 bit operations.

problem	arithmetic	order of growth
integer multiplication	$a \times b$	$M(N)$
integer division	$a / b,\ a \bmod b$	$M(N)$
integer square	a^2	$M(N)$
integer square root	$\lfloor \sqrt{a} \rfloor$	$M(N)$

integer arithmetic problems with the same complexity as integer multiplication

Q. Is brute-force algorithm optimal?

History of complexity of integer multiplication

year	algorithm	order of growth
?	**brute force**	N^2
1962	**Karatsuba**	$N^{1.585}$
1963	**Toom-3, Toom-4**	$N^{1.465}$, $N^{1.404}$
1966	**Toom-Cook**	$N^{1+\varepsilon}$
1971	**Schönhage-Strassen**	$N \log N \log \log N$
2007	**Fürer**	$N \log N\, 2^{\log^* N}$
?	?	N

number of bit operations to multiply two N-bit integers

used in Maple, Mathematica, gcc, cryptography, ...

Remark. GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.

GMP
«Arithmetic without limitations»

Numerical linear algebra reductions

Matrix multiplication. Given two N-by-N matrices, compute their product.

Brute force. N^3 flops.

$$\begin{array}{c|cccc} & & & & \\ & 0.1 & 0.2 & 0.8 & 0.1 \\ \textbf{row i} & 0.5 & 0.3 & 0.9 & 0.6 \\ & 0.1 & 0.0 & 0.7 & 0.4 \\ & 0.0 & 0.3 & 0.3 & 0.1 \end{array} \times \begin{array}{cccc} & & \textbf{column j} & \\ 0.4 & 0.3 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.0 & 0.6 \\ 0.0 & 0.0 & 0.4 & 0.5 \\ 0.8 & 0.4 & 0.1 & 0.9 \end{array} = \begin{array}{c|cccc} & & & \textbf{j} & \\ & 0.16 & 0.11 & 0.34 & 0.62 \\ \textbf{i} & 0.74 & 0.45 & 0.47 & 1.22 \\ & 0.36 & 0.19 & 0.33 & 0.72 \\ & 0.14 & 0.10 & 0.13 & 0.42 \end{array}$$

$0.5 \cdot 0.1 \ + \ 0.3 \cdot 0.0 \ + \ 0.9 \cdot 0.4 \ + \ 0.6 \cdot 0.1 = 0.47$

Numerical linear algebra reductions

Matrix multiplication. Given two N-by-N matrices, compute their product.

Brute force. N^3 flops.

problem	linear algebra	order of growth
matrix multiplication	$A \times B$	$MM(N)$
matrix inversion	A^{-1}	$MM(N)$
determinant	$\| A \|$	$MM(N)$
system of linear equations	$Ax = b$	$MM(N)$
LU decomposition	$A = L\,U$	$MM(N)$
least squares	$\min \|\|Ax - b\|\|_2$	$MM(N)$

numerical linear algebra problems with the same complexity as matrix multiplication

Q. Is brute-force algorithm optimal?

History of complexity of matrix multiplication

year	algorithm	order of growth
?	**brute force**	N^3
1969	**Strassen**	$N^{2.808}$
1978	**Pan**	$N^{2.796}$
1979	**Bini**	$N^{2.780}$
1981	**Schönhage**	$N^{2.522}$
1982	**Romani**	$N^{2.517}$
1982	**Coppersmith-Winograd**	$N^{2.496}$
1986	**Strassen**	$N^{2.479}$
1989	**Coppersmith-Winograd**	$N^{2.376}$
2010	**Strother**	$N^{2.3737}$
2011	**Williams**	$N^{2.3727}$
?	**?**	$N^{2+\varepsilon}$

number of floating-point operations to multiply two N-by-N matrices

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6.5 REDUCTIONS

Bird's-eye view

Def. A problem is intractable if it can't be solved in polynomial time.

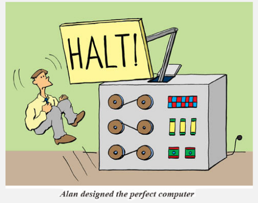
Desiderata. Prove that a problem is intractable.

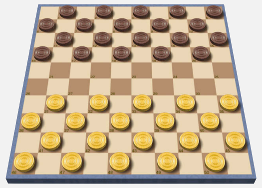
Two problems that provably require exponential time.

input size = c + lg K

- Given a constant-size program, does it halt in at most K steps?
- Given N-by-N checkers board position, can the first player force a win?

using forced capture rule





Frustrating news. Very few successes.

A core problem: satisfiability

SAT. Given a system of boolean equations, find a solution.

Ex.

$\neg x_1$	*or*	x_2	*or*	x_3			=	*true*
x_1	*or*	$\neg x_2$	*or*	x_3			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*	$\neg x_3$			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*		*or*	x_4	=	*true*
		$\neg x_2$	*or*	x_3	*or*	x_4	=	*true*

instance I

x_1	x_2	x_3	x_4
T	T	F	T

solution S

3-SAT. All equations of this form (with three variables per equation).

Key applications.

- Automatic verification systems for software.
- Mean field diluted spin glass model in physics.
- Electronic design automation (EDA) for hardware.
- ...

Satisfiability is conjectured to be intractable

Q. How to solve an instance of *3-SAT* with N variables?

A. Exhaustive search: try all 2^N truth assignments.

Q. Can we do anything substantially more clever?

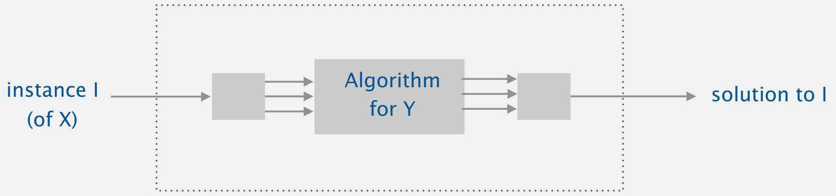
Conjecture (P ≠ NP). *3-SAT* is intractable (no poly-time algorithm).

consensus opinion

Polynomial-time reductions

Problem X poly-time (Cook) reduces to problem Y if X can be solved with:

- Polynomial number of standard computational steps.
- Polynomial number of calls to Y.



instance I (of X)

Algorithm for Y

solution to I

Algorithm for X

Establish intractability. If *3-SAT* poly-time reduces to Y, then Y is intractable. (assuming *3-SAT* is intractable)

Mentality.

- If I could solve Y in poly-time, then I could also solve *3-SAT* in poly-time.
- *3-SAT* is believed to be intractable.
- Therefore, so is Y.

Integer linear programming

ILP. Given a system of linear inequalities, find an integral solution.

$$\begin{aligned}3x_1 + 5x_2 + 2x_3 + x_4 + 4x_5 &\geq 10\\ 5x_1 + 2x_2 + 4x_4 + 1x_5 &\leq 7\\ x_1 + x_3 + 2x_4 &\leq 2\\ 3x_1 + 4x_3 + 7x_4 &\leq 7\\ x_1 + x_4 &\leq 1\\ x_1 + x_3 + x_5 &\leq 1\\ \text{all } x_i &= \{0, 1\}\end{aligned}$$

linear inequalities

integer variables

instance I

x_1	x_2	x_3	x_4	x_5
0	1	0	1	1

solution S

Context. Cornerstone problem in operations research.

Remark. Finding a real-valued solution is tractable (linear programming).

3-SAT poly-time reduces to ILP

3-SAT. Given a system of boolean equations, find a solution.

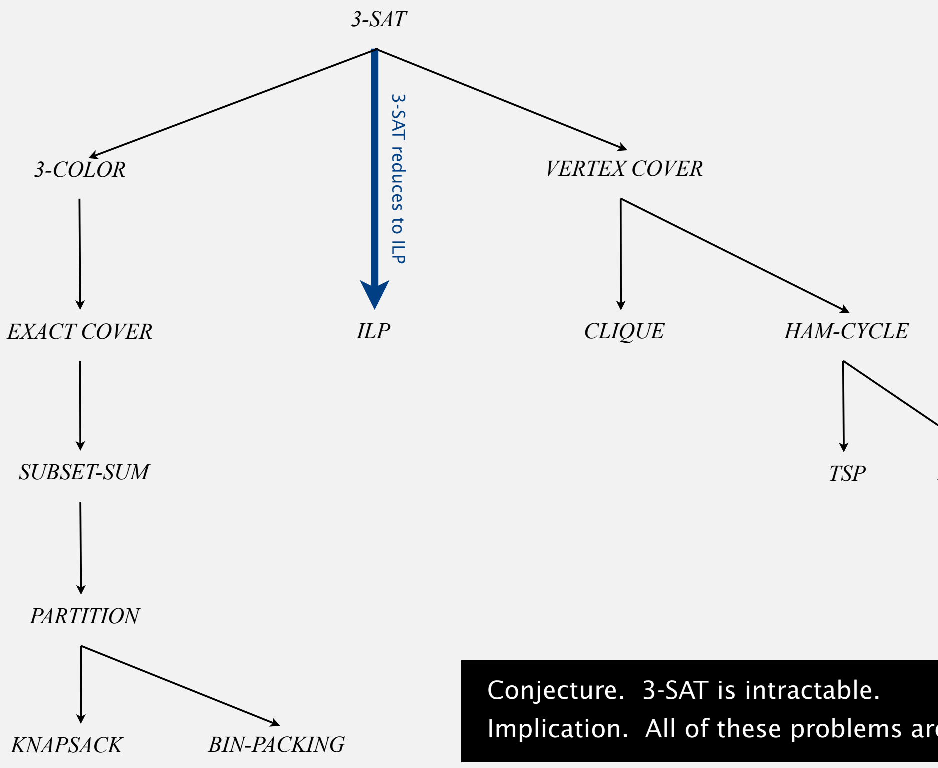
$\neg x_1$	*or*	x_2	*or*	x_3			=	*true*
x_1	*or*	$\neg x_2$	*or*	x_3			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*	$\neg x_3$			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*		*or*	x_4	=	*true*
		$\neg x_2$	*or*	x_3	*or*	x_4	=	*true*

ILP. Given a system of linear inequalities, find a 0-1 solution.

$(1-x_1)$	+	x_2	+	x_3			≥	1
x_1	+	$(1-x_2)$	+	x_3			≥	1
$(1-x_1)$	+	$(1-x_2)$	+	$(1-x_3)$			≥	1
$(1-x_1)$	+	$(1-x_2)$	+		+	x_4	≥	1
		$(1-x_2)$	+	x_3	+	x_4	≥	1

solution to this ILP instance gives solution to original 3-SAT instance

More poly-time reductions from 3-satisfiability



Dick Karp
'85 Turing award

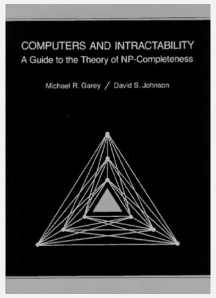
Conjecture. 3-SAT is intractable.
Implication. All of these problems are intractable.

Implications of poly-time reductions from 3-satisfiability

Establishing intractability through poly-time reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself that a new problem is (probably) intractable?
A1. [hard way] Long futile search for an efficient algorithm (as for *3-SAT*).
A2. [easy way] Reduction from *3-SAT*.

Caveat. Intricate reductions are common.



COMPUTERS AND INTRACTABILITY
A Guide to the Theory of NP-Completeness
Michael R. Garey / David S. Johnson

Search problems

Search problem. Problem where you can check a solution in poly-time.

Ex 1. *3-SAT*.

$\neg x_1$	*or*	x_2	*or*	x_3			=	*true*
x_1	*or*	$\neg x_2$	*or*	x_3			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*	$\neg x_3$			=	*true*
$\neg x_1$	*or*	$\neg x_2$	*or*		*or*	x_4	=	*true*
		$\neg x_2$	*or*	x_3	*or*	x_4	=	*true*

instance I

x_1	x_2	x_3	x_4
T	T	F	T

solution S

Ex 2. *FACTOR*. Given an *N*-bit integer *x*, find a nontrivial factor.

147573952589676412927

instance I

193707721

solution S

P vs. **NP**

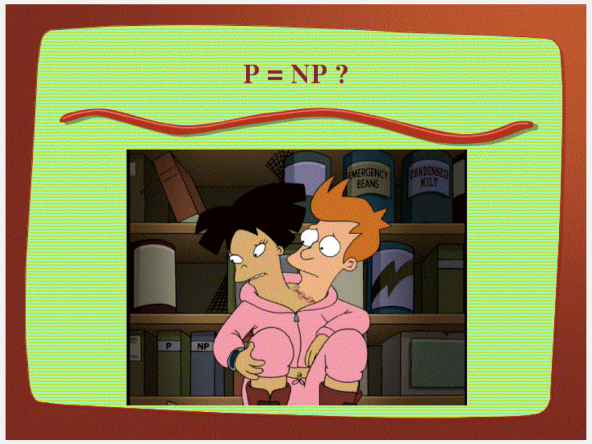
P. Set of search problems solvable in poly-time.

Importance. What scientists and engineers can compute feasibly.

NP. Set of search problems (checkable in poly-time).

Importance. What scientists and engineers aspire to compute feasibly.

Fundamental question.



Consensus opinion. No.

Cook-Levin theorem

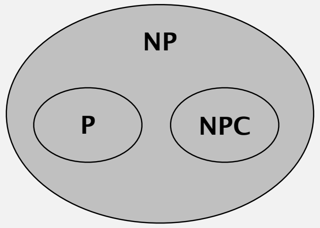
A problem is NP-COMPLETE if

- It is in **NP**.
- All problems in **NP** poly-time to reduce to it.

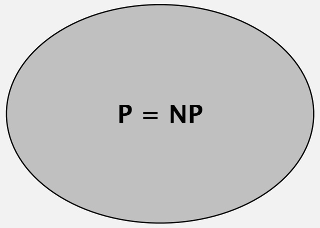
Cook-Levin theorem. *3-SAT* is **NP-COMPLETE**.

Corollary. *3-SAT* is tractable if and only if **P = NP**.

Two worlds.

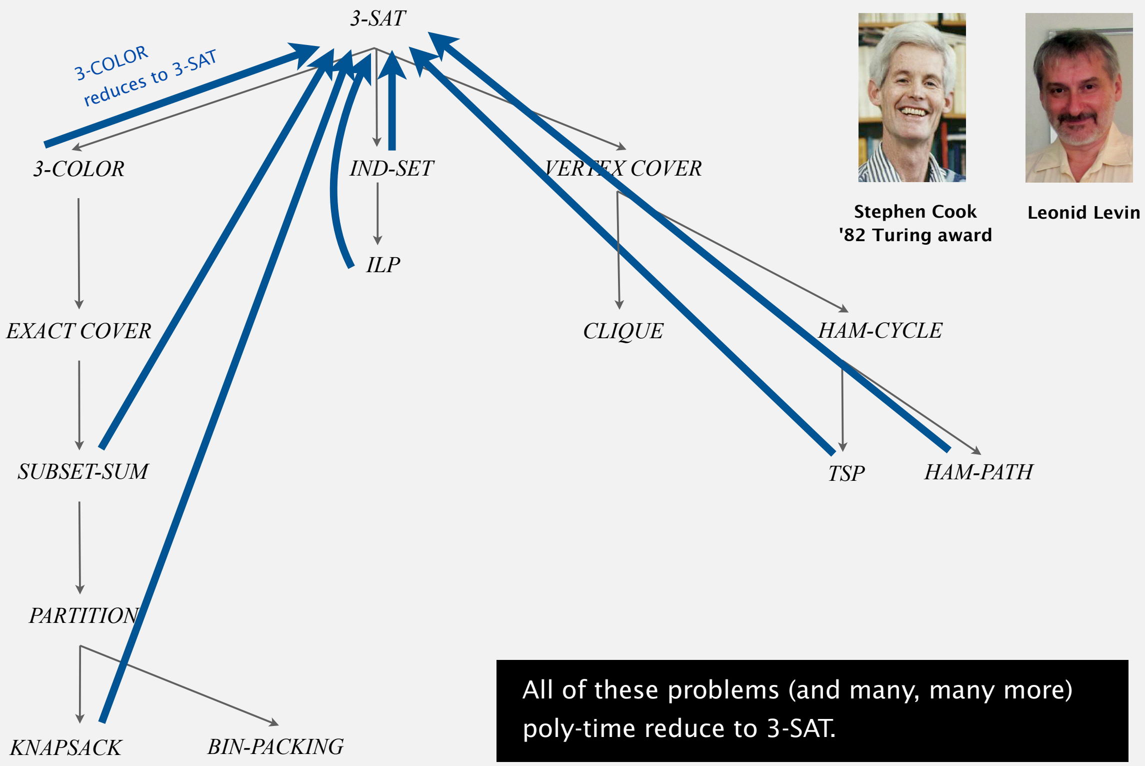


P ≠ NP



P = NP

Implications of Cook-Levin theorem

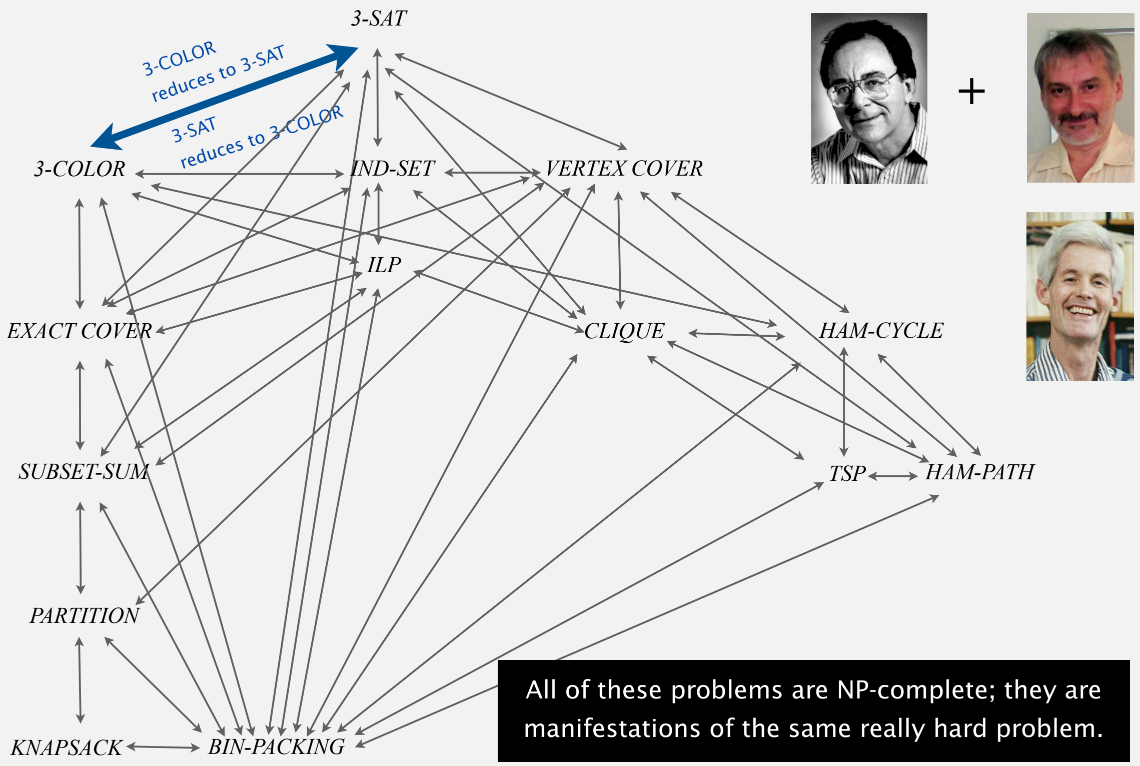


Stephen Cook
'82 Turing award

Leonid Levin

All of these problems (and many, many more) poly-time reduce to 3-SAT.

Implications of Karp + Cook-Levin



All of these problems are NP-complete; they are manifestations of the same really hard problem.

Birds-eye view: review

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	*min, max, median,* *Burrows-Wheeler transform, ...*
linearithmic	$N \log N$	*sorting, element distinctness, ...*
quadratic	N^2	?
⋮	⋮	⋮
exponential	c^N	?

Frustrating news. Huge number of problems have defied classification.

Birds-eye view: revised

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	*min, max, median,* *Burrows-Wheeler transform, ...*
linearithmic	$N \log N$	*sorting, element distinctness, ...*
M(N)	?	*integer multiplication,* *division, square root, ...*
MM(N)	?	*matrix multiplication, $Ax = b$,* *least square, determinant, ...*
⋮	⋮	⋮
NP-complete	*probably not N^b*	*3-SAT, IND-SET, ILP, ...*

Good news. Can put many problems into equivalence classes.

Complexity zoo

Complexity class. Set of problems sharing some computational property.

https://complexityzoo.uwaterloo.ca

Bad news. Lots of complexity classes (496 animals in zoo).

Summary

Reductions are important in theory to:

- Design algorithms.
- Establish lower bounds.
- Classify problems according to their computational requirements.

Reductions are important in practice to:

- Design algorithms.
- Design reusable software modules.
  - stacks, queues, priority queues, symbol tables, sets, graphs
  - sorting, regular expressions, suffix arrays
  - MST, shortest paths, maxflow, linear programming
- Determine difficulty of your problem and choose the right tool.