Algorithms
Robert Sedgewick | Kevin Wayne


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# 4.4 Shortest Paths

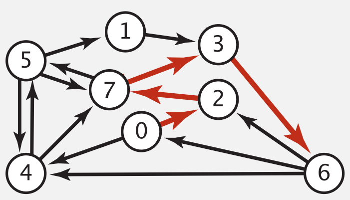
- *APIs*
- *shortest-paths properties*
- *Dijkstra's algorithm*
- *edge-weighted DAGs*
- *negative weights*

## Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from $s$ to $t$.

edge-weighted digraph

| | |
|---|---|
| 4->5 | 0.35 |
| 5->4 | 0.35 |
| 4->7 | 0.37 |
| 5->7 | 0.28 |
| 7->5 | 0.28 |
| 5->1 | 0.32 |
| 0->4 | 0.38 |
| 0->2 | 0.26 |
| 7->3 | 0.39 |
| 1->3 | 0.29 |
| 2->7 | 0.34 |
| 6->2 | 0.40 |
| 3->6 | 0.52 |
| 6->0 | 0.58 |
| 6->4 | 0.93 |

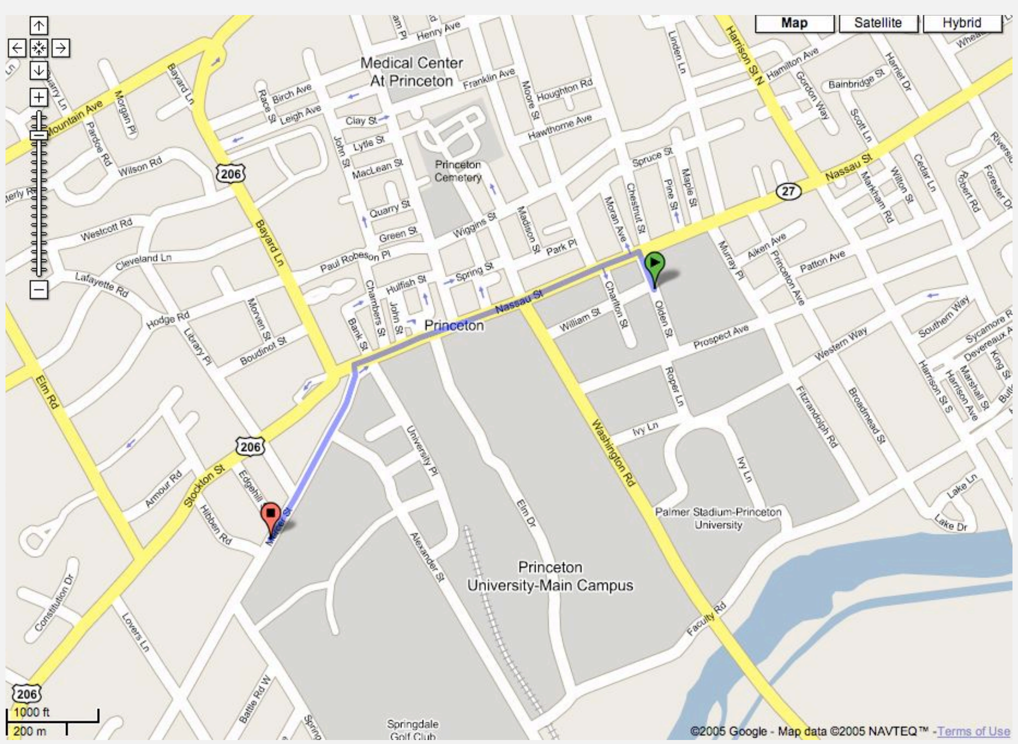



shortest path from 0 to 6

| | |
|---|---|
| 0->2 | 0.26 |
| 2->7 | 0.34 |
| 7->3 | 0.39 |
| 3->6 | 0.52 |



## Google maps






## Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Texture mapping.
- Robot navigation.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

http://en.wikipedia.org/wiki/Seam_carving

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

## Shortest path variants

**Which vertices?**

- Single source: from one vertex $s$ to every other vertex.
- Single sink: from every vertex to one vertex $t$.
- Source-sink: from one vertex $s$ to another $t$.
- All pairs: between all pairs of vertices.

**Restrictions on edge weights?**

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

**Cycles?**

- No directed cycles.
- No "negative cycles."


which variant?

**Simplifying assumption.** Shortest paths from $s$ to each vertex $v$ exist.

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## 4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights

## Weighted directed edge API

`public class DirectedEdge`

| | | |
|---|---|---|
| | `DirectedEdge(int v, int w, double weight)` | *weighted edge v→w* |
| `int` | `from()` | *vertex v* |
| `int` | `to()` | *vertex w* |
| `double` | `weight()` | *weight of this edge* |
| `String` | `toString()` | *string representation* |

$v \xrightarrow{\text{weight}} w$

Idiom for processing an edge `e`: `int v = e.from(), w = e.to();`



## Weighted directed edge: implementation in Java

Similar to `Edge` for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w;  }

   public int weight()
   {  return weight; }
}
```

from() and to() replace either() and other()

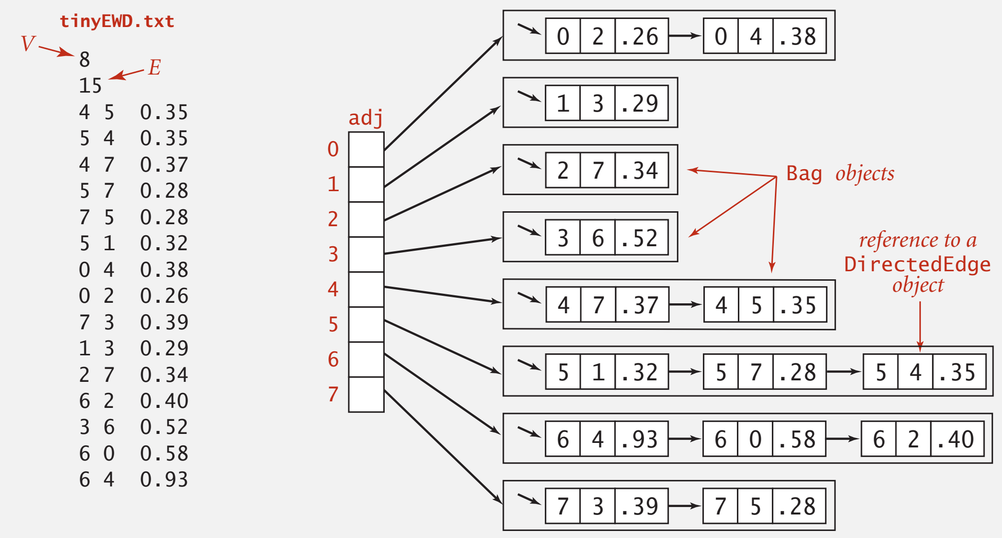
## Edge-weighted digraph API

| public class EdgeWeightedDigraph | | |
|---|---|---|
| | EdgeWeightedDigraph(int V) | *edge-weighted digraph with V vertices* |
| | EdgeWeightedDigraph(In in) | *edge-weighted digraph from input stream* |
| void | addEdge(DirectedEdge e) | *add weighted directed edge e* |
| Iterable<DirectedEdge> | adj(int v) | *edges pointing from v* |
| int | V() | *number of vertices* |
| int | E() | *number of edges* |
| Iterable<DirectedEdge> | edges() | *all edges* |
| String | toString() | *string representation* |

Conventions. Allow self-loops and parallel edges.



## Edge-weighted digraph: adjacency-lists representation






## Edge-weighted digraph: adjacency-lists implementation in Java

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```
public class EdgeWeightedDigraph
{
   private final int V;
   private final Bag<DirectedEdge>[] adj;

   public EdgeWeightedDigraph(int V)
   {
      this.V = V;
      adj = (Bag<DirectedEdge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
   }

   public void addEdge(DirectedEdge e)
   {
      int v = e.from();
      adj[v].add(e);
   }

   public Iterable<DirectedEdge> adj(int v)
   {  return adj[v];  }
}
```

add edge e = v→w to only v's adjacency list



## Single-source shortest paths API

Goal. Find the shortest path from *s* to every other vertex.

| public class SP | | |
|---|---|---|
| | SP(EdgeWeightedDigraph G, int s) | *shortest paths from s in graph G* |
| double | distTo(int v) | *length of shortest path from s to v* |
| Iterable <DirectedEdge> | pathTo(int v) | *shortest path from s to v* |
| boolean | hasPathTo(int v) | *is there a path from s to v?* |

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
   StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
   for (DirectedEdge e : sp.pathTo(v))
      StdOut.print(e + "  ");
   StdOut.println();
}
```

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| Iterable <DirectedEdge> | pathTo(int v) | *shortest path from s to v* |
| boolean | hasPathTo(int v) | *is there a path from s to v?* |

```
% java SP tinyEWD.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38  4->5 0.35  5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26  2->7 0.34  7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38  4->5 0.35
0 to 6 (1.51): 0->2 0.26  2->7 0.34  7->3 0.39  3->6 0.52
0 to 7 (0.60): 0->2 0.26  2->7 0.34
```




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## 4.4 Shortest Paths

- APIs
- *shortest-paths properties*
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights

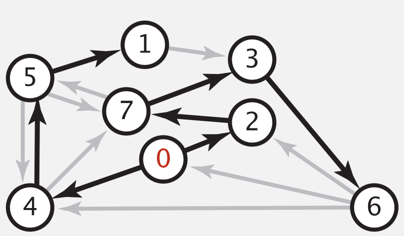
## Data structures for single-source shortest paths

Goal. Find the shortest path from $s$ to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- distTo[v] is length of shortest path from $s$ to $v$.
- edgeTo[v] is last edge on shortest path from $s$ to $v$.




shortest-paths tree from 0

| | edgeTo[] | distTo[] |
|---|---|---|
| 0 | null | 0 |
| 1 | 5->1 0.32 | 1.05 |
| 2 | 0->2 0.26 | 0.26 |
| 3 | 7->3 0.37 | 0.97 |
| 4 | 0->4 0.38 | 0.38 |
| 5 | 4->5 0.35 | 0.73 |
| 6 | 3->6 0.52 | 1.49 |
| 7 | 2->7 0.34 | 0.60 |

parent-link representation



## Data structures for single-source shortest paths

Goal. Find the shortest path from $s$ to every other vertex.

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Consequence. Can represent the SPT with two vertex-indexed arrays:

- distTo[v] is length of shortest path from $s$ to $v$.
- edgeTo[v] is last edge on shortest path from $s$ to $v$.

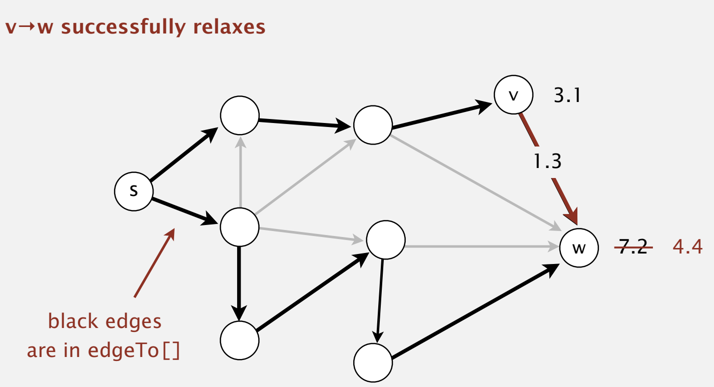
```
public double distTo(int v)
{  return distTo[v];  }

public Iterable<DirectedEdge> pathTo(int v)
{
   Stack<DirectedEdge> path = new Stack<DirectedEdge>();
   for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
      path.push(e);
   return path;
}
```

## Edge relaxation

Relax edge $e = v \to w$.

- `distTo[v]` is length of shortest known path from s to v.
- `distTo[w]` is length of shortest known path from s to w.
- `edgeTo[w]` is last edge on shortest known path from s to w.
- If e = v→w gives shorter path to w through v, update both `distTo[w]` and `edgeTo[w]`.



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Relax edge $e = v \to w$.

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- `edgeTo[w]` is last edge on shortest known path from s to w.
- If e = v→w gives shorter path to w through v, update both `distTo[w]` and `edgeTo[w]`.

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
      distTo[w] = distTo[v] + e.weight();
      edgeTo[w] = e;
   }
}
```



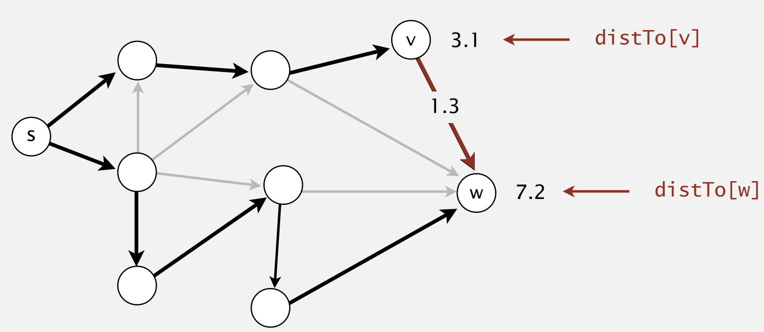
## Shortest-paths optimality conditions

Proposition. Let $G$ be an edge-weighted digraph.
Then `distTo[]` are the shortest path distances from s iff:

- `distTo[s]` = 0.
- For each vertex v, `distTo[v]` is the length of some path from s to v.
- For each edge e = v→w, `distTo[w]` ≤ `distTo[v]` + `e.weight()`.

Pf. ⇐ [ necessary ]

- Suppose that `distTo[w]` > `distTo[v]` + `e.weight()` for some edge e = v→w.
- Then, e gives a path from s to w (through v) of length less than `distTo[w]`.






## Shortest-paths optimality conditions

Proposition. Let $G$ be an edge-weighted digraph.
Then `distTo[]` are the shortest path distances from s iff:

- `distTo[s]` = 0.
- For each vertex v, `distTo[v]` is the length of some path from s to v.
- For each edge e = v→w, `distTo[w]` ≤ `distTo[v]` + `e.weight()`.

Pf. ⇒ [ sufficient ]

- Suppose that $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k = w$ is a shortest path from $s$ to $w$.
- Then,

$$\begin{aligned}
\texttt{distTo}[v_1] &\le \texttt{distTo}[v_0] + e_1\texttt{.weight()} \\
\texttt{distTo}[v_2] &\le \texttt{distTo}[v_1] + e_2\texttt{.weight()} \\
&\;\;\ldots \\
\texttt{distTo}[v_k] &\le \texttt{distTo}[v_{k-1}] + e_k\texttt{.weight()}
\end{aligned}$$

$e_i$ = $i^{th}$ edge on shortest path from s to w

- Add inequalities; simplify; and substitute `distTo[v₀] = distTo[s] = 0`:

$$\texttt{distTo}[w] = \texttt{distTo}[v_k] \le \underbrace{e_1\texttt{.weight()} + e_2\texttt{.weight()} + \ldots + e_k\texttt{.weight()}}_{\text{weight of shortest path from s to w}}$$

- Thus, `distTo[w]` is the weight of shortest path to $w$. ▪

weight of some path from s to w

## Generic shortest-paths algorithm

**Generic algorithm (to compute SPT from s)**

**Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.**

**Repeat until optimality conditions are satisfied:**
- **Relax any edge.**

Proposition. Generic algorithm computes SPT (if it exists) from `s`.

Pf sketch.
- The entry `distTo[v]` is always the length of a simple path from `s` to `v`.
- Each successful relaxation decreases `distTo[v]` for some `v`.
- The entry `distTo[v]` can decrease at most a finite number of times. ▪



## Generic shortest-paths algorithm

**Generic algorithm (to compute SPT from s)**

**Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.**

**Repeat until optimality conditions are satisfied:**
- **Relax any edge.**

Efficient implementations. How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman-Ford algorithm (no negative cycles).

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## 4.4 Shortest Paths

- APIs
- shortest-paths properties
- *Dijkstra's algorithm*
- edge-weighted DAGs
- negative weights

## Edsger W. Dijkstra: select quotes

*" Do only what only you can do. "*

*" In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. "*

*" The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. "*

*" It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. "*

*" APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. "*

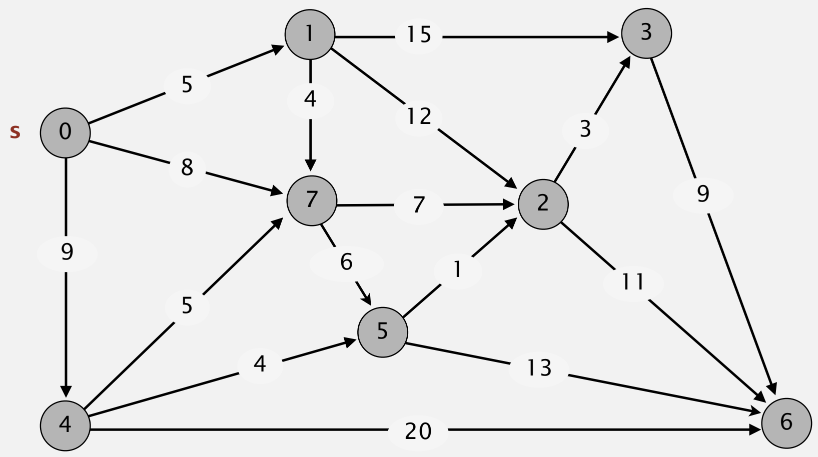
Edsger W. Dijkstra
Turing award 1972

## Edsger W. Dijkstra: select quotes




## Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s
  (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.




an edge-weighted digraph

| | |
|---|---|
| 0→1 | 5.0 |
| 0→4 | 9.0 |
| 0→7 | 8.0 |
| 1→2 | 12.0 |
| 1→3 | 15.0 |
| 1→7 | 4.0 |
| 2→3 | 3.0 |
| 2→6 | 11.0 |
| 3→6 | 9.0 |
| 4→5 | 4.0 |
| 4→6 | 20.0 |
| 4→7 | 5.0 |
| 5→2 | 1.0 |
| 5→6 | 13.0 |
| 7→5 | 6.0 |
| 7→2 | 7.0 |



## Dijkstra's algorithm demo

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  (non-tree vertex with the lowest `distTo[]` value).
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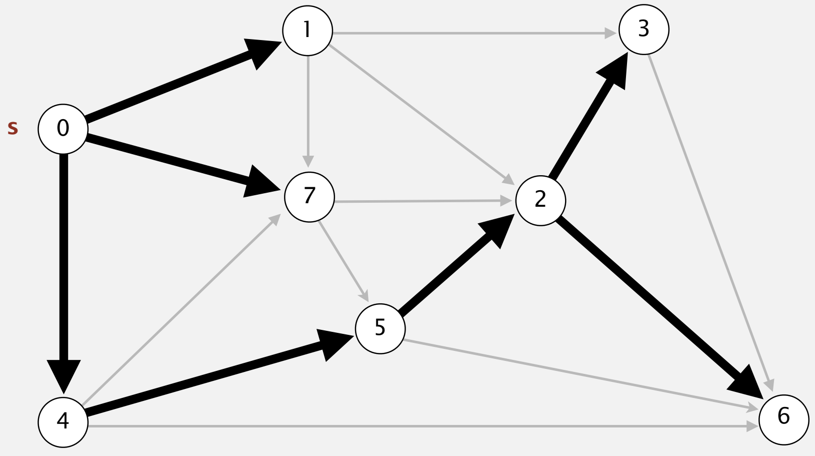
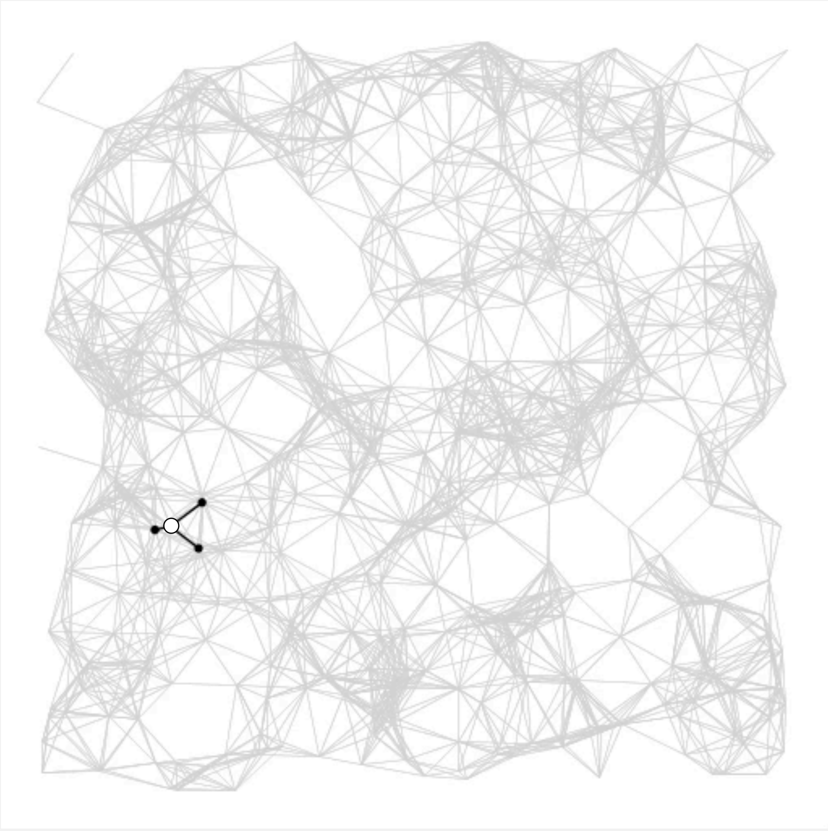



| v | distTo[] | edgeTo[] |
|---|---|---|
| 0 | 0.0 | - |
| 1 | 5.0 | 0→1 |
| 2 | 14.0 | 5→2 |
| 3 | 17.0 | 2→3 |
| 4 | 9.0 | 0→4 |
| 5 | 13.0 | 4→5 |
| 6 | 25.0 | 2→6 |
| 7 | 8.0 | 0→7 |

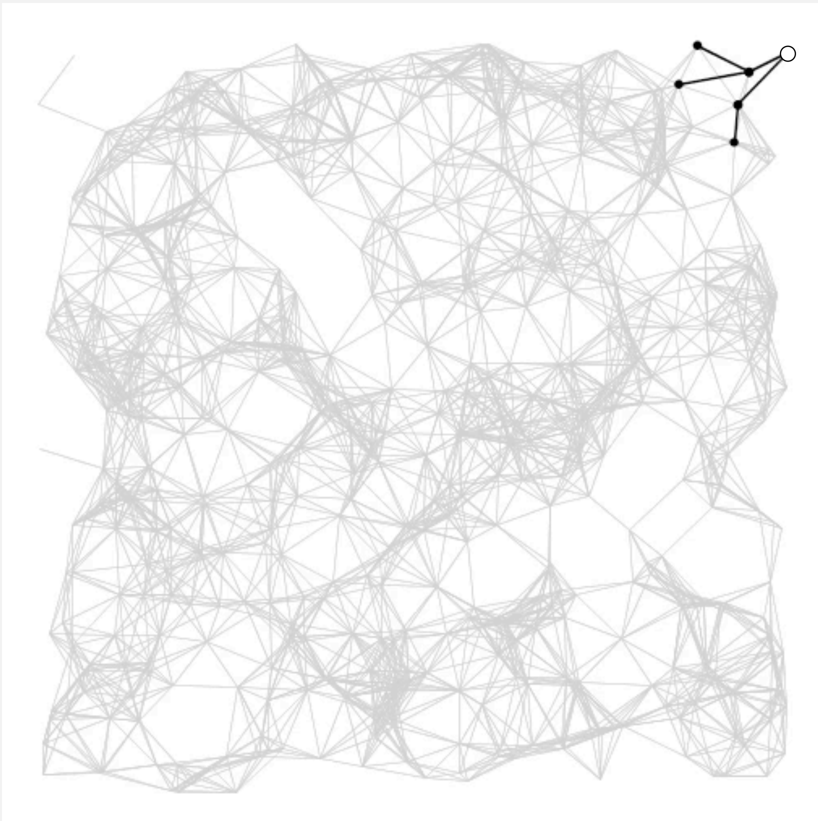
shortest-paths tree from vertex s



## Dijkstra's algorithm visualization



## Dijkstra's algorithm visualization





## Dijkstra's algorithm: correctness proof

Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge e = v→w is relaxed exactly once (when vertex v is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← we choose lowest `distTo[]` value at each step (and edge weights are nonnegative)
- Thus, upon termination, shortest-paths optimality conditions hold. ▪



## Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;
   private IndexMinPQ<Double> pq;

   public DijkstraSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new IndexMinPQ<Double>(G.V());

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      pq.insert(s, 0.0);
      while (!pq.isEmpty())
      {
          int v = pq.delMin();
          for (DirectedEdge e : G.adj(v))
             relax(e);
      }
   }
}
```

relax vertices in order of distance from s



## Dijkstra's algorithm: Java implementation

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
       if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
       else                pq.insert     (w, distTo[w]);
   }
}
```

update PQ

## Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: $V$ insert, $V$ delete-min, $E$ decrease-key.

| PQ implementation | insert | delete-min | decrease-key | total |
|---|---|---|---|---|
| **unordered array** | $1$ | $V$ | $1$ | $V^2$ |
| **binary heap** | $\log V$ | $\log V$ | $\log V$ | $E \log V$ |
| **d-way heap** | $\log_d V$ | $d \log_d V$ | $\log_d V$ | $E \log_{E/V} V$ |
| **Fibonacci heap** | $1$ † | $\log V$ † | $1$ † | $E + V \log V$ |

† amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.



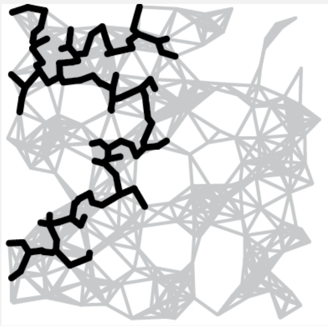
## Computing a spanning tree in a graph

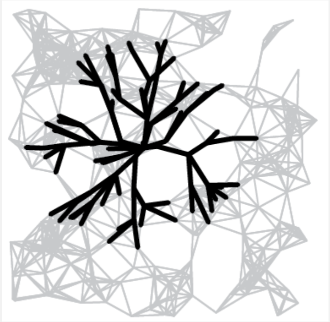
Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a spanning tree.

Main distinction: Rule used to choose next vertex for the tree.

- Prim: Closest vertex to the tree (via an undirected edge).
- Dijkstra: Closest vertex to the source (via a directed path).





Note: DFS and BFS are also in this family of algorithms.

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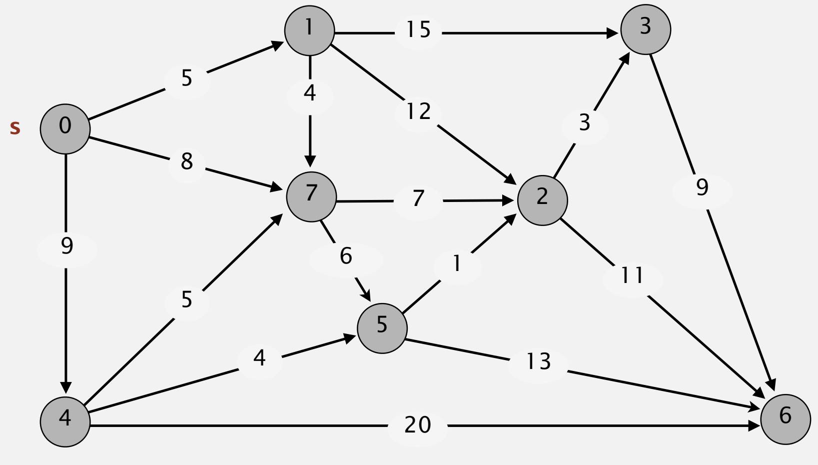
## 4.4 Shortest Paths



## Acyclic edge-weighted digraphs

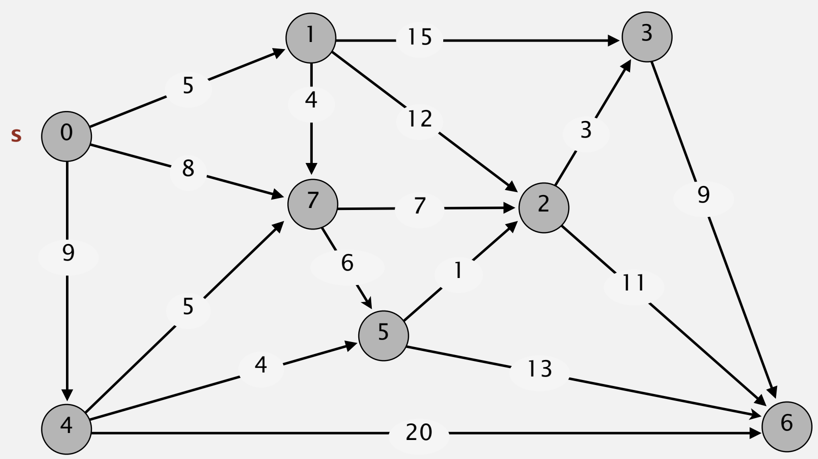
Q. Suppose that an edge-weighted digraph has no directed cycles.
Is it easier to find shortest paths than in a general digraph?




A. Yes!

# Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.




an edge-weighted DAG

| | |
|---|---|
| 0→1 | 5.0 |
| 0→4 | 9.0 |
| 0→7 | 8.0 |
| 1→2 | 12.0 |
| 1→3 | 15.0 |
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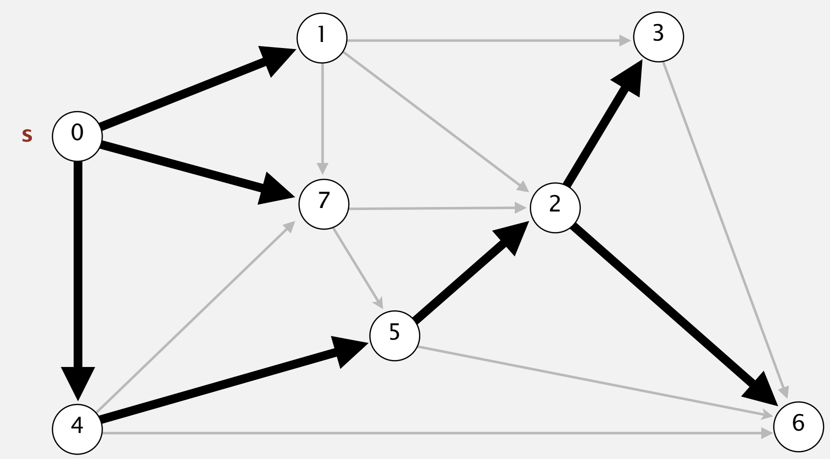
| 4→7 | 5.0 |
| 5→2 | 1.0 |
| 5→6 | 13.0 |
| 7→5 | 6.0 |
| 7→2 | 7.0 |



# Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.




shortest-paths tree from vertex s

0 1 4 7 5 2 3 6

| v | distTo[] | edgeTo[] |
|---|---|---|
| 0 | 0.0 | - |
| 1 | 5.0 | 0→1 |
| 2 | 14.0 | 5→2 |
| 3 | 17.0 | 2→3 |
| 4 | 9.0 | 0→4 |
| 5 | 13.0 | 4→5 |
| 6 | 25.0 | 2→6 |
| 7 | 8.0 | 0→7 |



# Shortest paths in edge-weighted DAGs

Proposition. Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

edge weights can be negative!

Pf.

- Each edge `e = v→w` is relaxed exactly once (when vertex `v` is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← because of topological order, no edge pointing to v will be relaxed after v is relaxed
- Thus, upon termination, shortest-paths optimality conditions hold. ▪



# Shortest paths in edge-weighted DAGs

```
public class AcyclicSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;

   public AcyclicSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      Topological topological = new Topological(G);
      for (int v : topological.order())
         for (DirectedEdge e : G.adj(v))
            relax(e);
   }
}
```

topological order

## Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.


http://www.youtube.com/watch?v=vIFCV2spKtg



## Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

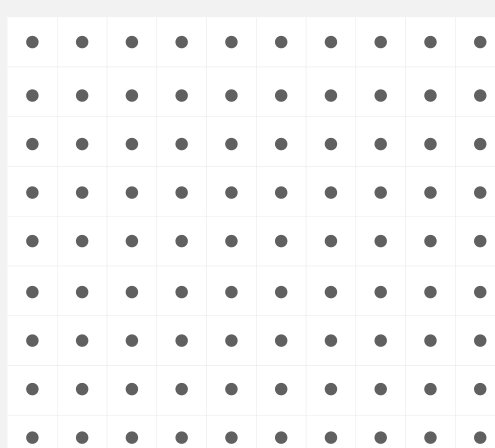
In the wild. Photoshop CS 5, Imagemagick, GIMP, ...



## Content-aware resizing

To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.





## Content-aware resizing

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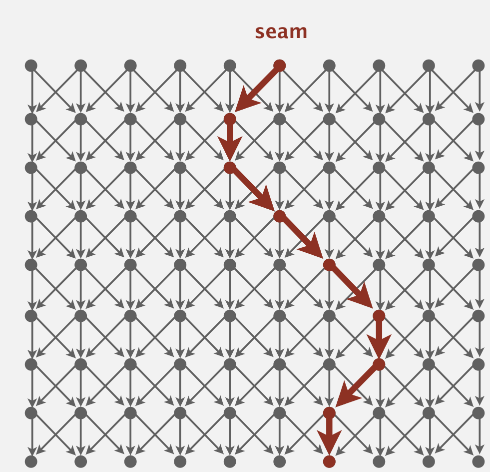
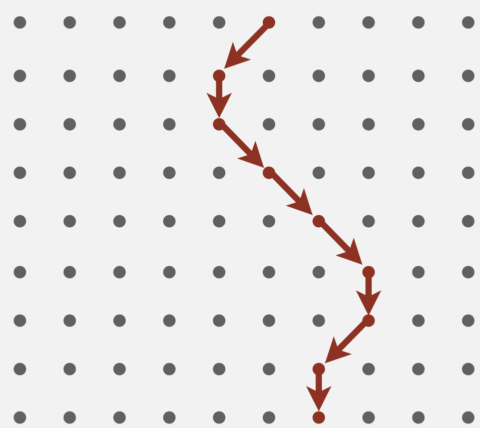


## Content-aware resizing

To remove vertical seam:

- Delete pixels on seam (one in each row).






## Content-aware resizing

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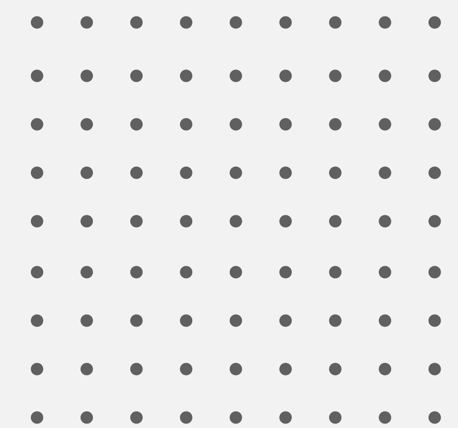




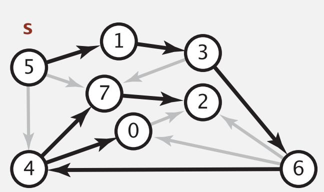
## Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

- Negate all weights.
- Find shortest paths.
- Negate weights in result.

equivalent: reverse sense of equality in `relax()`

| longest paths input | | shortest paths input | |
|---|---|---|---|
| 5->4 | 0.35 | 5->4 | -0.35 |
| 4->7 | 0.37 | 4->7 | -0.37 |
| 5->7 | 0.28 | 5->7 | -0.28 |
| 5->1 | 0.32 | 5->1 | -0.32 |
| 4->0 | 0.38 | 4->0 | -0.38 |
| 0->2 | 0.26 | 0->2 | -0.26 |
| 3->7 | 0.39 | 3->7 | -0.39 |
| 1->3 | 0.29 | 1->3 | -0.29 |
| 7->2 | 0.34 | 7->2 | -0.34 |
| 6->2 | 0.40 | 6->2 | -0.40 |
| 3->6 | 0.52 | 3->6 | -0.52 |
| 6->0 | 0.58 | 6->0 | -0.58 |
| 6->4 | 0.93 | 6->4 | -0.93 |



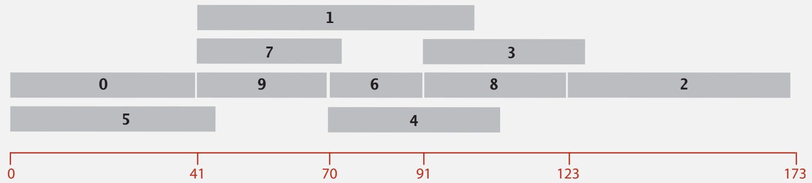

Key point. Topological sort algorithm works even with negative weights.



## Longest paths in edge-weighted DAGs: application

Parallel job scheduling. Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

| *job* | *duration* | *must complete before* |
|---|---|---|
| 0 | 41.0 | 1 7 9 |
| 1 | 51.0 | 2 |
| 2 | 50.0 | |
| 3 | 36.0 | |
| 4 | 38.0 | |
| 5 | 45.0 | |
| 6 | 21.0 | 3 8 |
| 7 | 32.0 | 3 8 |
| 8 | 32.0 | 2 |
| 9 | 29.0 | 4 6 |



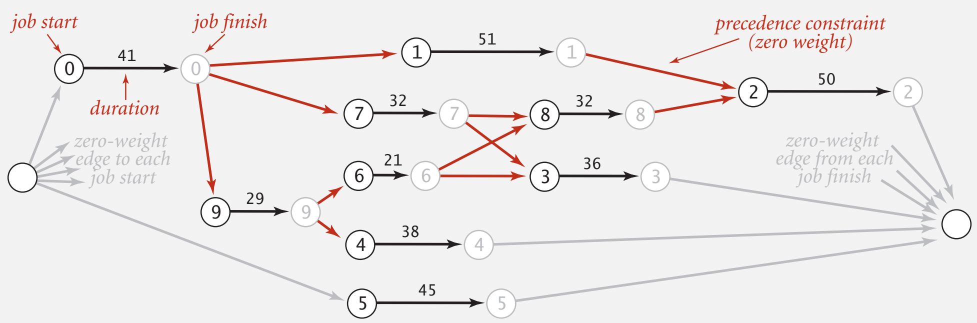

Parallel job scheduling solution

## Critical path method

CPM. To solve a parallel job-scheduling problem, create edge-weighted DAG:

- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

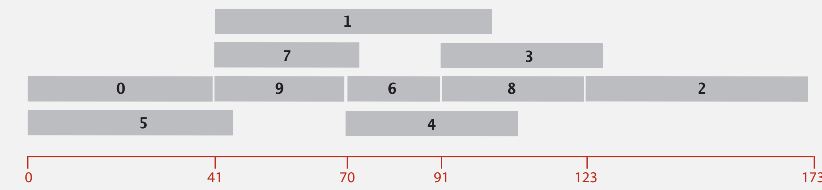
| *job* | *duration* | *must complete before* |
|---|---|---|
| 0 | 41.0 | 1 7 9 |
| 1 | 51.0 | 2 |
| 2 | 50.0 | |
| 3 | 36.0 | |
| 4 | 38.0 | |
| 5 | 45.0 | |
| 6 | 21.0 | 3 8 |
| 7 | 32.0 | 3 8 |
| 8 | 32.0 | 2 |
| 9 | 29.0 | 4 6 |



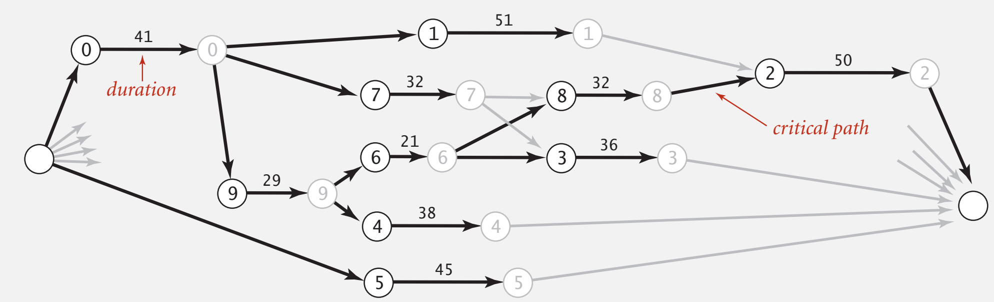



## Critical path method

CPM. Use longest path from the source to schedule each job.




Parallel job scheduling solution







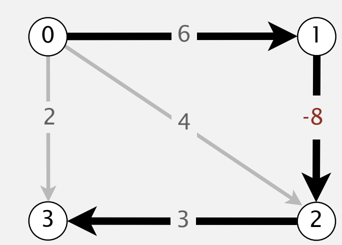
http://algs4.cs.princeton.edu

## 4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- *negative weights*

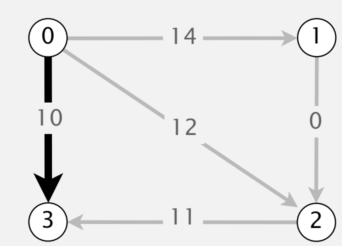
## Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.




Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight doesn't work.




Adding 8 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

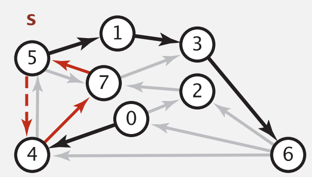
Conclusion. Need a different algorithm.

## Negative cycles

Def. A negative cycle is a directed cycle whose sum of edge weights is negative.

digraph

```
4->5  0.35
5->4 -0.66
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```




negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5

shortest path from 0 to 6
0->4->7->5->4->7->5...->1->3->6

Proposition. A SPT exists iff no negative cycles.

assuming all vertices reachable from s



## Bellman-Ford algorithm

**Bellman-Ford algorithm**

**Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.**

**Repeat V times:**
- **Relax each edge.**

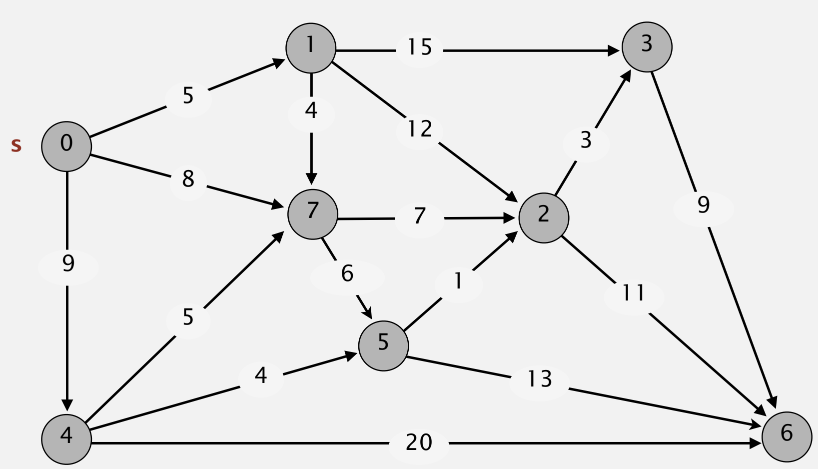
```
for (int i = 0; i < G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v))
         relax(e);
```

pass i (relax each edge)



## Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.



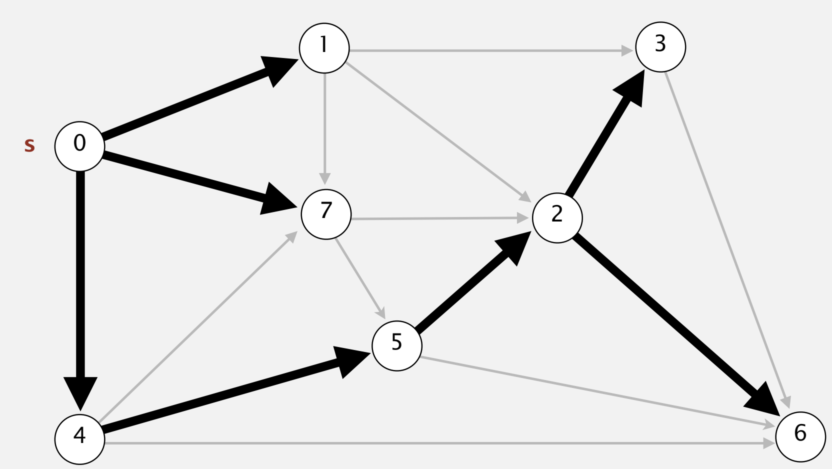

an edge-weighted digraph

| | |
|---|---|
| 0→1 | 5.0 |
| 0→4 | 9.0 |
| 0→7 | 8.0 |
| 1→2 | 12.0 |
| 1→3 | 15.0 |
| 1→7 | 4.0 |
| 2→3 | 3.0 |
| 2→6 | 11.0 |
| 3→6 | 9.0 |
| 4→5 | 4.0 |
| 4→6 | 20.0 |
| 4→7 | 5.0 |
| 5→2 | 1.0 |
| 5→6 | 13.0 |
| 7→5 | 6.0 |
| 7→2 | 7.0 |



## Bellman-Ford algorithm demo

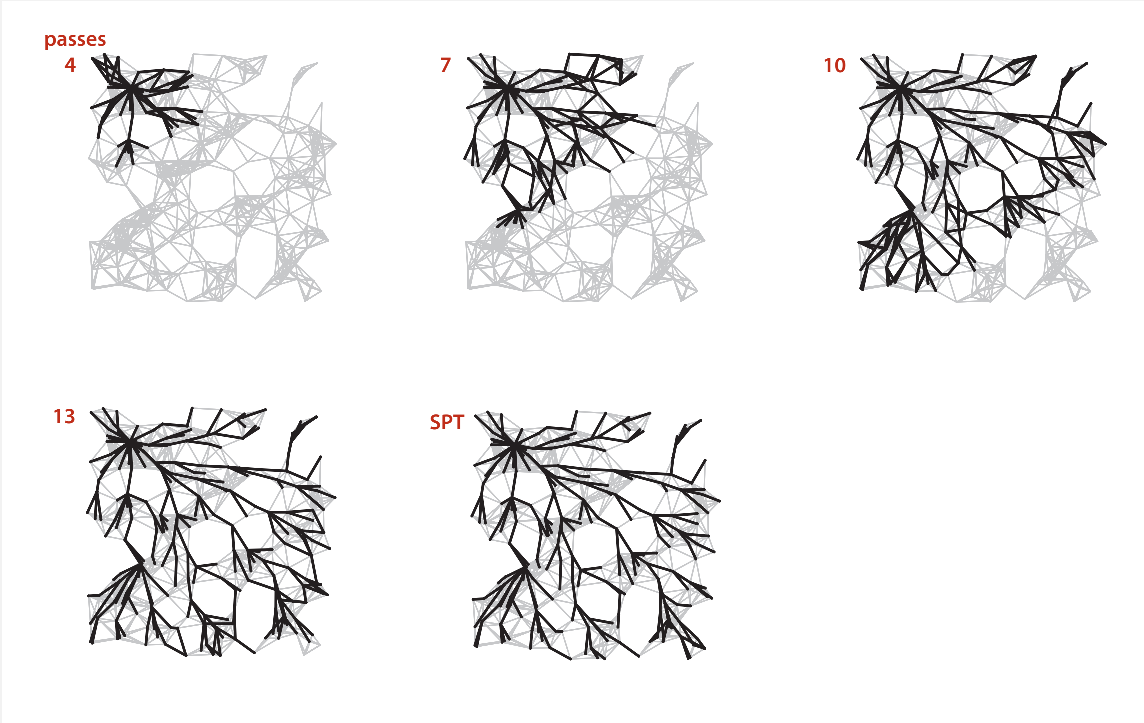
Repeat $V$ times: relax all $E$ edges.




shortest-paths tree from vertex s

| v | distTo[] | edgeTo[] |
|---|---|---|
| 0 | 0.0 | - |
| 1 | 5.0 | 0→1 |
| 2 | 14.0 | 5→2 |
| 3 | 17.0 | 2→3 |
| 4 | 9.0 | 0→4 |
| 5 | 13.0 | 4→5 |
| 6 | 25.0 | 2→6 |
| 7 | 8.0 | 0→7 |

## Bellman-Ford algorithm: visualization






## Bellman-Ford algorithm: analysis

**Bellman-Ford algorithm**

**Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.**

**Repeat V times:**

**– Relax each edge.**

Proposition. Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

Pf idea. After pass `i`, found path that is at least as short as any shortest path containing `i` (or fewer) edges.



## Bellman-Ford algorithm: practical improvement

Observation. If `distTo[v]` does not change during pass `i`, no need to relax any edge pointing from `v` in pass `i+1`.

FIFO implementation. Maintain queue of vertices whose `distTo[]` changed.

be careful to keep at most one copy of each vertex on queue (why?)

Overall effect.

- The running time is still proportional to $E \times V$ in worst case.
- But much faster than that in practice.



## Single source shortest-paths implementation: cost summary

| algorithm | restriction | typical case | worst case | extra space |
|---|---|---|---|---|
| **topological sort** | no directed cycles | $E + V$ | $E + V$ | $V$ |
| **Dijkstra (binary heap)** | no negative weights | $E \log V$ | $E \log V$ | $V$ |
| **Bellman-Ford** | no negative cycles | $E V$ | $E V$ | $V$ |
| **Bellman-Ford (queue-based)** | | $E + V$ | $E V$ | $V$ |

Remark 1. Directed cycles make the problem harder.
Remark 2. Negative weights make the problem harder.
Remark 3. Negative cycles makes the problem intractable.

## Finding a negative cycle

Negative cycle. Add two method to the API for SP.

| | | |
|---|---|---|
| boolean | hasNegativeCycle() | *is there a negative cycle?* |
| Iterable <DirectedEdge> | negativeCycle() | *negative cycle reachable from s* |

digraph

```
4->5  0.35
5->4 -0.66
4->7  0.37
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7->5  0.28
5->1  0.32
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3->6  0.52
6->0  0.58
6->4  0.93
```
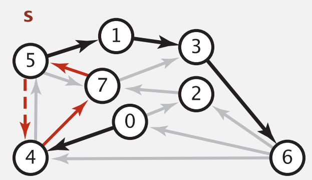
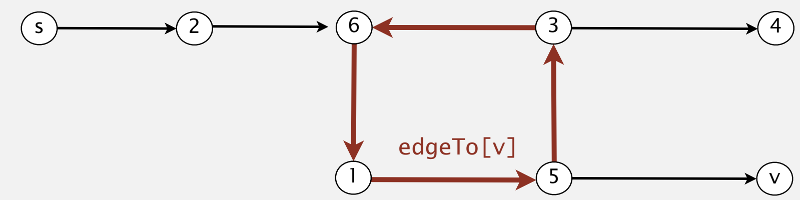




negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5



## Finding a negative cycle

Observation. If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.




Proposition. If any vertex `v` is updated in pass `V`, there exists a negative cycle (and can trace back `edgeTo[v]` entries to find it).

In practice. Check for negative cycles more frequently.



## Negative cycle application: arbitrage detection

Problem. Given table of exchange rates, is there an arbitrage opportunity?

| | USD | EUR | GBP | CHF | CAD |
|---|---|---|---|---|---|
| USD | 1 | 0.741 | 0.657 | 1.061 | 1.011 |
| EUR | 1.350 | 1 | 0.888 | 1.433 | 1.366 |
| GBP | 1.521 | 1.126 | 1 | 1.614 | 1.538 |
| CHF | 0.943 | 0.698 | 0.620 | 1 | 0.953 |
| CAD | 0.995 | 0.732 | 0.650 | 1.049 | 1 |

Ex. \$1,000 ⇒ 741 Euros ⇒ 1,012.206 Canadian dollars ⇒ \$1,007.14497.

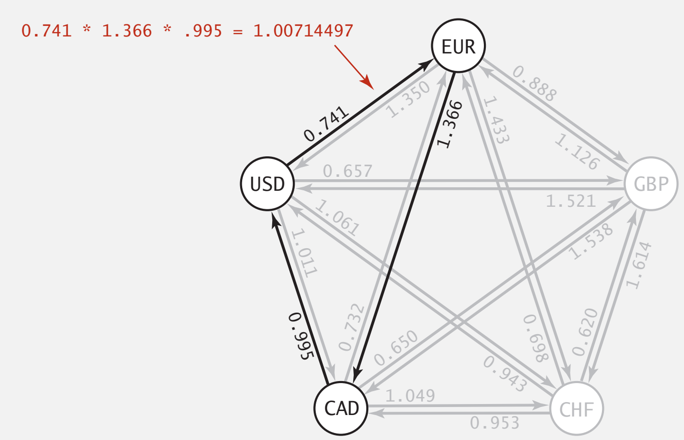
$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$



## Negative cycle application: arbitrage detection

Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is $> 1$.

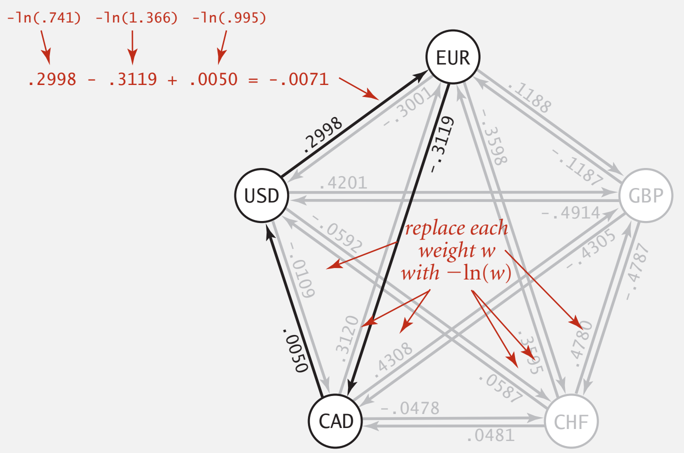



Challenge. Express as a negative cycle detection problem.

## Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logs.

- Let weight of edge $v \rightarrow w$ be $-ln$ (exchange rate from currency $v$ to $w$).
- Multiplication turns to addition; $> 1$ turns to $< 0$.
- Find a directed cycle whose sum of edge weights is $< 0$ (negative cycle).




Remark. Fastest algorithm is extraordinarily valuable!



## Shortest paths summary

Nonnegative weights.

- Arises in many application.
- Dijkstra's algorithm is nearly linear-time.

Acyclic edge-weighted digraphs.

- Arise in some applications.
- Topological sort algorithm is linear time.
- Edge weights can be negative.

Negative weights and negative cycles.

- Arise in some applications.
- If no negative cycles, can find shortest paths via Bellman-Ford.
- If negative cycles, can find one via Bellman-Ford.

Shortest-paths is a broadly useful problem-solving model.