



<http://algs4.cs.princeton.edu>

## 4.1 UNDIRECTED GRAPHS

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- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *challenges*



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## 4.1 UNDIRECTED GRAPHS

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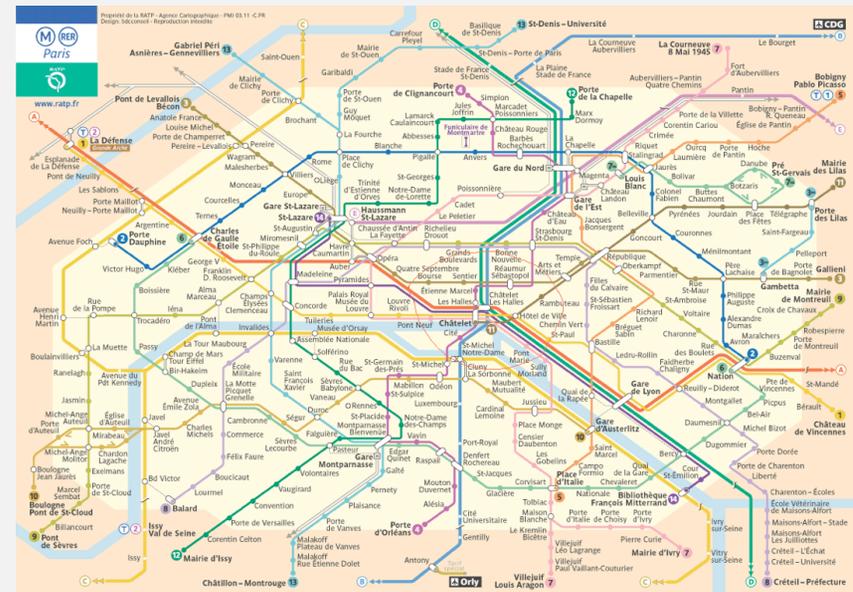
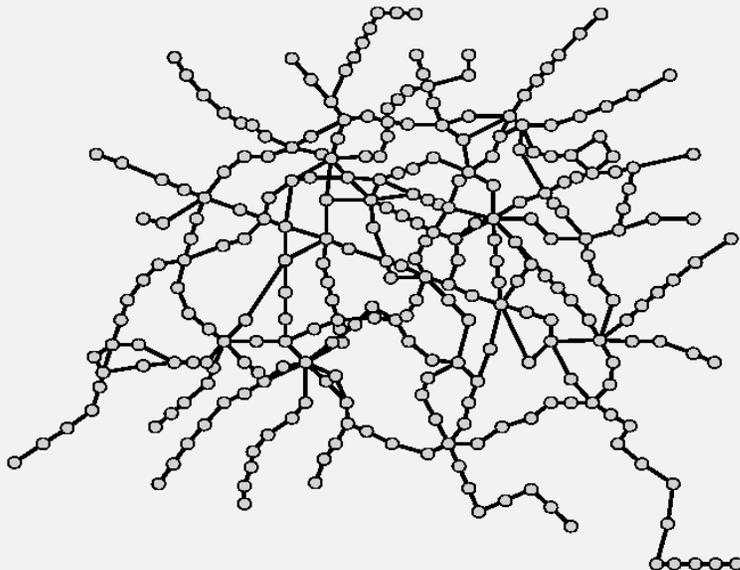
- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *challenges*

# Undirected graphs

Graph. Set of **vertices** connected pairwise by **edges**.

Why study graph algorithms?

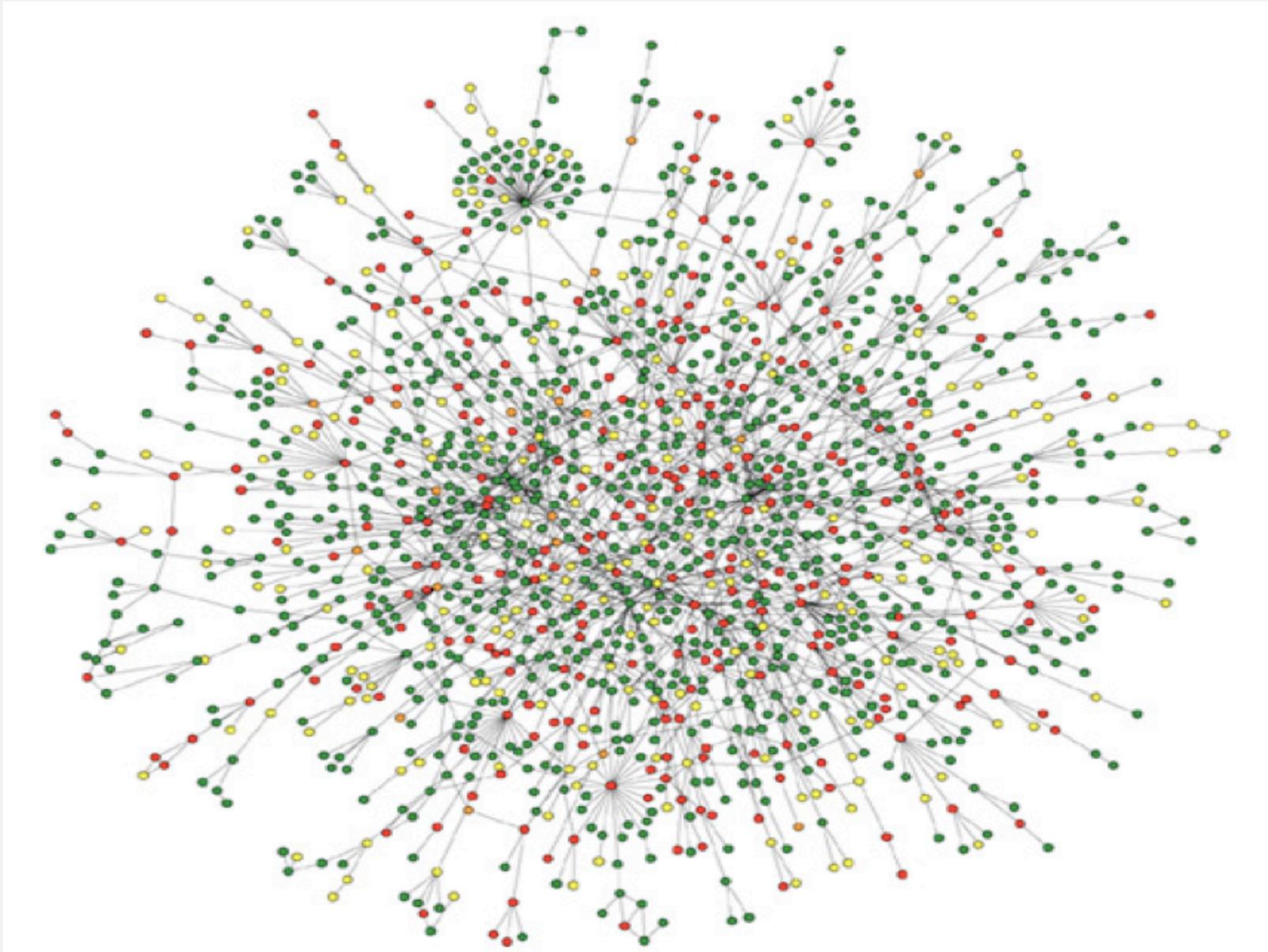
- Thousands of practical applications.
- Hundreds of graph algorithms known.
- Interesting and broadly useful abstraction.
- Challenging branch of computer science and discrete math.





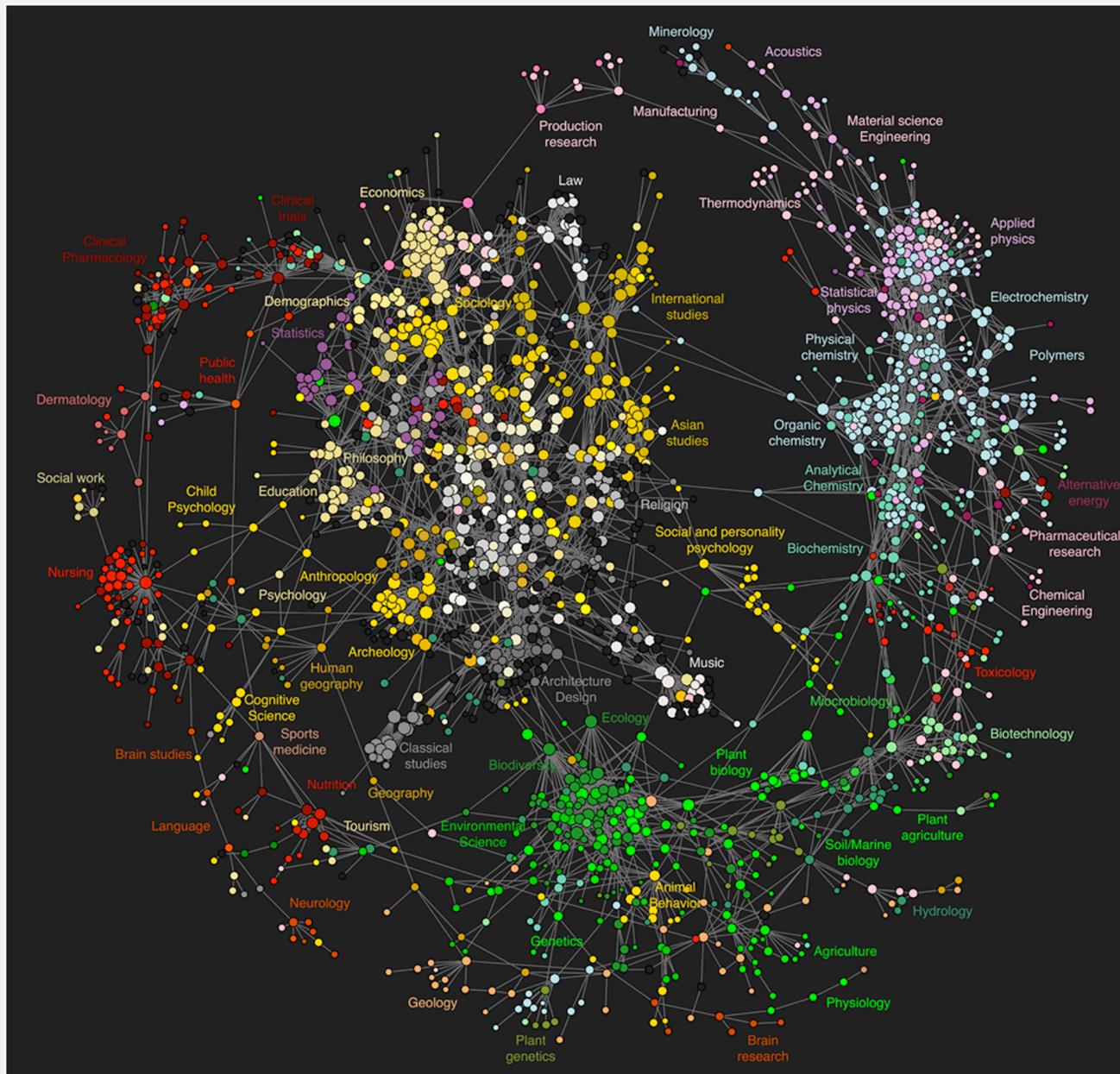
# Protein-protein interaction network

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Reference: Jeong et al, Nature Review | Genetics

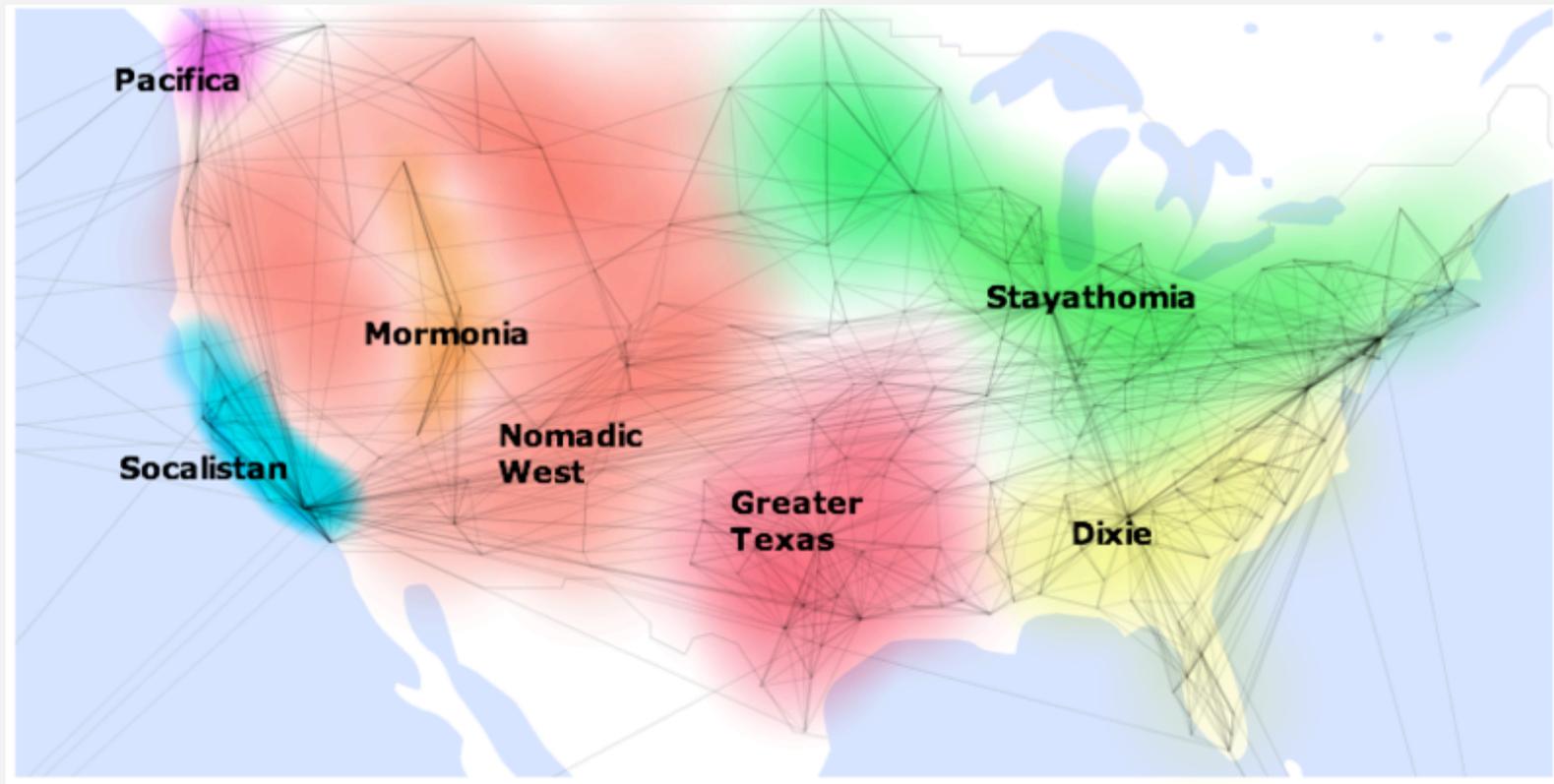
# Map of science clickstreams



<http://www.plosone.org/article/info:doi/10.1371/journal.pone.0004803>

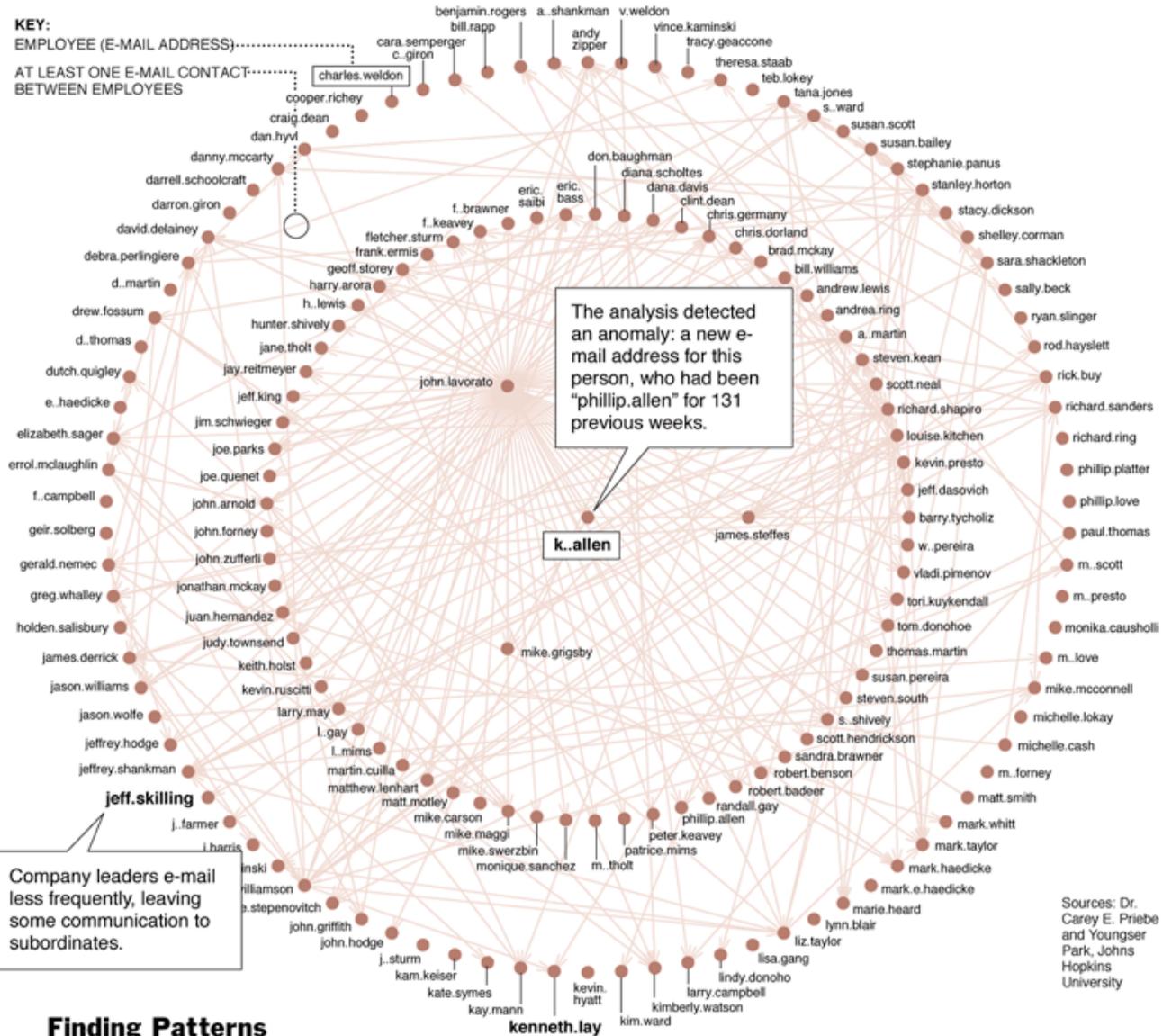
# America according to the Facebook graph

---



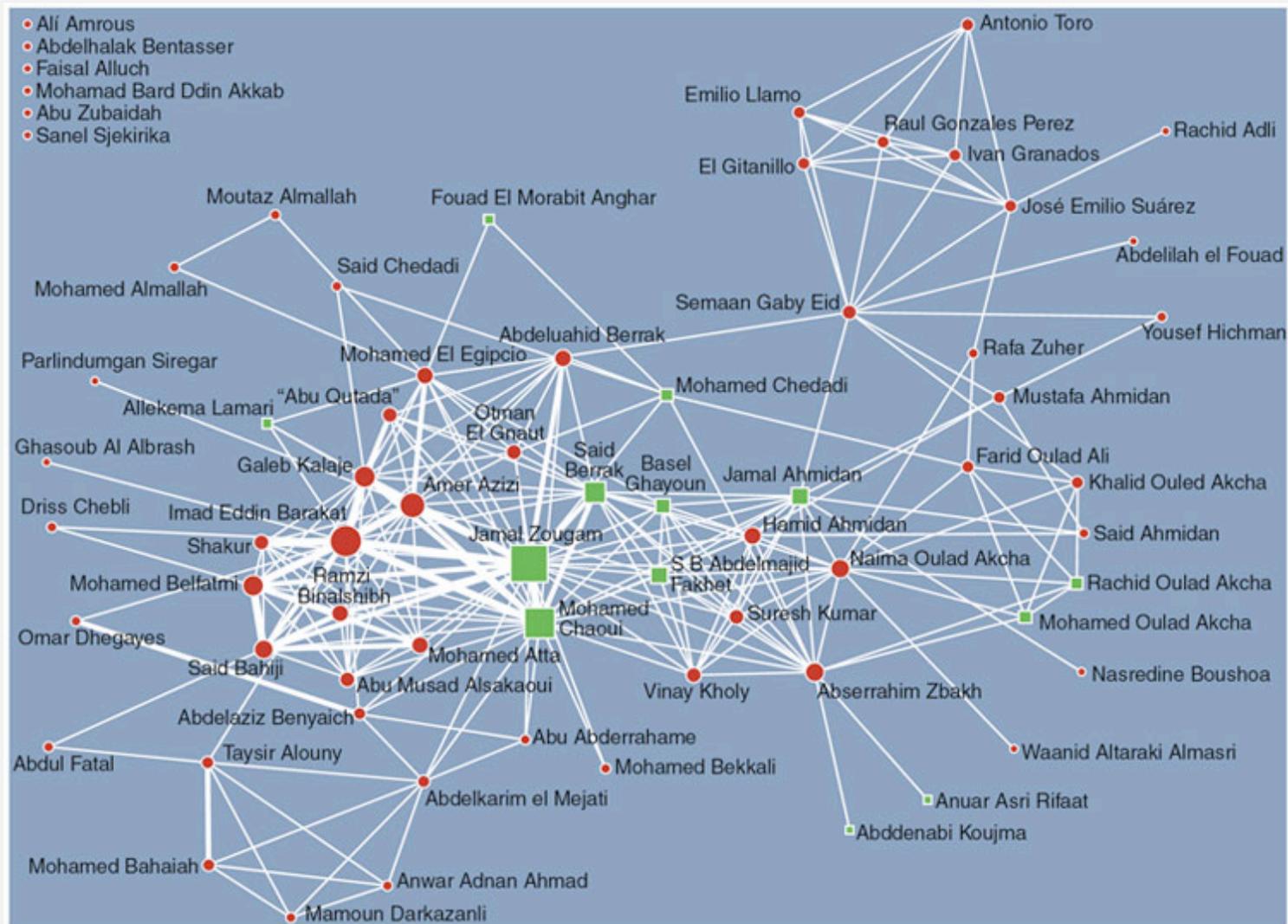
"How to split up the US" by Pete Warden

# One week of Enron emails

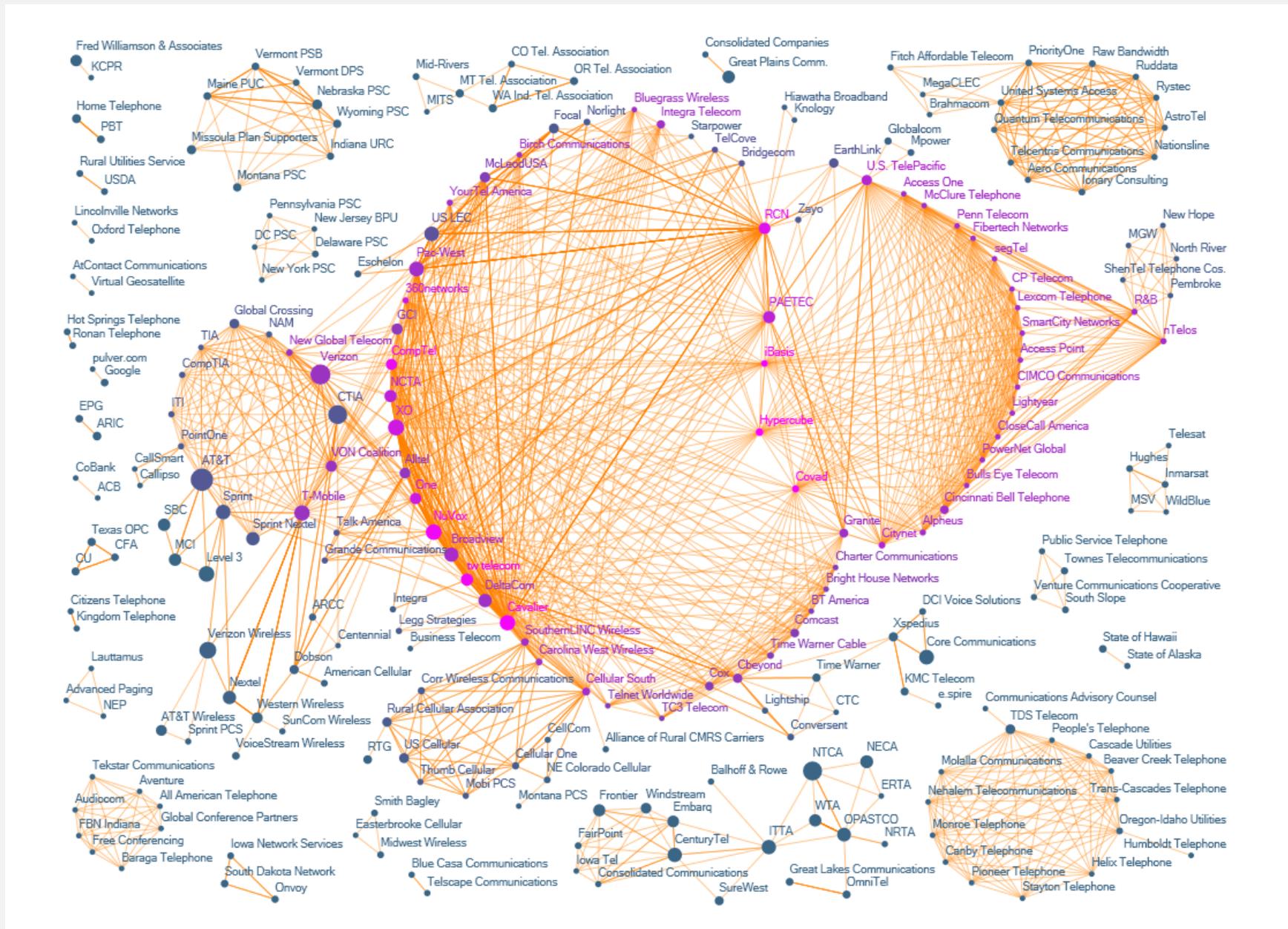


**Finding Patterns  
 In Corporate Chatter**

# Terrorist network

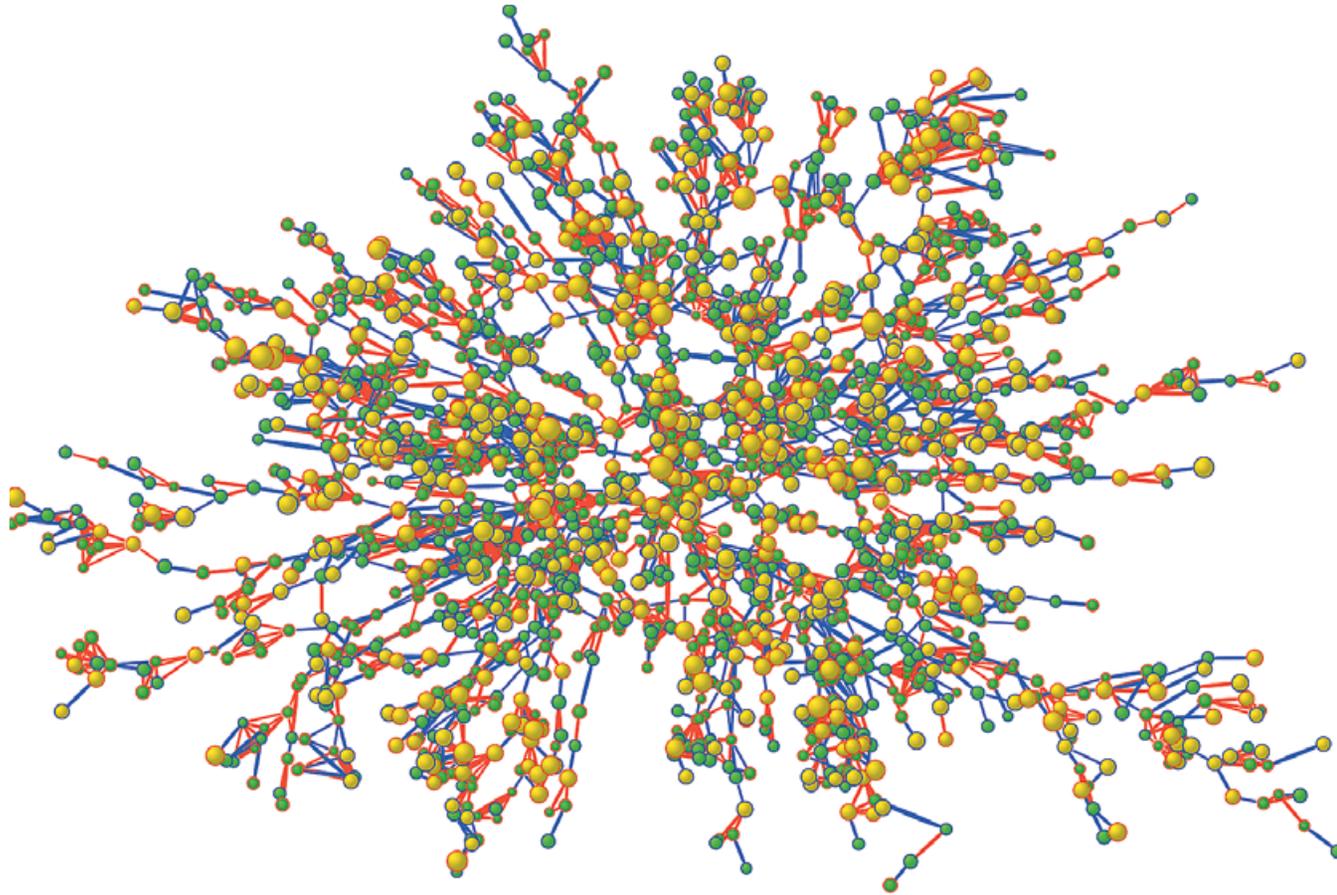


# The evolution of FCC lobbying coalitions



"The Evolution of FCC Lobbying Coalitions" by Pierre de Vries in JoSS Visualization Symposium 2010

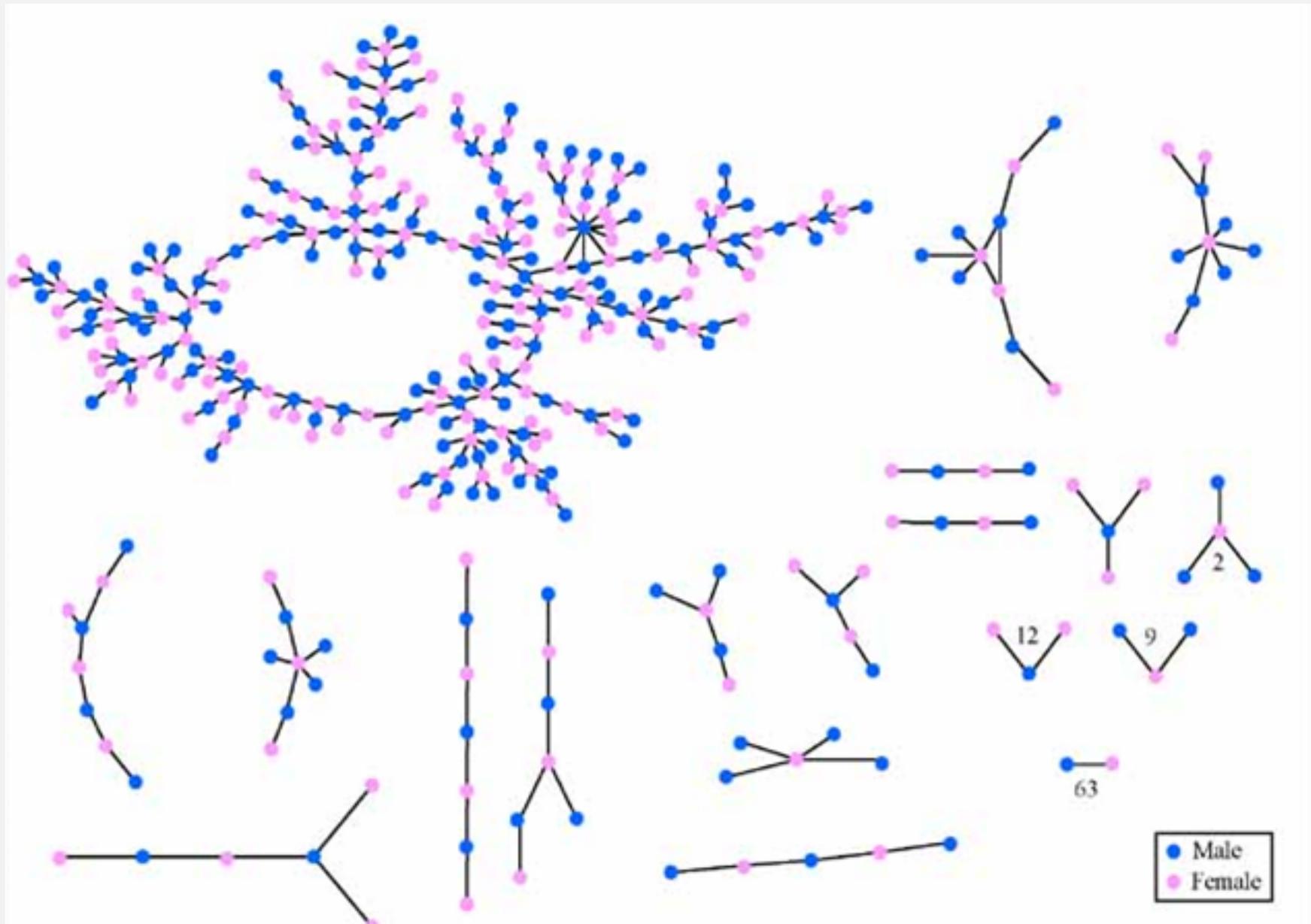
# Framingham heart study



**Figure 1.** Largest Connected Subcomponent of the Social Network in the Framingham Heart Study in the Year 2000.

Each circle (node) represents one person in the data set. There are 2200 persons in this subcomponent of the social network. Circles with red borders denote women, and circles with blue borders denote men. The size of each circle is proportional to the person's body-mass index. The interior color of the circles indicates the person's obesity status: yellow denotes an obese person (body-mass index,  $\geq 30$ ) and green denotes a nonobese person. The colors of the ties between the nodes indicate the relationship between them: purple denotes a friendship or marital tie and orange denotes a familial tie.

# Sexual/romantic network of a high school



# Graph applications

---

graph	vertex	edge
communication	telephone, computer	fiber optic cable
circuit	gate, register, processor	wire
mechanical	joint	rod, beam, spring
financial	stock, currency	transactions
transportation	street intersection, airport	highway, airway route
internet	class C network	connection
game	board position	legal move
social relationship	person, actor	friendship, movie cast
neural network	neuron	synapse
protein network	protein	protein-protein interaction
chemical compound	molecule	bond

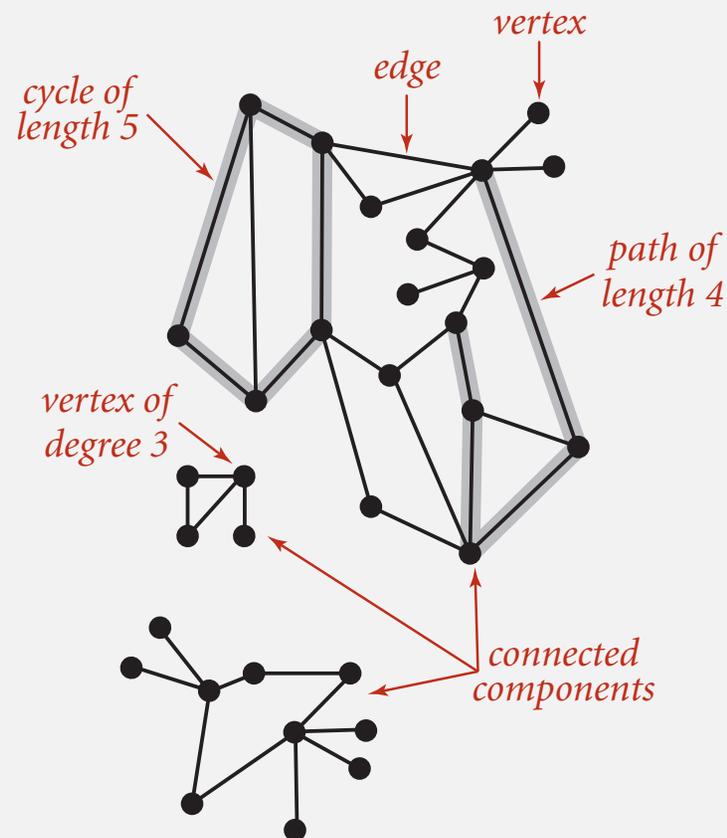
# Graph terminology

---

**Path.** Sequence of vertices connected by edges.

**Cycle.** Path whose first and last vertices are the same.

Two vertices are **connected** if there is a path between them.



## Some graph-processing problems

---

**Path.** Is there a path between  $s$  and  $t$ ?

**Shortest path.** What is the shortest path between  $s$  and  $t$ ?

**Cycle.** Is there a cycle in the graph?

**Euler tour.** Is there a cycle that uses each edge exactly once?

**Hamilton tour.** Is there a cycle that uses each vertex exactly once.

**Connectivity.** Is there a way to connect all of the vertices?

**MST.** What is the best way to connect all of the vertices?

**Biconnectivity.** Is there a vertex whose removal disconnects the graph?

**Planarity.** Can you draw the graph in the plane with no crossing edges

**Graph isomorphism.** Do two adjacency lists represent the same graph?

**Challenge.** Which of these problems are easy? difficult? intractable?



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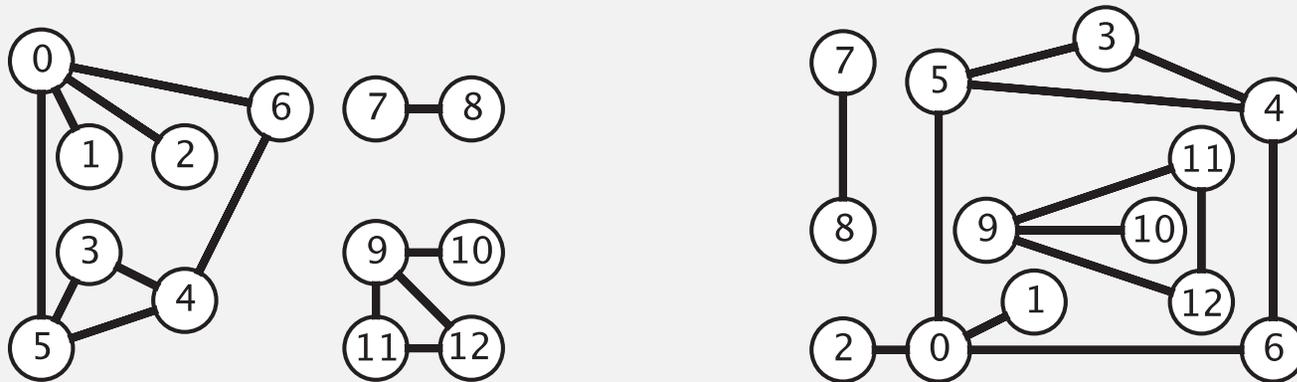
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- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *challenges*

# Graph representation

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**Graph drawing.** Provides intuition about the structure of the graph.



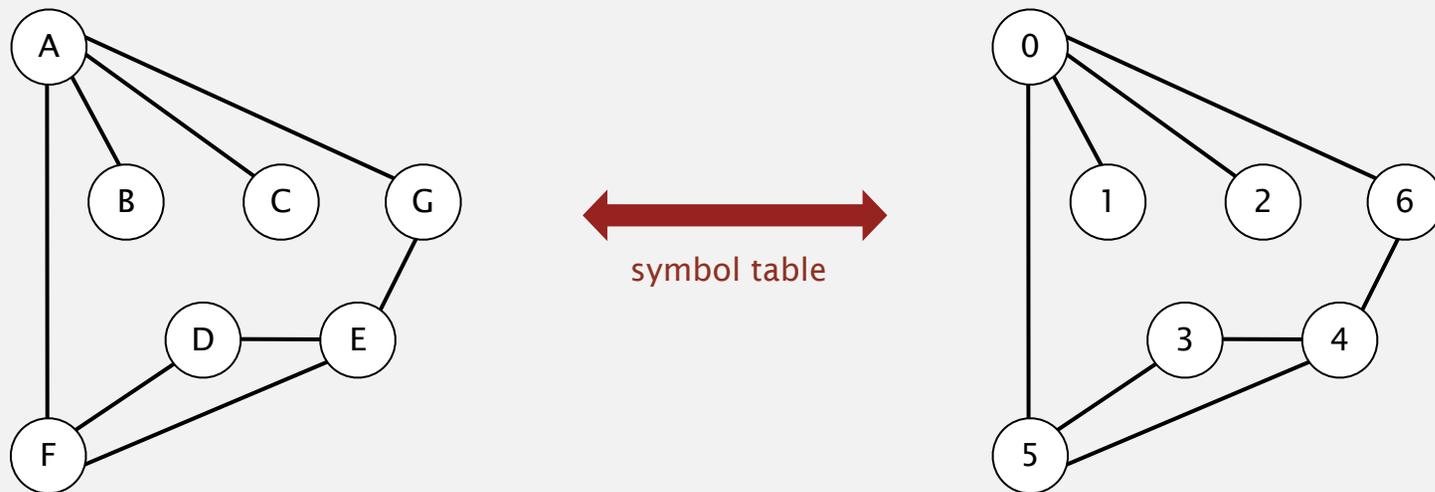
two drawings of the same graph

**Caveat.** Intuition can be misleading.

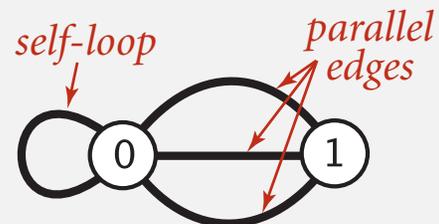
# Graph representation

## Vertex representation.

- This lecture: use integers between 0 and  $V-1$ .
- Applications: convert between names and integers with symbol table.



## Anomalies.



# Graph API

---

```
public class Graph
```

```
    Graph(int V)
```

*create an empty graph with V vertices*

```
    Graph(In in)
```

*create a graph from input stream*

```
    void addEdge(int v, int w)
```

*add an edge v-w*

```
    Iterable<Integer> adj(int v)
```

*vertices adjacent to v*

```
    int V()
```

*number of vertices*

```
    int E()
```

*number of edges*

```
In in = new In(args[0]);  
Graph G = new Graph(in);
```

← read graph from  
input stream

```
for (int v = 0; v < G.V(); v++)  
    for (int w : G.adj(v))  
        StdOut.println(v + "-" + w);
```

← print out each  
edge (twice)

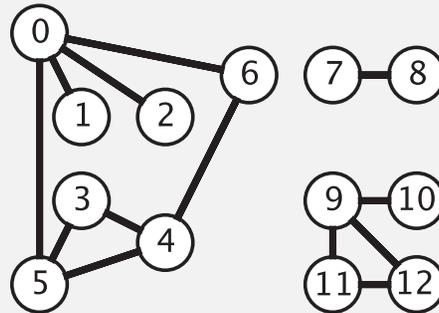
# Graph API: sample client

Graph input format.

**tinyG.txt**

*V* → 13  
13 ← *E*

```
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3
```



```
% java Test tinyG.txt
0-6
0-2
0-1
0-5
1-0
2-0
3-5
3-4
...
12-11
12-9
```

```
In in = new In(args[0]);
Graph G = new Graph(in);

for (int v = 0; v < G.V(); v++)
    for (int w : G.adj(v))
        StdOut.println(v + "-" + w);
```

← read graph from  
input stream

← print out each  
edge (twice)

# Typical graph-processing code

---

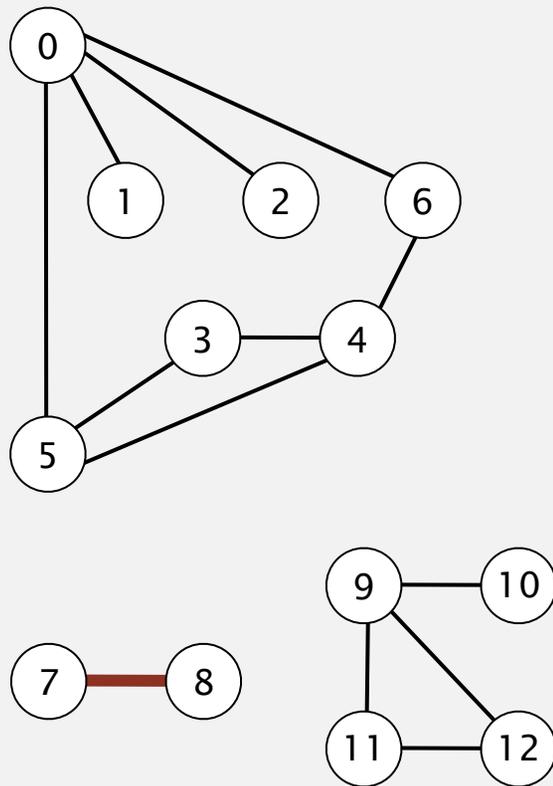
```
public class Graph  
  
    Graph(int V)                create an empty graph with V vertices  
  
    Graph(In in)                create a graph from input stream  
  
    void addEdge(int v, int w)   add an edge v-w  
  
    Iterable<Integer> adj(int v) vertices adjacent to v  
  
    int V()                      number of vertices  
  
    int E()                      number of edges
```

```
// degree of vertex v in graph G  
public static int degree(Graph G, int v)  
{  
    int degree = 0;  
    for (int w : G.adj(v))  
        degree++;  
    return degree;  
}
```

# Set-of-edges graph representation

---

Maintain a list of the edges (linked list or array).

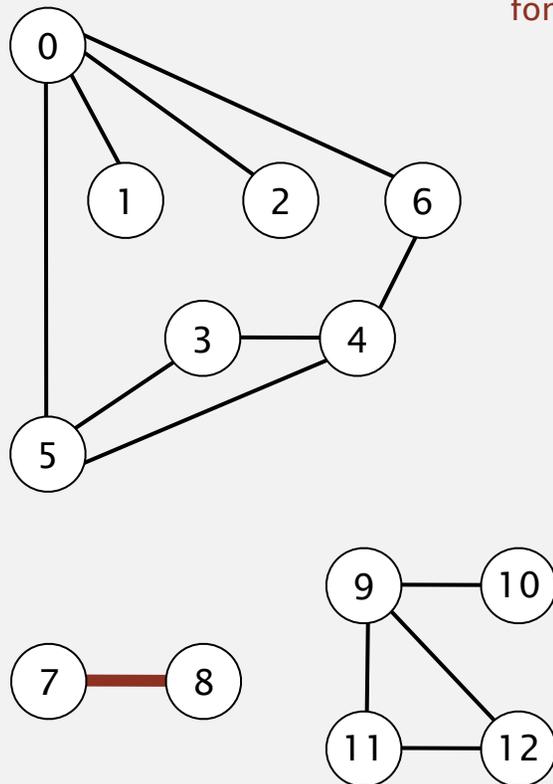


0	1
0	2
0	5
0	6
3	4
3	5
4	5
4	6
7	8
9	10
9	11
9	12
11	12

# Adjacency-matrix graph representation

Maintain a two-dimensional  $V$ -by- $V$  boolean array;

for each edge  $v-w$  in graph:  $\text{adj}[v][w] = \text{adj}[w][v] = \text{true}$ .

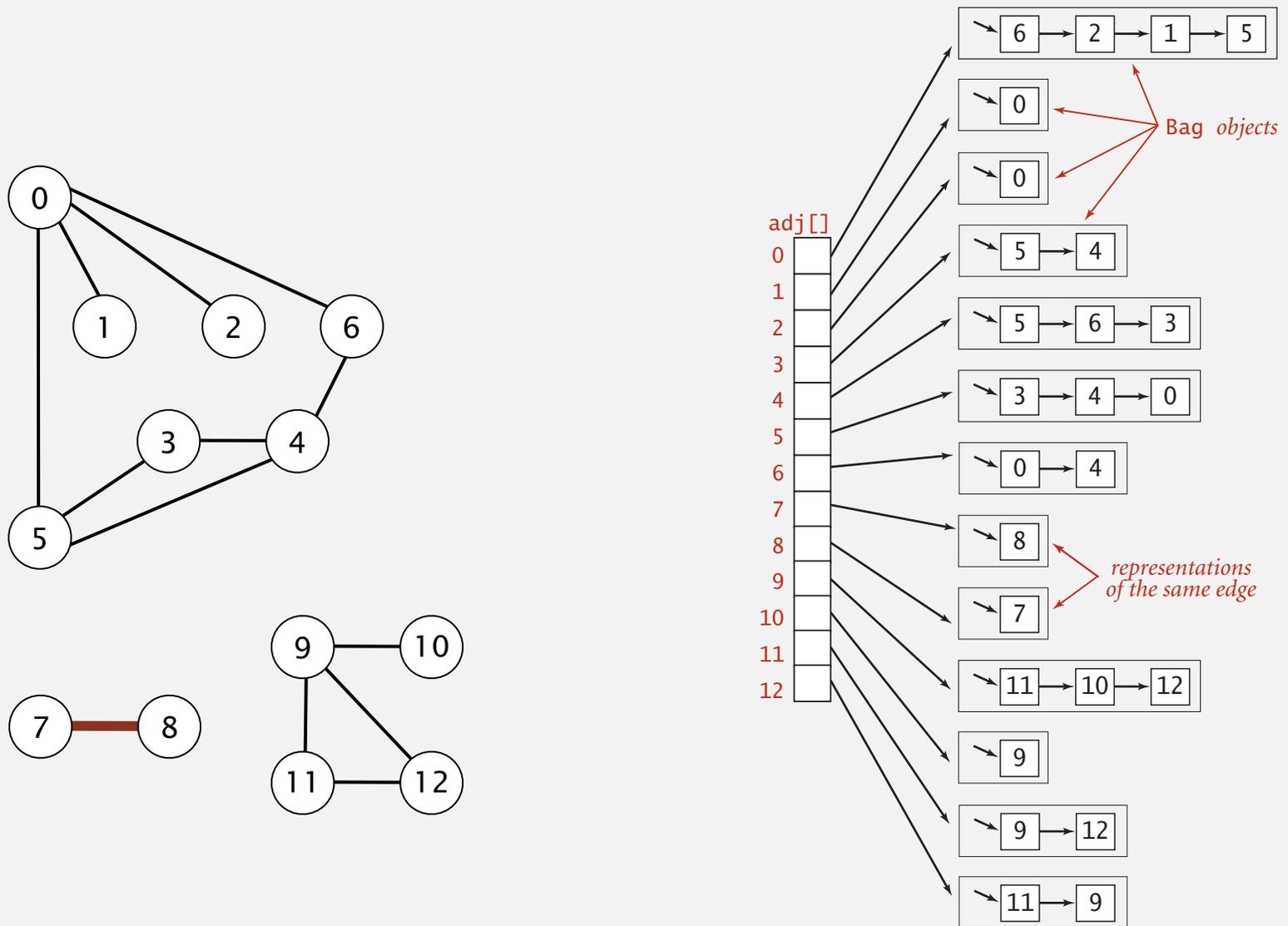


two entries  
for each edge

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0
4	0	0	0	1	0	1	1	0	0	0	0	0	0
5	1	0	0	1	1	0	0	0	0	0	0	0	0
6	1	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	1
10	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	0	1
12	0	0	0	0	0	0	0	0	0	1	0	1	0

# Adjacency-list graph representation

Maintain vertex-indexed array of lists.



# Adjacency-list graph representation: Java implementation

---

```
public class Graph  
{
```

```
    private final int V;  
    private Bag<Integer>[] adj;
```

← adjacency lists  
( using Bag data type )

```
    public Graph(int V  
    {
```

```
        this.V = V;  
        adj = (Bag<Integer>[]) new Bag[V];  
        for (int v = 0; v < V; v++)  
            adj[v] = new Bag<Integer>();
```

← create empty graph  
with V vertices

```
    public void addEdge(int v, int w  
    {
```

```
        adj[v].add(w);  
        adj[w].add(v);
```

← add edge v-w  
(parallel edges and  
self-loops allowed)

```
    public Iterable<Integer> adj(int v  
    { return adj[v]; }
```

← iterator for vertices adjacent to v

```
}
```

# Graph representations

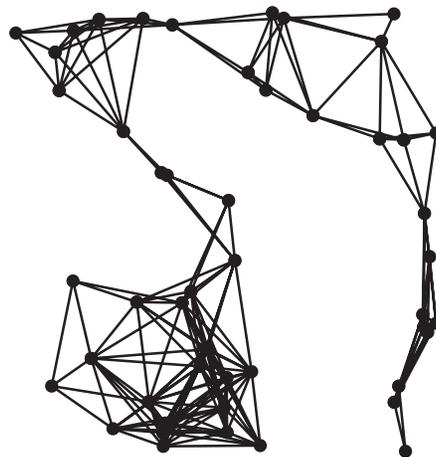
---

**In practice.** Use adjacency-lists representation.

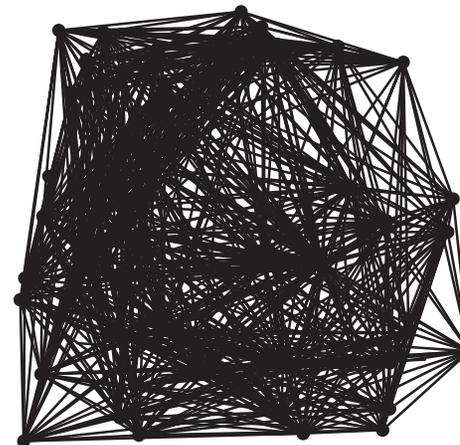
- Algorithms based on iterating over vertices adjacent to  $v$ .
- Real-world graphs tend to be **sparse**.

huge number of vertices,  
small average vertex degree

sparse ( $E = 200$ )



dense ( $E = 1000$ )



Two graphs ( $V = 50$ )

# Graph representations

---

**In practice.** Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent to  $v$ .
- Real-world graphs tend to be **sparse**.

huge number of vertices,  
small average vertex degree



representation	space	add edge	edge between $v$ and $w$ ?	iterate over vertices adjacent to $v$ ?
list of edges	$E$	1	$E$	$E$
adjacency matrix	$V^2$	1 *	1	$V$
adjacency lists	$E + V$	1	degree( $v$ )	degree( $v$ )

\* disallows parallel edges



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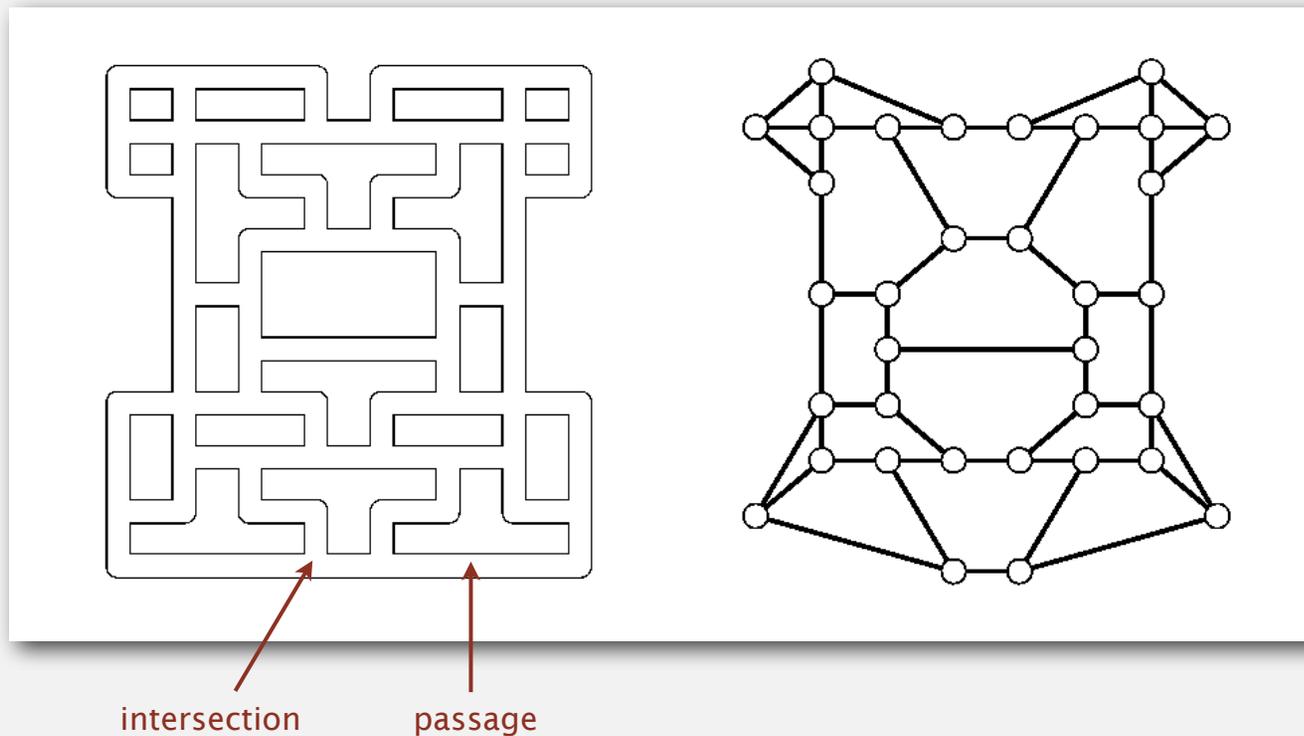
- ▶ *introduction*
- ▶ *graph API*
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- ▶ *breadth-first search*
- ▶ *challenges*

# Maze exploration

---

## Maze graph.

- Vertex = intersection.
- Edge = passage.



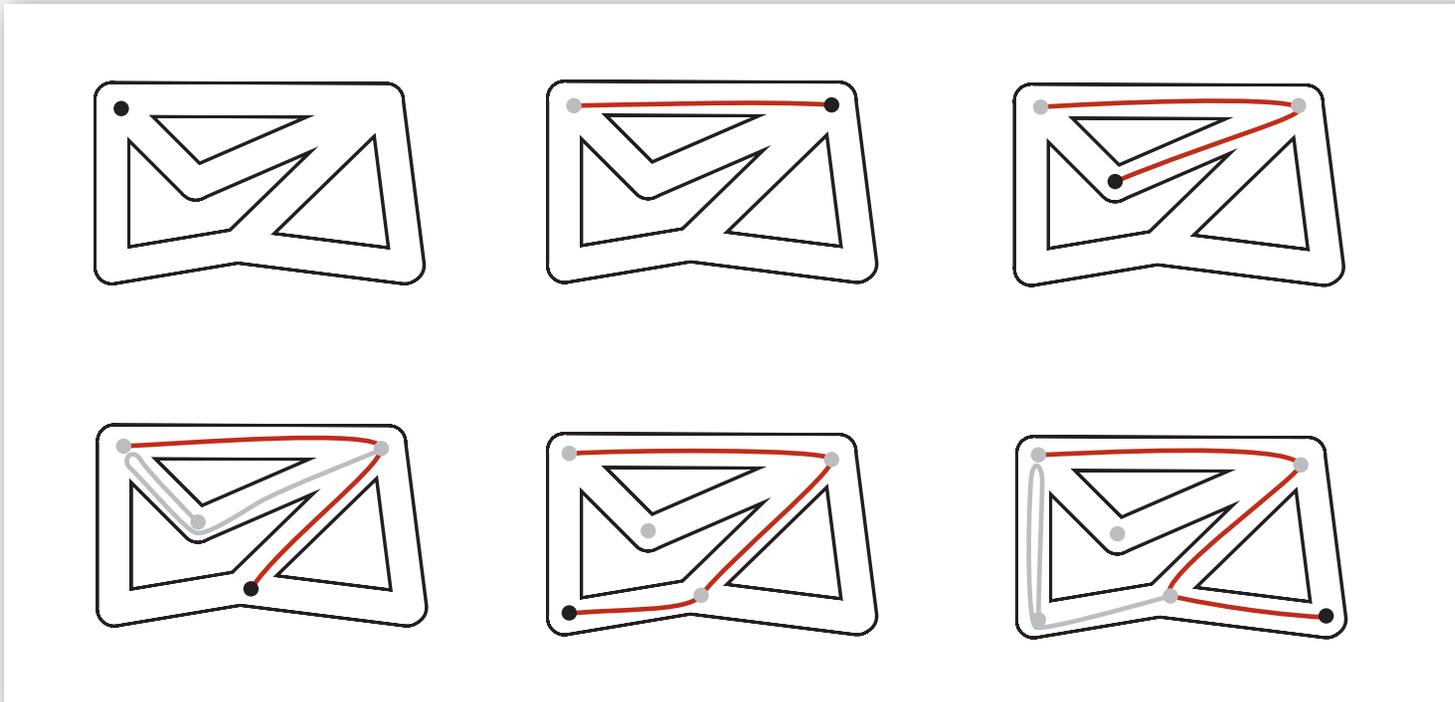
**Goal.** Explore every intersection in the maze.

# Trémaux maze exploration

---

## Algorithm.

- Unroll a ball of string behind you.
- Mark each visited intersection and each visited passage.
- Retrace steps when no unvisited options.



# Trémaux maze exploration

---

## Algorithm.

- Unroll a ball of string behind you.
- Mark each visited intersection and each visited passage.
- Retrace steps when no unvisited options.

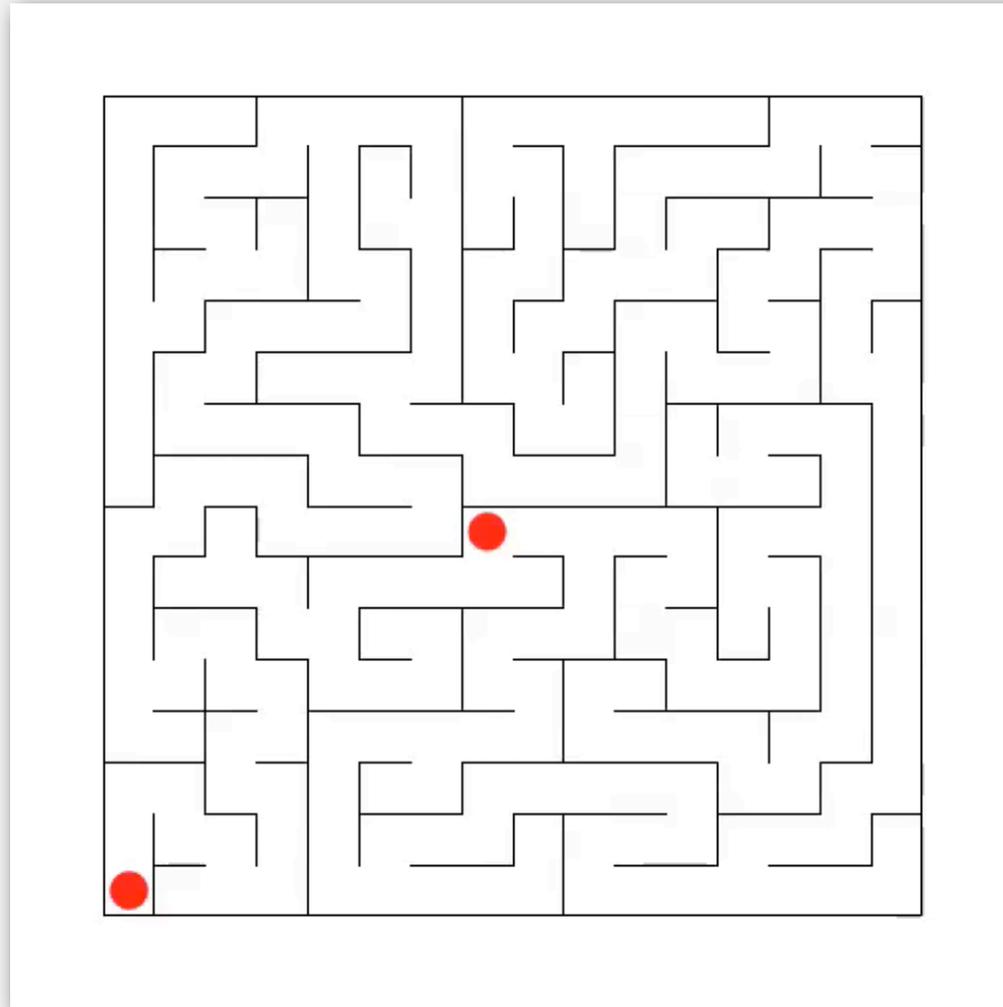
**First use?** Theseus entered Labyrinth to kill the monstrous Minotaur; Ariadne instructed Theseus to use a ball of string to find his way back out.



Claude Shannon (with Theseus mouse)

# Maze exploration

---



# Depth-first search

---

**Goal.** Systematically search through a graph.

**Idea.** Mimic maze exploration.

**DFS (to visit a vertex  $v$ )**

---

**Mark  $v$  as visited.**

**Recursively visit all unmarked  
vertices  $w$  adjacent to  $v$ .**

---

**Typical applications.**

- Find all vertices connected to a given source vertex.
- Find a path between two vertices.

**Design challenge.** How to implement?

# Design pattern for graph processing

---

**Design pattern.** Decouple graph data type from graph processing.

- Create a Graph object.
- Pass the Graph to a graph-processing routine.
- Query the graph-processing routine for information.

```
public class Paths
```

```
    Paths(Graph G, int s)           find paths in G from source s
```

```
    boolean hasPathTo(int v)       is there a path from s to v?
```

```
    Iterable<Integer> pathTo(int v) path from s to v; null if no such path
```

```
Paths paths = new Paths(G, s);  
for (int v = 0; v < G.V(); v++)  
    if (paths.hasPathTo(v))  
        StdOut.println(v);
```

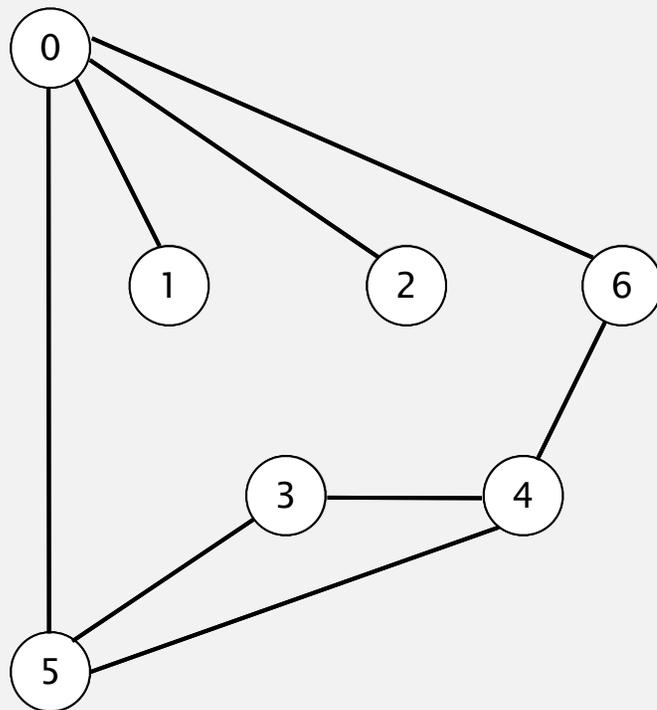
← print all vertices  
connected to s

# Depth-first search demo

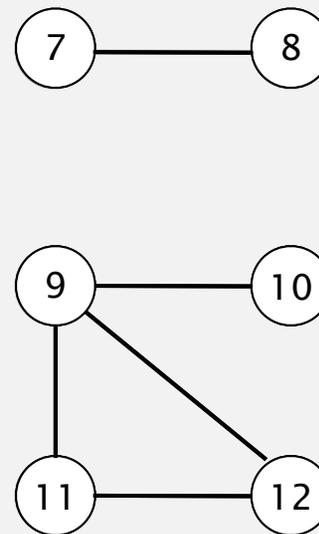
To visit a vertex  $v$ :



- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



graph G



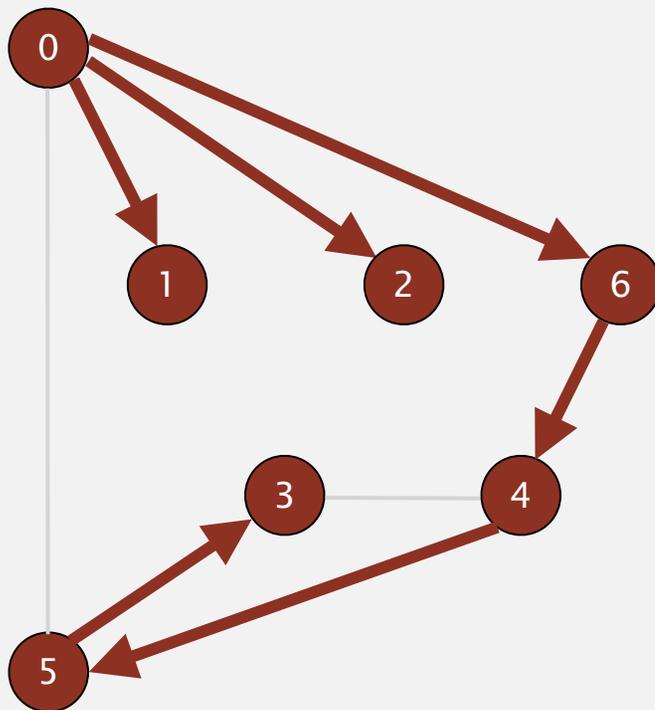
**tinyG.txt**

```
V → 13
13 ← E
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3
```

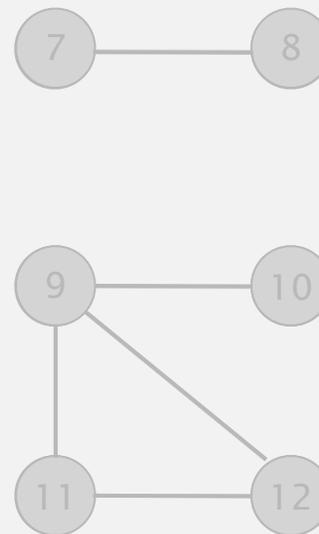
# Depth-first search demo

To visit a vertex  $v$  :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



vertices reachable from 0



$v$	marked[]	edgeTo[v]
0	T	-
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	-
8	F	-
9	F	-
10	F	-
11	F	-
12	F	-

# Depth-first search

---

**Goal.** Find all vertices connected to  $s$  (and a corresponding path).

**Idea.** Mimic maze exploration.

**Algorithm.**

- Use recursion (ball of string).
- Mark each visited vertex (and keep track of edge taken to visit it).
- Return (retrace steps) when no unvisited options.

**Data structures.**

- `boolean[] marked` to mark visited vertices.
- `int[] edgeTo` to keep tree of paths.  
(`edgeTo[w] == v`) means that edge  $v-w$  taken to visit  $w$  for first time

# Depth-first search

---

```
public class DepthFirstPaths
{
    private boolean[] marked;
    private int[] edgeTo;
    private int s;
```

marked[v] = true  
if v connected to s  
edgeTo[v] = previous  
vertex on path from s to v

```
    public DepthFirstSearch(Graph G, int s)
    {
        ...
        dfs(G, s);
    }
```

initialize data structures  
find vertices connected to s

```
    private void dfs(Graph G, int v)
    {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w])
            {
                dfs(G, w);
                edgeTo[w] = v;
            }
    }
}
```

recursive DFS does the work

# Depth-first search properties

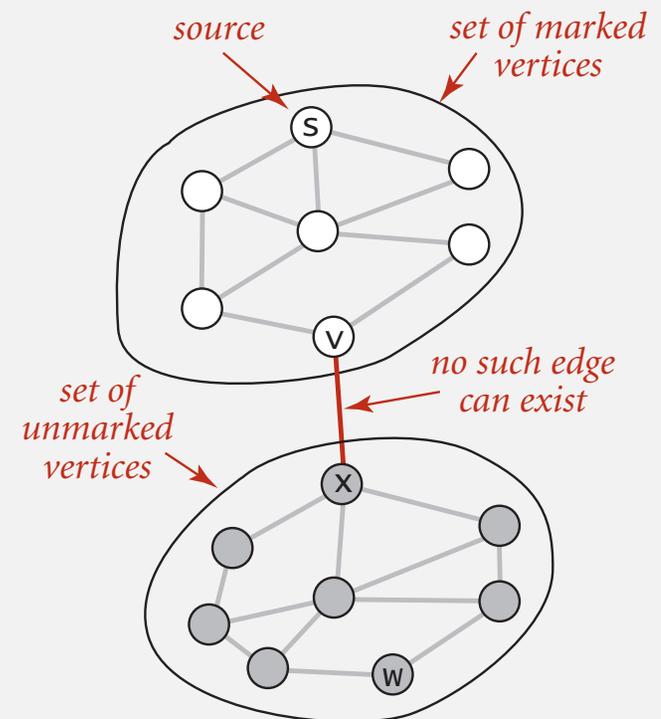
**Proposition.** DFS marks all vertices connected to  $s$  in time proportional to the sum of their degrees.

**Pf.** [correctness]

- If  $w$  marked, then  $w$  connected to  $s$  (why?)
- If  $w$  connected to  $s$ , then  $w$  marked.  
(if  $w$  unmarked, then consider last edge on a path from  $s$  to  $w$  that goes from a marked vertex to an unmarked one).

**Pf.** [running time]

Each vertex connected to  $s$  is visited once.



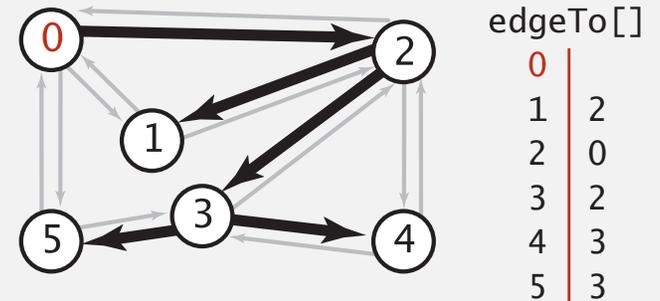
# Depth-first search properties

**Proposition.** After DFS, can find vertices connected to  $s$  in constant time and can find a path to  $s$  (if one exists) in time proportional to its length.

**Pf.** `edgeTo[]` is parent-link representation of a tree rooted at  $s$ .

```
public boolean hasPathTo(int v)
{ return marked[v]; }

public Iterable<Integer> pathTo(int v)
{
    if (!hasPathTo(v)) return null;
    Stack<Integer> path = new Stack<Integer>();
    for (int x = v; x != s; x = edgeTo[x])
        path.push(x);
    path.push(s);
    return path;
}
```



# Depth-first search application: flood fill

---

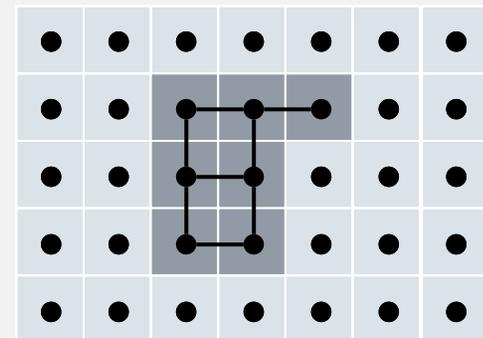
**Challenge.** Flood fill (Photoshop magic wand).

**Assumptions.** Picture has millions to billions of pixels.



**Solution.** Build a **grid graph**.

- Vertex: pixel.
- Edge: between two adjacent gray pixels.
- Blob: all pixels connected to given pixel.





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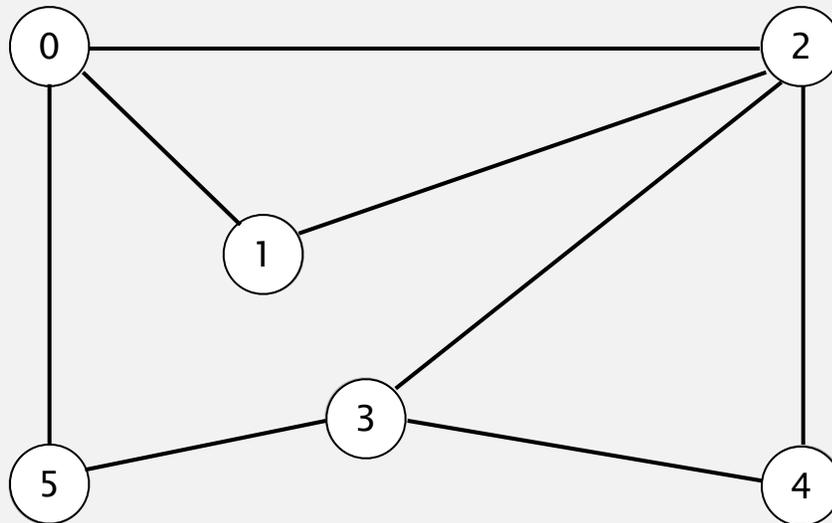
---

- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *challenges*

# Breadth-first search demo

Repeat until queue is empty:

- Remove vertex  $v$  from queue.
- Add to queue all unmarked vertices adjacent to  $v$  and mark them.



tinyCG.txt

$V \rightarrow$  6  
8  $\leftarrow E$   
0 5  
2 4  
2 3  
1 2  
0 1  
3 4  
3 5  
0 2

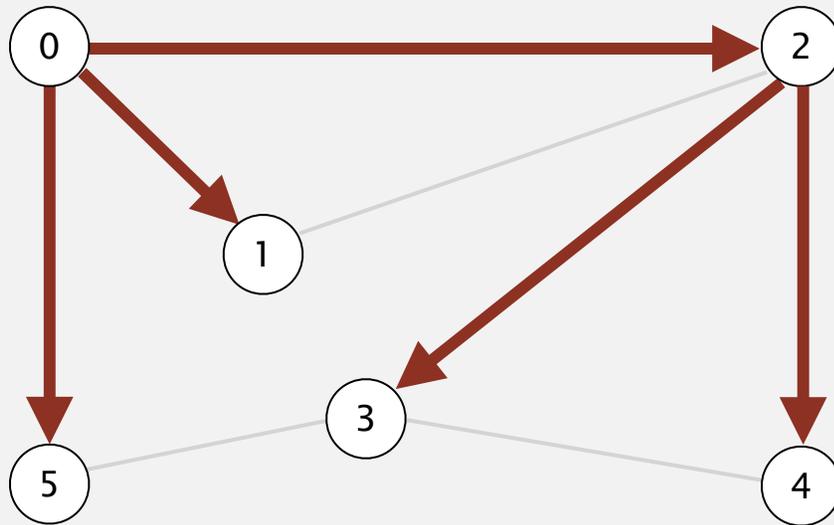
graph G

## Breadth-first search demo

---

Repeat until queue is empty:

- Remove vertex  $v$  from queue.
- Add to queue all unmarked vertices adjacent to  $v$  and mark them.



$v$	edgeTo[]	distTo[]
0	-	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

done

# Breadth-first search

---

**Depth-first search.** Put unvisited vertices on a **stack**.

**Breadth-first search.** Put unvisited vertices on a **queue**.

**Shortest path.** Find path from  $s$  to  $t$  that uses **fewest number of edges**.

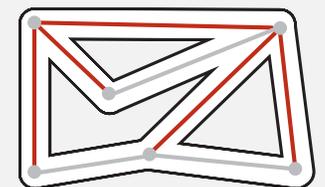
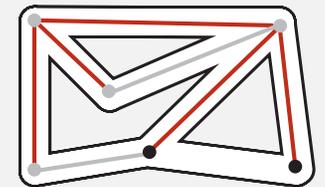
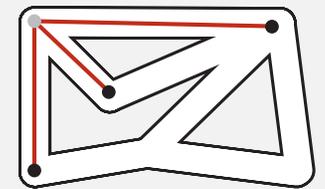
## **BFS** (from source vertex $s$ )

---

Put  $s$  onto a FIFO queue, and mark  $s$  as visited.

Repeat until the queue is empty:

- remove the least recently added vertex  $v$
  - add each of  $v$ 's unvisited neighbors to the queue, and mark them as visited.
- 



**Intuition.** BFS examines vertices in increasing distance from  $s$ .

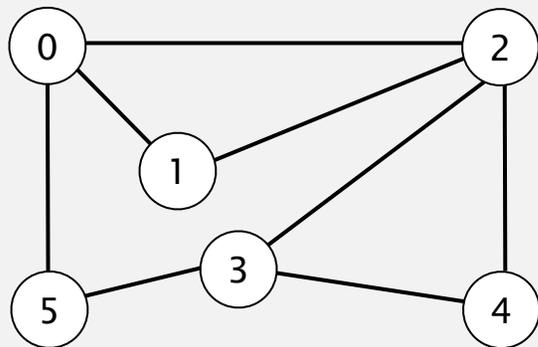
## Breadth-first search properties

---

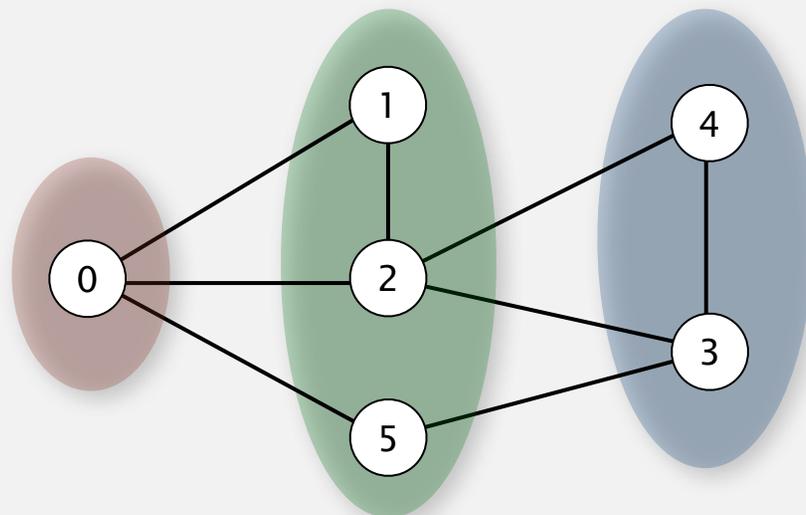
**Proposition.** BFS computes shortest paths (fewest number of edges) from  $s$  to all other vertices in a graph in time proportional to  $E + V$ .

**Pf.** [correctness] Queue always consists of zero or more vertices of distance  $k$  from  $s$ , followed by zero or more vertices of distance  $k + 1$ .

**Pf.** [running time] Each vertex connected to  $s$  is visited once.



graph



dist = 0

dist = 1

dist = 2

# Breadth-first search

---

```
public class BreadthFirstPaths
{
    private boolean[] marked;
    private int[] edgeTo;
    private int[] distTo;
    ...

```

```
    private void bfs(Graph G, int s) {
        Queue<Integer> q = new Queue<Integer>();
        q.enqueue(s);
        marked[s] = true;
        distTo[s] = 0;

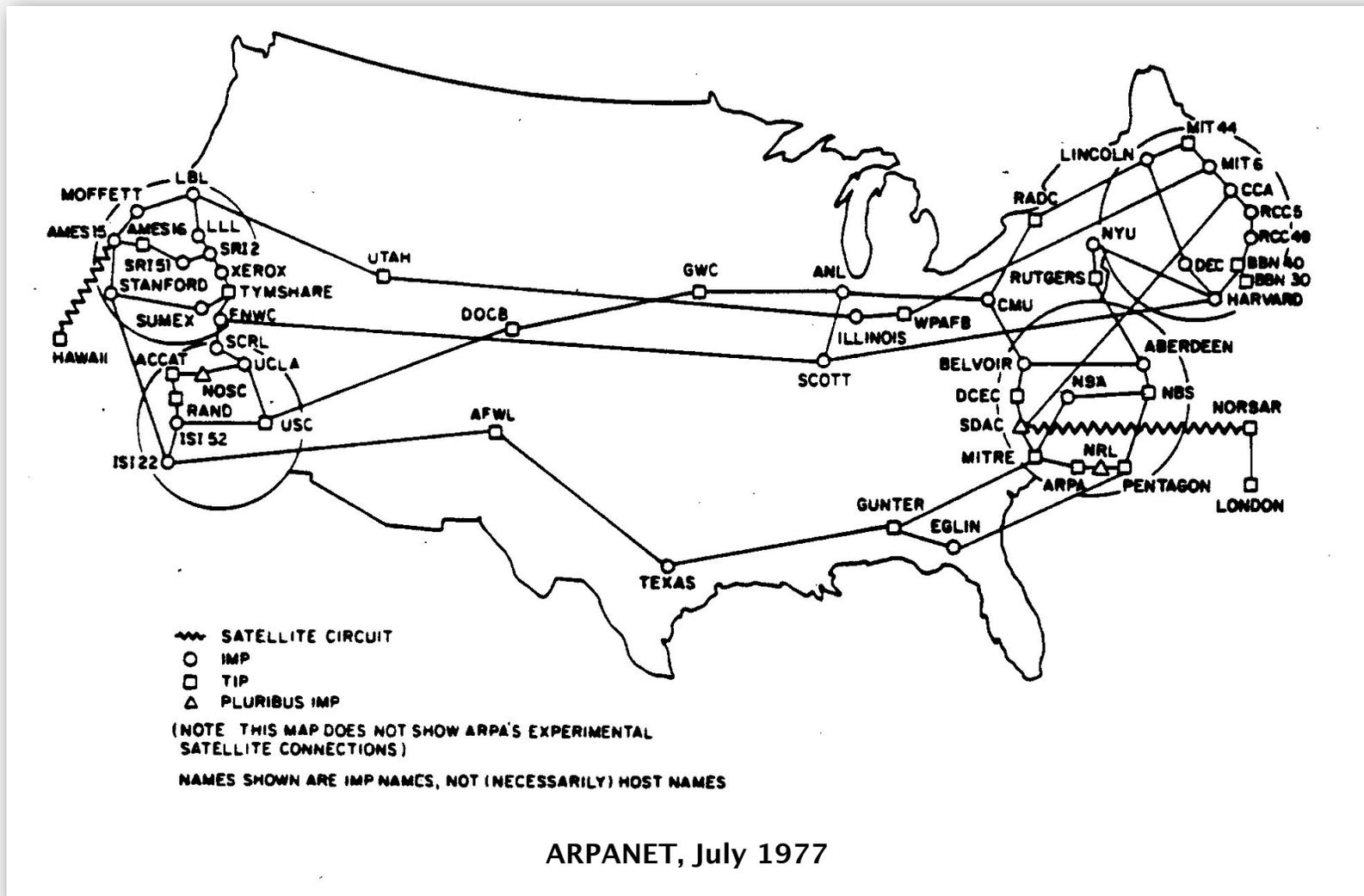
        while (!q.isEmpty()) {
            int v = q.dequeue();
            for (int w : G.adj(v)) {
                if (!marked[w]) {
                    q.enqueue(w);
                    marked[w] = true;
                    edgeTo[w] = v;
                    distTo[w] = distTo[v] + 1;
                }
            }
        }
    }
}
```

← initialize FIFO queue of  
vertices to explore

← found new vertex w  
via edge v-w

# Breadth-first search application: routing

Fewest number of hops in a communication network.



# Breadth-first search application: Kevin Bacon numbers

Kevin Bacon numbers.

The Oracle of Bacon

http://www.oracleofbacon.org/cgi-bin/movieinks?game=0&firstname=Kevin+Baco

THE ORACLE OF BACON

Help  
Credits  
How it Works  
Contact Us  
Other games >

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Buzz Mauro  
↓ with  
Sweet Dreams (2005)  
↓ with  
Tatiana Ramirez  
↓ with  
Interior de un silencio, El (2005)  
↓ with  
Andres Suarez  
↓ with  
Carlita's Secret (2004)  
↓ with  
Paula Lemes (I)  
↓ with  
Frost/Nixon (2008)  
↓ with  
Kevin Bacon

Kevin Bacon to Buzz Mauro Find link More options >>

<http://oracleofbacon.org>



Endless Games board game

New 2 Degrees

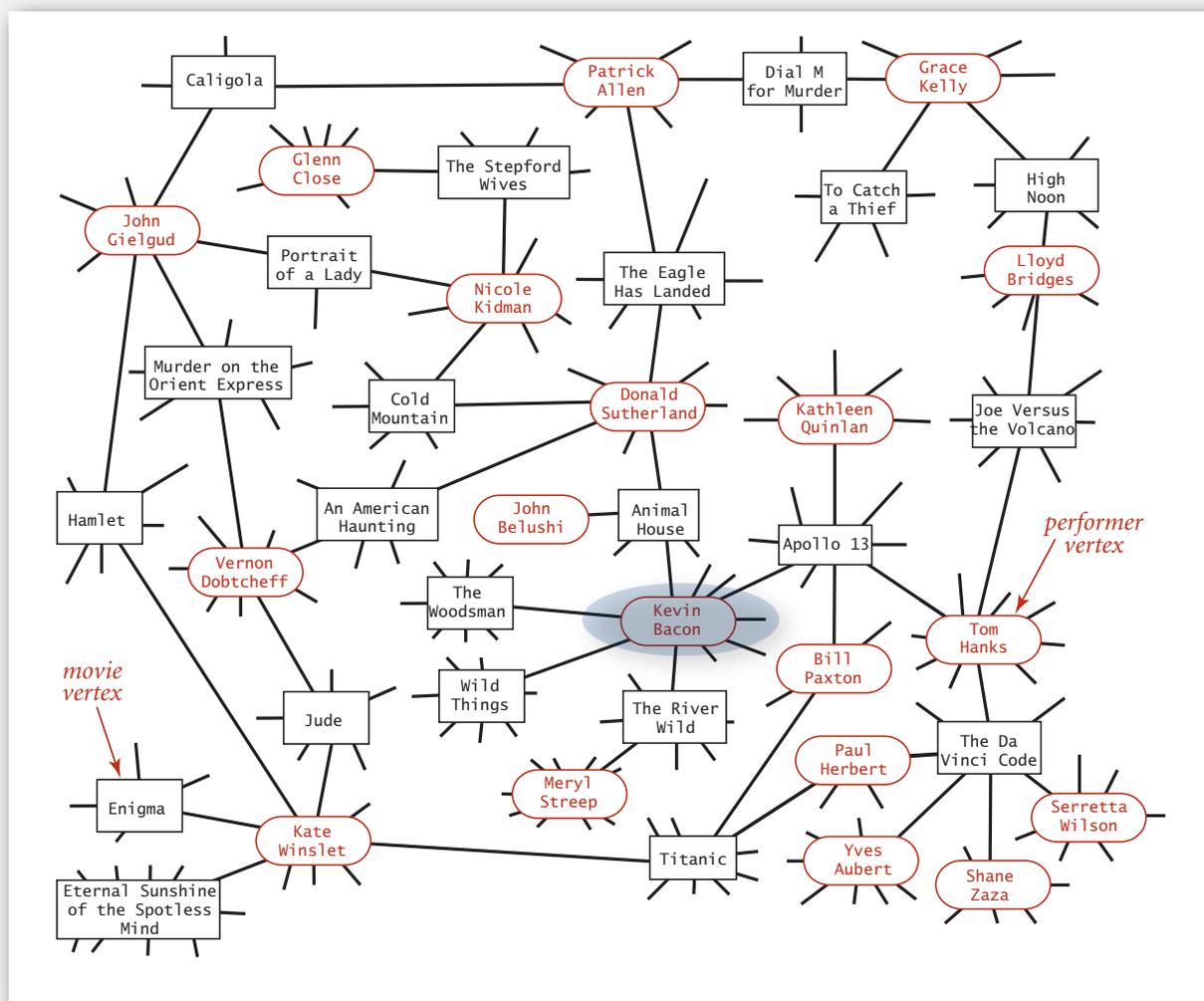
Uma Thurman  
acted in  
Be Cool (2005) 1°  
with  
Scott Adsit  
who acted in  
The Informant! (2009) 2°  
with  
Matt Damon

Lookup Trivia # Guess Degrees Scoreboard

SixDegrees iPhone App

# Kevin Bacon graph

- Include one vertex for each performer **and** one for each movie.
- Connect a movie to all performers that appear in that movie.
- Compute shortest path from  $s = \text{Kevin Bacon}$ .







<http://algs4.cs.princeton.edu>

## 4.1 UNDIRECTED GRAPHS

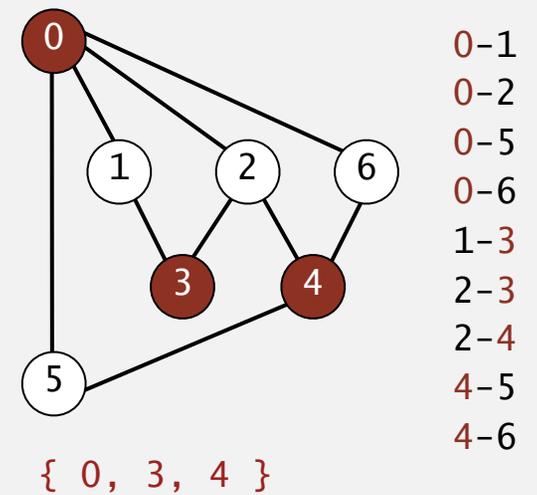
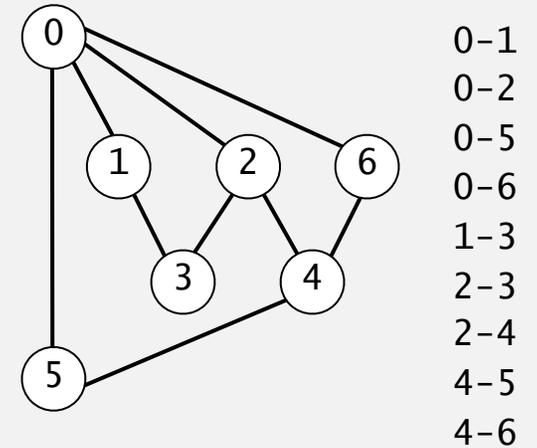
---

- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ ***challenges***

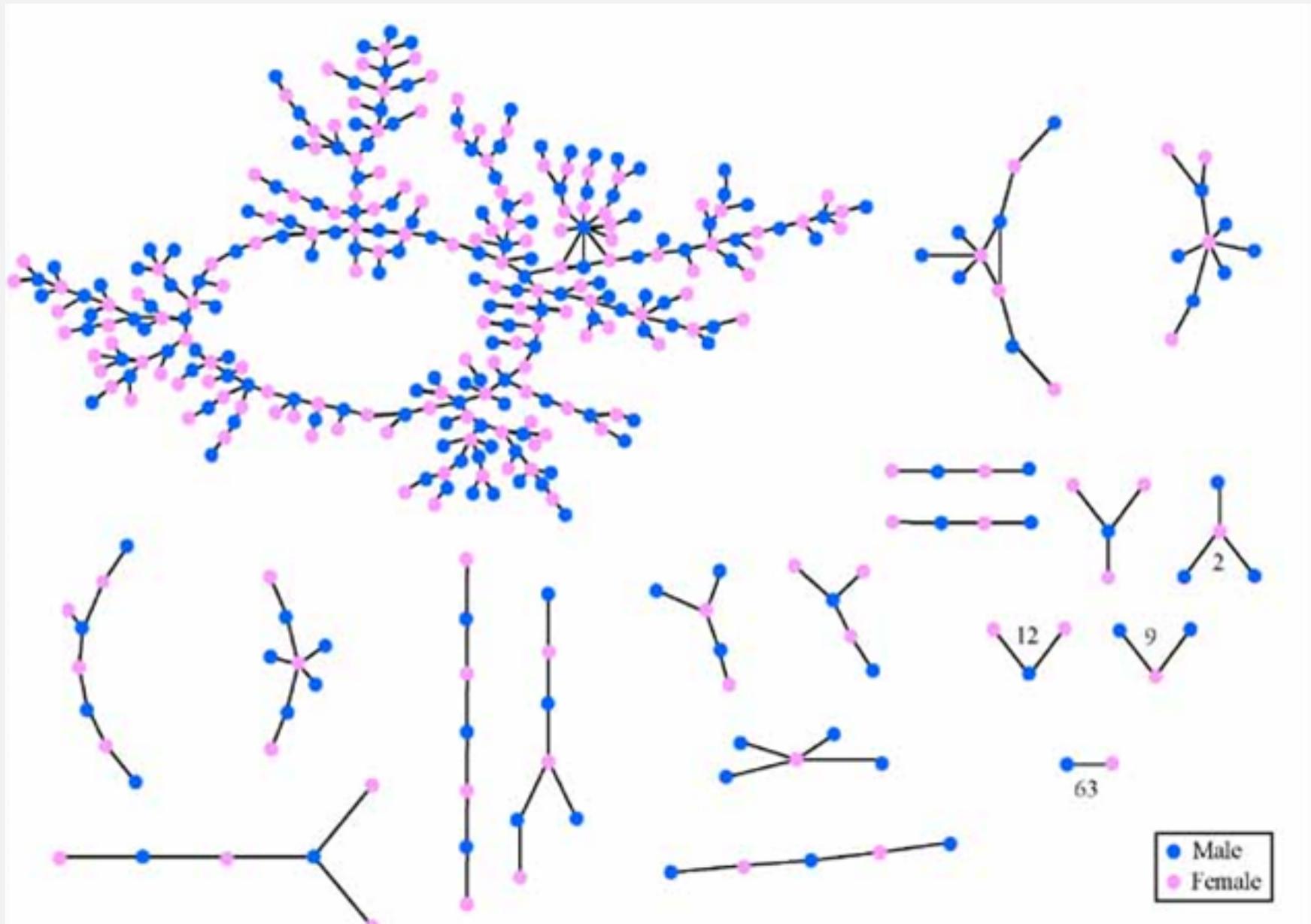
# Graph-processing challenge 1

---

**Problem.** Is a graph bipartite?



# Intuition: think about it as a dating graph



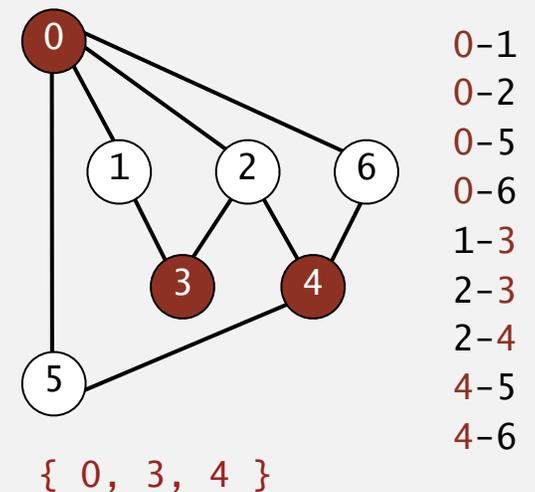
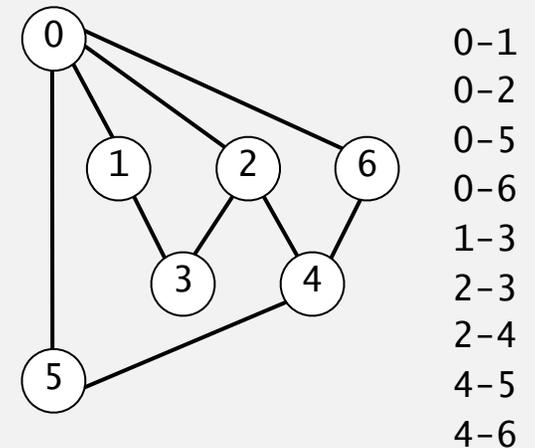
# Graph-processing challenge 1

**Problem.** Is a graph bipartite?

**How difficult?**

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

simple DFS-based solution  
(see textbook)



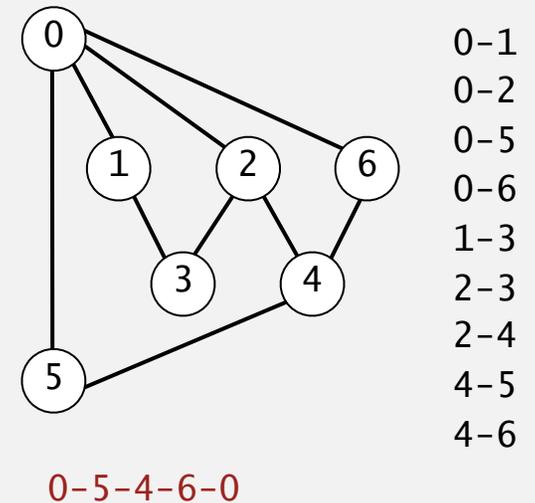
# Graph-processing challenge 2

**Problem.** Find a cycle.

**How difficult?**

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

simple DFS-based solution  
(see textbook)

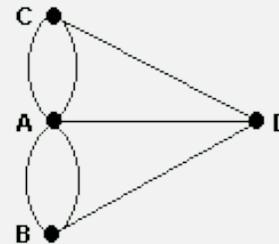
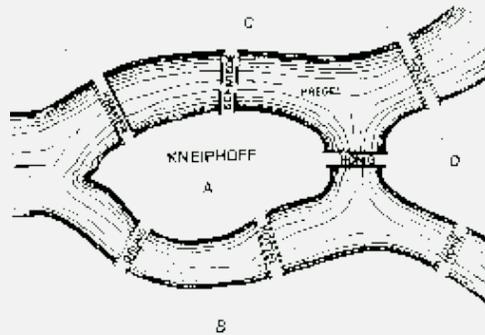


# Bridges of Königsberg

---

## The Seven Bridges of Königsberg. [Leonhard Euler 1736]

*“ ... in Königsberg in Prussia, there is an island A, called the Kneiphof; the river which surrounds it is divided into two branches ... and these branches are crossed by seven bridges. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he could cross each bridge once and only once. ”*



**Euler tour.** Is there a (general) cycle that uses each edge exactly once?

**Answer.** Yes iff connected and all vertices have **even** degree.

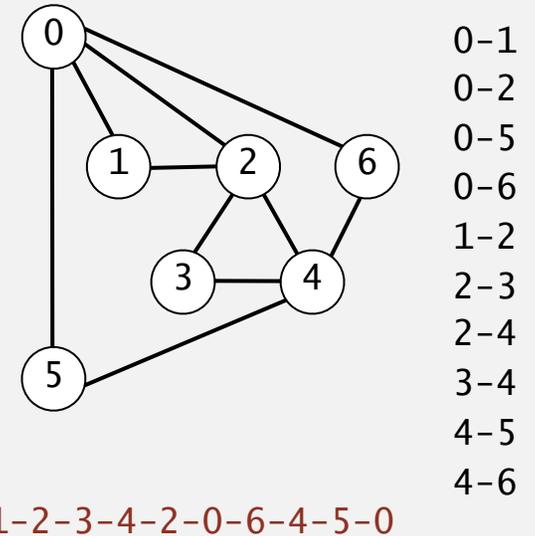
# Graph-processing challenge 3

**Problem.** Find a (general) cycle that uses every edge exactly once.

## How difficult?

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

Eulerian tour  
(classic graph-processing problem)

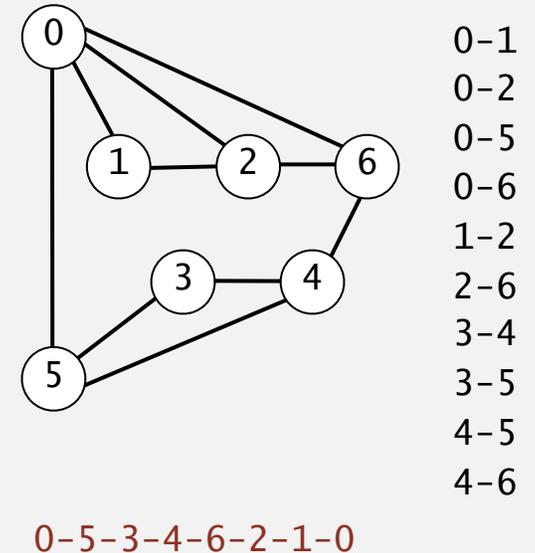


# Graph-processing challenge 4

**Problem.** Find a cycle that visits every vertex exactly once.

How difficult?

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- Hire an expert.
- ✓ • Intractable. ← Hamiltonian cycle  
(classical NP-complete problem)
- No one knows.
- Impossible.



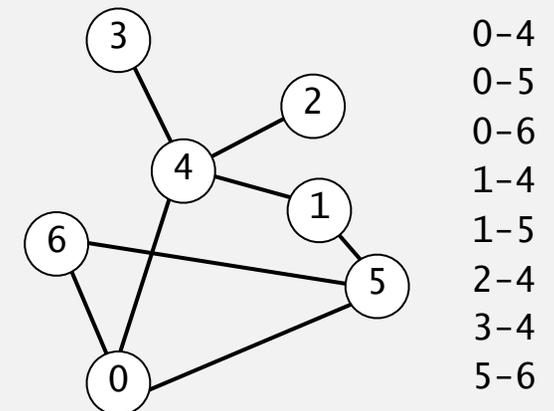
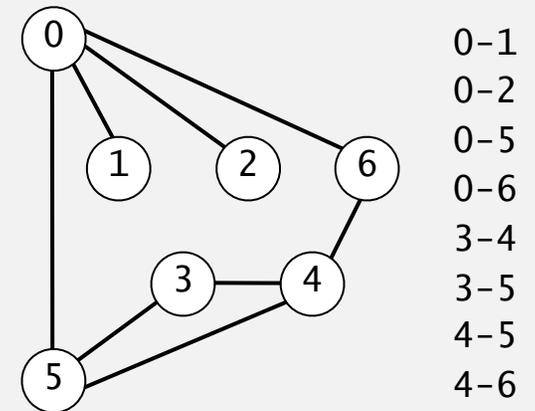
# Graph-processing challenge 5

**Problem.** Are two graphs identical except for vertex names?

How difficult?

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- ✓ • No one knows.
- Impossible.

graph isomorphism is  
longstanding open problem



$0 \leftrightarrow 4, 1 \leftrightarrow 3, 2 \leftrightarrow 2, 3 \leftrightarrow 6, 4 \leftrightarrow 5, 5 \leftrightarrow 0, 6 \leftrightarrow 1$

# Graph-processing challenge 6

**Problem.** Lay out a graph in the plane without crossing edges?

## How difficult?

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- ✓ • Hire an expert.
- Intractable.
- No one knows.
- Impossible.

linear-time DFS-based planarity algorithm  
discovered by Tarjan in 1970s  
(too complicated for most practitioners)

