9. Scientific Computing

Applications of Scientific Computing

Science and engineering challenges.

- Fluid dynamics.
- Seismic surveys.
- Plasma dynamics.
- Ocean circulation.
- Electronics design.
- Pharmaceutical design.
- Human genome project.
- Vehicle crash simulation.
- Global climate simulation.
- Nuclear weapons simulation.
- Molecular dynamics simulation.

Commercial applications.

- Web search.
- Financial modeling.
- Computer graphics.
- Digital audio and video.
- Natural language processing.
- Architecture walk-throughs.
- Medical diagnostics (MRI, CAT).

Common features.

- Problems tend to be continuous instead of discrete.
- Algorithms often need to scale to handle huge problems.

Representing Real Numbers

Challenge: use fixed size words to represent, e.g.,

- `2.1`
- `0.00000000000000000000000000000000345878778`
- `-1020455.000322`
- `365090807000000000000000000000000.0`

We appear to need:

- A sign bit
- An exponent, which might need to be negative
- A "significand" or "mantissa"
- AND a way to cram all this into 32 or 64 bits.

Floating Point

IEEE 754 representation.

- Used by all modern computers.
- Scientific notation, but in binary.
- Single precision: `float` = 32 bits.
- Double precision: `double` = 64 bits.

Ex. Single precision representation of `-0.453125`.

sign bit | exponent | significand

`1` | `0 1 1 1 1 1 0 1` | `1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0`

`-1` | `125` | `1/2 + 1/4 + 1/16 = 0.8125`

(bias, hidden bit)

$$-1 \times 2^{125-127} \times 1.8125 = -0.453125$$

Floating Point

Remark. Most real numbers are not representable, including π and 1/10.

Roundoff error. When result of calculation is not representable.
Consequence. Non-intuitive behavior for uninitiated.

```
if (0.1 + 0.2 == 0.3) { // NO  }
if (0.1 + 0.3 == 0.4) { // YES }
```

Financial computing. Calculate 9% sales tax on a 50¢ phone call.
Banker's rounding. Round to nearest integer, to even integer if tie.

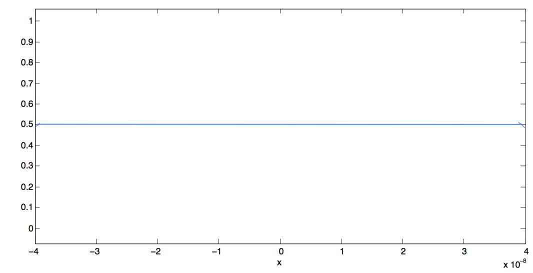
```
double a1 = 1.14 * 75;       // 85.49999999999999
double a2 = Math.round(a1);  // 85    ← you lost 1¢

double b1 = 1.09 * 50;       // 54.50000000000001
double b2 = Math.round(b1);  // 55    ← SEC violation (!)
```


Catastrophic Cancellation

A simple function. $f(x) = \frac{1 - \cos x}{x^2}$

Goal. Plot f(x) for $-4 \cdot 10^{-8} \le x \le 4 \cdot 10^{-8}$.

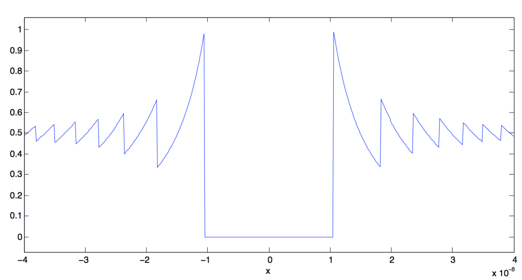


Exact answer

Catastrophic Cancellation

A simple function. $f(x) = \frac{1 - \cos x}{x^2}$

Goal. Plot f(x) for $-4 \cdot 10^{-8} \le x \le 4 \cdot 10^{-8}$.



IEEE 754 double precision answer

Catastrophic Cancellation

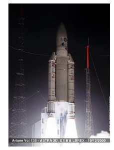
```
public static double fl(double x) {
   return (1.0 - Math.cos(x)) / (x * x);
}
```

Ex. Evaluate `fl(x)` for `x = 1.1e-8`.

- `Math.cos(x) = 0.99999999999999988897769753748434595763683319091796875.`
  - nearest floating point value agrees with exact answer to 16 decimal places.
- `(1.0 - Math.cos(x)) = 1.1102e-16`
  - inaccurate estimate of exact answer ($6.05 \cdot 10^{-17}$)
- `(1.0 - Math.cos(x)) / (x * x) = 0.9175`
  - 80% larger than exact answer (about 0.5)

Catastrophic cancellation. Devastating loss of precision when small numbers are computed from large numbers, which themselves are subject to roundoff error.

Numerical Catastrophes



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Ariane 5 rocket. [June 4, 1996]

- 10 year, $7 billion ESA project exploded after launch.
- 64-bit float converted to 16 bit signed int.
- Unanticipated overflow.

Vancouver stock exchange. [November, 1983]

- Index undervalued by 44%.
- Recalculated index after each trade by adding change in price.
- 22 months of accumulated truncation error.

Patriot missile accident. [February 25, 1991]

- Failed to track scud; hit Army barracks, killed 28.
- Inaccuracy in measuring time in 1/20 of a second since using 24 bit binary floating point.

Gaussian Elimination

Linear System of Equations

Linear system of equations. N linear equations in N unknowns.

$$\begin{aligned} 0\,x_0 + 1\,x_1 + 1\,x_2 &= 4 \\ 2\,x_0 + 4\,x_1 - 2\,x_2 &= 2 \\ 0\,x_0 + 3\,x_1 + 15\,x_2 &= 36 \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$$

matrix notation: find x such that Ax = b

Fundamental problems in science and engineering.

- Chemical equilibrium.
- Linear and nonlinear optimization.
- Kirchoff's current and voltage laws.
- Hooke's law for finite element methods.
- Leontief's model of economic equilibrium.
- Numerical solutions to differential equations.
- …

Chemical Equilibrium

Ex. Combustion of propane.

$$x_0 C_3H_8 + x_1 O_2 \Rightarrow x_2 CO_2 + x_3 H_2O$$

Stoichiometric constraints.

- Carbon: $3x_0 = x_2$.
- Hydrogen: $8x_0 = 2x_3$.
- Oxygen: $2x_1 = 2x_2 + x_3$.
- Normalize: $x_0 = 1$.

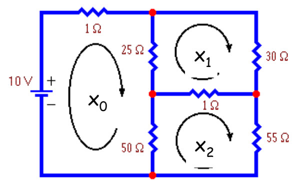
(Carbon, Hydrogen, Oxygen: conservation of mass)

$$C_3H_8 + 5O_2 \Rightarrow 3CO_2 + 4H_2O$$

Remark. Stoichiometric coefficients tend to be small integers; among first hints suggesting the atomic nature of matter.

Kirchoff's Current Law

Ex. Find current flowing in each branch of a circuit.



Kirchoff's current law.

- $10 = 1x_0 + 25(x_0 - x_1) + 50\,(x_0 - x_2)$.
- $0 = 25(x_1 - x_0) + 30x_1 + 1(x_1 - x_2)$.
- $0 = 50(x_2 - x_0) + 1(x_2 - x_1) + 55x_2$.

} conservation of electrical charge

Solution. $x_0 = 0.2449$, $x_1 = 0.1114$, $x_2 = 0.1166$.

Upper Triangular System of Equations

Upper triangular system. $a_{ij} = 0$ for $i > j$.

$$
\begin{array}{rcrcrcr}
2\,x_0 & + & 4\,x_1 & - & 2\,x_2 & = & 2 \\
0\,x_0 & + & 1\,x_1 & + & 1\,x_2 & = & 4 \\
0\,x_0 & + & 0\,x_1 & + & 12\,x_2 & = & 24
\end{array}
$$

Back substitution. Solve by examining equations in reverse order.

- Equation 2: $x_2 = 24/12 = 2$.
- Equation 1: $x_1 = 4 - x_2 = 2$.
- Equation 0: $x_0 = (2 - 4x_1 + 2x_2) / 2 = -1$.

```
for (int i = N-1; i >= 0; i--) {
   double sum = 0.0;
   for (int j = i+1; j < N; j++)
      sum += A[i][j] * x[j];
   x[i] = (b[i] - sum) / A[i][i];
}
```

$$x_i = \frac{1}{a_{ii}}\left[b_i - \sum_{j=i+1}^{N-1} a_{ij}\, x_j \right]$$

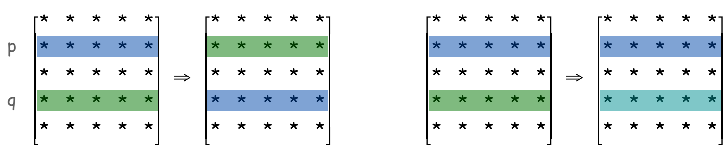
Gaussian Elimination

Gaussian elimination.

- Among oldest and most widely used solutions.
- Repeatedly apply row operations to make system upper triangular.
- Solve upper triangular system by back substitution.

Elementary row operations.

- Exchange row p and row q.
- Add a multiple α of row p to row q.



Key invariant. Row operations preserve solutions.

Gaussian Elimination: Row Operations

Elementary row operations.

$$
\begin{array}{rcrcrcr}
0\,x_0 & + & 1\,x_1 & + & 1\,x_2 & = & 4 \\
2\,x_0 & + & 4\,x_1 & - & 2\,x_2 & = & 2 \\
0\,x_0 & + & 3\,x_1 & + & 15\,x_2 & = & 36
\end{array}
$$

⬇ (interchange row 0 and 1)

$$
\begin{array}{rcrcrcr}
2\,x_0 & + & 4\,x_1 & - & 2\,x_2 & = & 2 \\
0\,x_0 & + & 1\,x_1 & + & 1\,x_2 & = & 4 \\
0\,x_0 & + & 3\,x_1 & + & 15\,x_2 & = & 36
\end{array}
$$

⬇ (subtract 3x row 1 from row 2)

$$
\begin{array}{rcrcrcr}
2\,x_0 & + & 4\,x_1 & - & 2\,x_2 & = & 2 \\
0\,x_0 & + & 1\,x_1 & + & 1\,x_2 & = & 4 \\
0\,x_0 & + & 0\,x_1 & + & 12\,x_2 & = & 24
\end{array}
$$

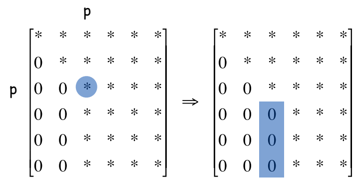
Gaussian Elimination: Forward Elimination

Forward elimination. Apply row operations to make upper triangular.

Pivot. Zero out entries below pivot a_{pp}.

$$a_{ij} = a_{ij} - \frac{a_{ip}}{a_{pp}} a_{pj}$$

$$b_i = b_i - \frac{a_{ip}}{a_{pp}} b_p$$

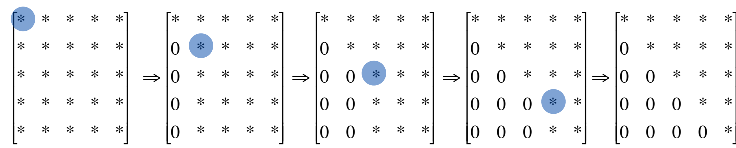



```
for (int i = p + 1; i < N; i++) {
   double alpha = A[i][p] / A[p][p];
   b[i] -= alpha * b[p];
   for (int j = p; j < N; j++)
      A[i][j] -= alpha * A[p][j];
}
```


Gaussian Elimination: Forward Elimination

Forward elimination. Apply row operations to make upper triangular.

Pivot. Zero out entries below pivot a_{pp}.



```
for (int p = 0; p < N; p++) {
   for (int i = p + 1; i < N; i++) {
      double alpha = A[i][p] / A[p][p];
      b[i] -= alpha * b[p];
      for (int j = p; j < N; j++)
         A[i][j] -= alpha * A[p][j];
   }
}
```


Gaussian Elimination Example

$1\,x_0$	+	$0\,x_1$	+	$1\,x_2$	+	$4\,x_3$	=	1
$2\,x_0$	+	$-1\,x_1$	+	$1\,x_2$	+	$7\,x_3$	=	2
$-2\,x_0$	+	$1\,x_1$	+	$0\,x_2$	+	$-6\,x_3$	=	3
$1\,x_0$	+	$1\,x_1$	+	$1\,x_2$	+	$9\,x_3$	=	4

Gaussian Elimination Example

$1\,x_0$	+	$0\,x_1$	+	$1\,x_2$	+	$4\,x_3$	=	1
$0\,x_0$	+	$-1\,x_1$	+	$-1\,x_2$	+	$-1\,x_3$	=	0
$0\,x_0$	+	$1\,x_1$	+	$2\,x_2$	+	$2\,x_3$	=	5
$0\,x_0$	+	$1\,x_1$	+	$0\,x_2$	+	$5\,x_3$	=	3

Gaussian Elimination Example

$1\ x_0$	+	$0\ x_1$	+	$1\ x_2$	+	$4\ x_3$	=	1
$0\ x_0$	+	$-1\ x_1$	+	$-1\ x_2$	+	$-1\ x_3$	=	0
$0\ x_0$	+	$0\ x_1$	+	$1\ x_2$	+	$1\ x_3$	=	5
$0\ x_0$	+	$0\ x_1$	+	$-1\ x_2$	+	$4\ x_3$	=	3

Gaussian Elimination Example

$1\ x_0$	+	$0\ x_1$	+	$1\ x_2$	+	$4\ x_3$	=	1
$0\ x_0$	+	$-1\ x_1$	+	$-1\ x_2$	+	$-1\ x_3$	=	0
$0\ x_0$	+	$0\ x_1$	+	$1\ x_2$	+	$1\ x_3$	=	5
$0\ x_0$	+	$0\ x_1$	+	$0\ x_2$	+	$5\ x_3$	=	8

Gaussian Elimination Example

$1\ x_0$	+	$0\ x_1$	+	$1\ x_2$	+	$4\ x_3$	=	1
$0\ x_0$	+	$-1\ x_1$	+	$-1\ x_2$	+	$-1\ x_3$	=	0
$0\ x_0$	+	$0\ x_1$	+	$1\ x_2$	+	$1\ x_3$	=	5
$0\ x_0$	+	$0\ x_1$	+	$0\ x_2$	+	$5\ x_3$	=	8

$$x_3 = 8/5$$
$$x_2 = 5 - x_3 = 17/5$$
$$x_1 = 0 - x_2 - x_3 = -25/5$$
$$x_0 = 1 - x_2 - 4x_3 = -44/5$$

Gaussian Elimination: Partial Pivoting

Remark. Previous code fails spectacularly if pivot $a_{pp} = 0$.

$1\ x_0$	+	$1\ x_1$	+	$0\ x_3$	=	1
$2\ x_0$	+	$2\ x_1$	+	$-2\ x_3$	=	-2
$0\ x_0$	+	$3\ x_1$	+	$15\ x_3$	=	33

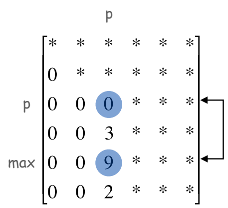
$1\ x_0$	+	$1\ x_1$	+	$0\ x_3$	=	1
$0\ x_0$	+	$0\ x_1$	+	$-2\ x_3$	=	-4
$0\ x_0$	+	$3\ x_1$	+	$15\ x_3$	=	33

$1\ x_0$	+	$1\ x_1$	+	$0\ x_3$	=	1
$0\ x_0$	+	$0\ x_1$	+	$-2\ x_3$	=	-4
$0\ x_0$	+	Nan x_1	+	Inf x_3	=	Inf

Gaussian Elimination: Partial Pivoting

Partial pivoting. Swap row p with the row that has largest entry in column p among rows i below the diagonal.

```
// find pivot row
int max = p;
for (int i = p + 1; i < N; i++)
   if (Math.abs(A[i][p]) > Math.abs(A[max][p])) max = i;

// swap rows p and max
double[] T = A[p]; A[p] = A[max]; A[max] = T;
double   t = b[p]; b[p] = b[max]; b[max] = t;
```



Q. What if pivot $a_{pp} = 0$ while partial pivoting?
A. System has no solutions or infinitely many solutions.

Gaussian Elimination with Partial Pivoting

```
public static double[] lsolve(double[][] A, double[] b) {
   int N = b.length;
   // Gaussian elimination
   for (int p = 0; p < N; p++) {
      // partial pivot
      int max = p;
      for (int i = p+1; i < N; i++)
         if (Math.abs(A[i][p]) > Math.abs(A[max][p]))
            max = i;
      double[] T = A[p]; A[p] = A[max]; A[max] = T;
      double   t = b[p]; b[p] = b[max]; b[max] = t;

      // zero out entries of A and b using pivot A[p][p]
      for (int i = p+1; i < N; i++) {
         double alpha = A[i][p] / A[p][p];
         b[i] -= alpha * b[p];
         for (int j = p; j < N; j++)
            A[i][j] -= alpha * A[p][j];
      }
   }
   // back substitution
   double[] x = new double[N];
   for (int i = N-1; i >= 0; i--) {
      double sum = 0.0;
      for (int j = i+1; j < N; j++)
         sum += A[i][j] * x[j];
      x[i] = (b[i] - sum) / A[i][i];
   }
   return x;
}
```

~ $N^3/3$ additions, ~ $N^3/3$ multiplications

~ $N^2/2$ additions, ~ $N^2/2$ multiplications

Stability and Conditioning

Numerically Unstable Algorithms

Stability. Algorithm `fl(x)` for computing f(x) is numerically stable if `fl(x)` ≈ f(x+ε) for some small perturbation ε.

Nearly the right answer to nearly the right problem.

Ex 1. Numerically unstable way to compute $f(x) = \frac{1 - \cos x}{x^2}$

```
public static double fl(double x) {
   return (1.0 - Math.cos(x)) / (x * x);
}
```

- `fl(1.1e-8) = 0.9175.`

true answer ≈ 1/2.

$$f(x) = \frac{2\sin^2(x/2)}{x^2}$$

a numerically stable formula

Numerically Unstable Algorithms

Stability. Algorithm `fl(x)` for computing f(x) is numerically stable if `fl(x)` ≈ f(x+ε) for some small perturbation ε.

Nearly the right answer to nearly the right problem.

Ex 2. Gaussian elimination (w/o partial pivoting) can fail spectacularly.

$a = 10^{-17}$

$$\begin{aligned} a\,x_0 + 1\,x_1 &= 1 \\ 1\,x_0 + 2\,x_1 &= 3 \end{aligned}$$

Algorithm	x_0	x_1
no pivoting	0.0	1.0
partial pivoting	1.0	1.0
exact	$\frac{1}{1-2a} \approx 1$	$\frac{1-3a}{1-2a} \approx 1$

Theorem. Partial pivoting improves numerical stability.

Ill-Conditioned Problems

Conditioning. Problem is well-conditioned if f(x) ≈ f(x+ε) for all small perturbation ε.

Solution varies gradually as problem varies.

Ex. Hilbert matrix.
- Tiny perturbation to H_n makes it singular.
- Cannot solve $H_{12}\, x = b$ using floating point.

$$H_4 = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

Hilbert matrix

Matrix condition number. [Turing, 1948] Widely-used concept for detecting ill-conditioned linear systems.

Stability and Conditioning

Accuracy depends on both stability and conditioning.
- Danger: apply unstable algorithm to well-conditioned problem.
- Danger: apply stable algorithm to ill-conditioned problem.
- Safe: apply stable algorithm to well-conditioned problem.

Numerical analysis. Art and science of designing numerically stable algorithms for well-conditioned problems.

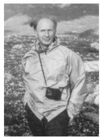
Lesson 1. Some algorithms are unsuitable for floating point solutions.
Lesson 2. Some problems are unsuitable to floating point solutions.

Numerically Solving an Initial Value ODE

Lorenz attractor.
- Idealized atmospheric model to describe turbulent flow.
- Convective rolls: warm fluid at bottom, rises to top, cools off, and falls down.

$$\begin{aligned} \frac{dx}{dt} &= -10(x-y) \\ \frac{dy}{dt} &= -xz + 28x - y \\ \frac{dz}{dt} &= xy - \frac{8}{3} z \end{aligned}$$

x = fluid flow velocity
y = ∇ temperature between ascending and descending currents
z = distortion of vertical temperature profile from linearity



Edward Lorenz

Solution. No closed form solution for x(t), y(t), z(t).
Approach. Numerically solve ODE.

Euler's Method

Euler's method. [to numerically solve initial value ODE]

- Choose Δt sufficiently small.
- Approximate function at time t by tangent line at t.
- Estimate value of function at time t + Δt according to tangent line.
- Increment time to t + Δt.
- Repeat.

$$
\begin{aligned}
x_{t+\Delta t} &= x_t + \Delta t \, \tfrac{dx}{dt}(x_t, y_t, z_t) \\
y_{t+\Delta t} &= y_t + \Delta t \, \tfrac{dy}{dt}(x_t, y_t, z_t) \\
z_{t+\Delta t} &= z_t + \Delta t \, \tfrac{dz}{dt}(x_t, y_t, z_t)
\end{aligned}
$$

Advanced methods. Use less computation to achieve desired accuracy.

- 4th order Runge-Kutta: evaluate slope four times per step.
- Variable time step: automatically adjust timescale Δt.
- See COS 323.

Lorenz Attractor: Java Implementation

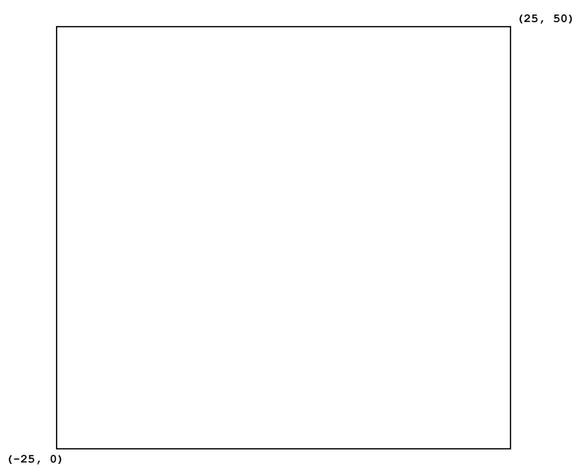
```
public class Lorenz {
   public static double dx(double x, double y, double z)
   { return -10*(x - y);      }

   public static double dy(double x, double y, double z)
   { return -x*z + 28*x - y; }

   public static double dz(double x, double y, double z)
   { return x*y - 8*z/3;      }

   public static void main(String[] args) {
      double x = 0.0, y = 20.0, z = 25.0;
      double dt = 0.001;
      StdDraw.setXscale(-25, 25);
      StdDraw.setYscale(  0, 50);

      while (true) {
         double xnew = x + dt * dx(x, y, z);
         double ynew = y + dt * dy(x, y, z);
         double znew = z + dt * dz(x, y, z);
         x = xnew; y = ynew; z = znew;
         StdDraw.point(x, z);
      }
   }
}
```

Euler's method

plot x vs. z

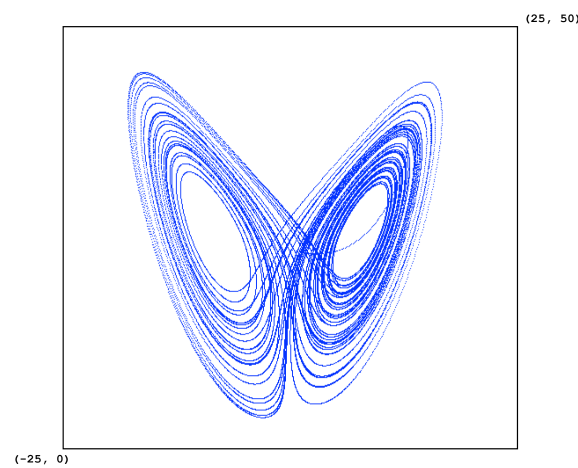
The Lorenz Attractor

```
% java Lorenz
```



The Lorenz Attractor

```
% java Lorenz
```



Butterfly Effect

Experiment.

- Initialize y = 20.01 instead of y = 20.
- Plot original trajectory in blue, perturbed one in magenta.
- What happens?

Ill-conditioning.

- Sensitive dependence on initial conditions.
- Property of system, not of numerical solution approach.

> Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas? - Title of 1972 talk by Edward Lorenz