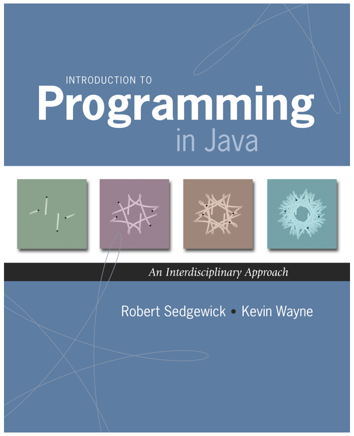
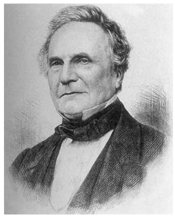
4.1, 4.2 Performance and Sorting

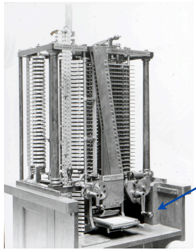


Running Time

> *"As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time?"* – *Charles Babbage*



Charles Babbage (1864)



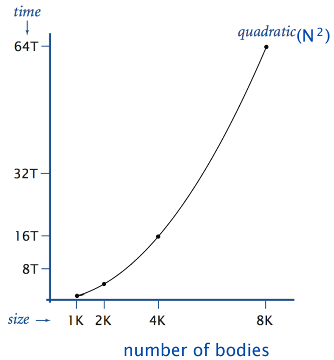
Analytic Engine

how many times do you have to turn the crank?

Algorithmic Successes

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.



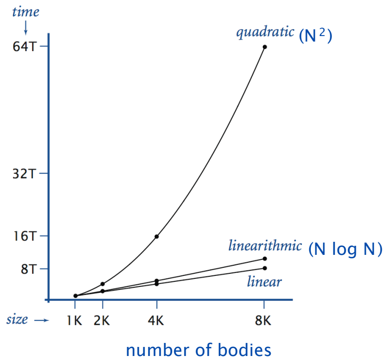
)

Algorithmic Successes

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut: N log N steps, enables new research.

Andrew Appel
PU '81

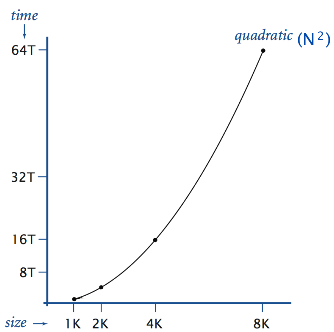


Algorithmic Successes

Freidrich Gauss
1805

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: N^2 steps.

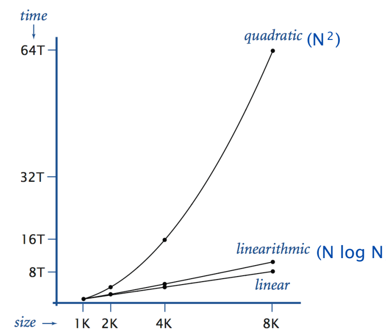


Algorithmic Successes

John Tukey
1965

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: N^2 steps.
- FFT algorithm: N log N steps, enables new technology.



Sorting

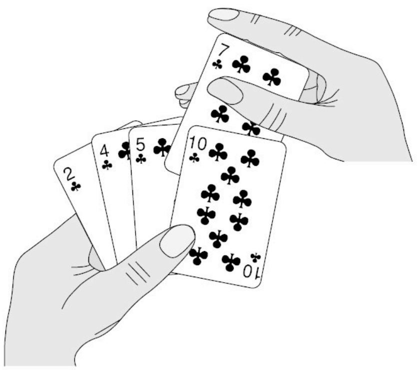
Sorting

Sorting problem. Rearrange N items in ascending order.

Applications. Binary search, statistics, databases, data compression, bioinformatics, computer graphics, scientific computing, (too numerous to list) ...

	→	
Hauser		Hanley
Hong		Haskell
Hsu		Hauser
Hayes		Hayes
Haskell		Hong
Hanley		Hornet
Hornet		Hsu

Insertion Sort



Insertion Sort

Insertion sort.

- Brute-force sorting solution.
- Move left-to-right through array.
- Insert each element into final position by exchanging it with larger elements to its left, one-by-one.

i	j	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
6	6	and	had	him	his	was	you	the	but
6	5	and	had	him	his	was	the	you	but
6	4	and	had	him	his	the	was	you	but
		and	had	him	his	the	was	you	but

Inserting `a[6]` *into position by exchanging with larger entries to its left*

Insertion Sort

Insertion sort.

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

i	j	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
		was	had	him	and	you	his	the	but
1	0	had	was	him	and	you	his	the	but
2	1	had	him	was	and	you	his	the	but
3	0	and	had	him	was	you	his	the	but
4	4	and	had	him	was	you	his	the	but
5	3	and	had	him	his	was	you	the	but
6	4	and	had	him	his	the	was	you	but
7	1	and	but	had	him	his	the	was	you
		and	but	had	him	his	the	was	you

Inserting `a[1]` *through* `a[N-1]` *into position (insertion sort)*

Insertion Sort

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	2.78	7.42	0.56	1.12	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 0: step 0.

Insertion Sort

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	3.14	4.42	6.21	7.42	7.71

Iteration 10: DONE.

Insertion Sort: Java Implementation

```
public class Insertion
{

   public static void sort(double[] a)
   {
      int N = a.length;
      for (int i = 1; i < N; i++)
         for (int j = i; j > 0; j--)
            if (a[j-1] > a[j])
               exch(a, j-1, j);
            else break;            // see text p. 70
   }


   private static void exch(double[] a, int i, int j)
   {
      double swap = a[i];
      a[i] = a[j];
      a[j] = swap;
   }
}
```


Insertion Sort: Observation

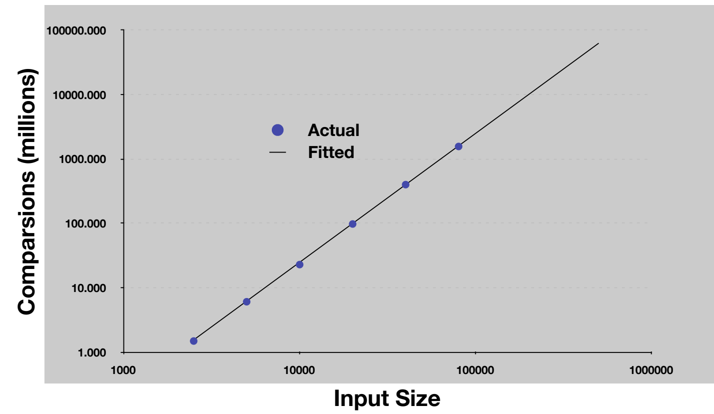
Observe and tabulate running time for various values of N.

- Data source: N random numbers between 0 and 1.
- Machine: Apple G5 1.8GHz with 1.5GB memory running OS X.
- Timing: Skagen wristwatch.

N	Comparisons	Time
5,000	6.2 million	0.13 seconds
10,000	25 million	0.43 seconds
20,000	99 million	1.5 seconds
40,000	400 million	5.6 seconds
80,000	1600 million	23 seconds

Insertion Sort: Empirical Analysis

Data analysis. Plot # comparisons vs. input size on log-log scale.



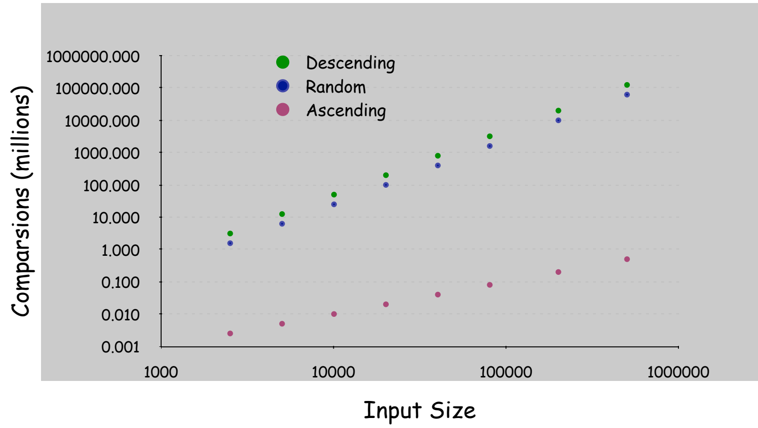
Hypothesis. # comparisons grows quadratically with input size $\sim N^2 / 4$.

slide

Insertion Sort: Empirical Analysis

Observation. Number of compares depends on input family.

- Descending: $\sim N^2 / 2$.
- Random: $\sim N^2 / 4$.
- Ascending: $\sim N$.



Analysis: Empirical vs. Mathematical

Empirical analysis.

- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

Mathematical analysis.

- Analyze algorithm to estimate # ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

Critical difference. Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.

Insertion Sort: Mathematical Analysis

Worst case. [descending]

- Iteration i requires i comparisons.
- Total = $(0 + 1 + 2 + ... + N-1) \sim N^2 / 2$ compares.

E	F	G	H	I	J	D	C	B	A
						i			

Average case. [random]

- Iteration i requires $i / 2$ comparisons on average.
- Total = $(0 + 1 + 2 + ... + N-1) / 2 \sim N^2 / 4$ compares

A	C	D	F	H	J	E	B	I	G
						i			

Insertion Sort: Lesson

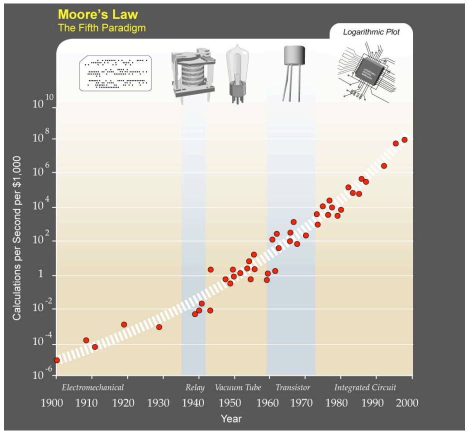
Lesson. Supercomputer can't rescue a bad algorithm.

Computer	Comparisons Per Second	Thousand	Million	Billion
laptop	10^7	instant	1 day	3 centuries
super	10^{12}	instant	1 second	2 weeks

Moore's Law

Moore's law. Transistor density on a chip doubles every 2 years.

Variants. Memory, disk space, bandwidth, computing power per $.



http://en.wikipedia.org/wiki/Moore's_law

Moore's Law and Algorithms

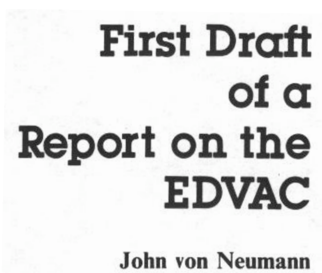
Quadratic algorithms do not scale with technology.

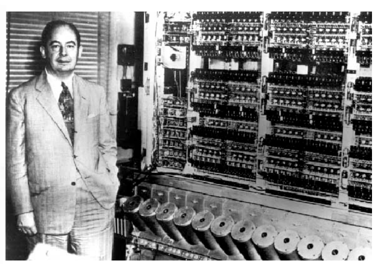
- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

"Software inefficiency can always outpace Moore's Law. Moore's Law isn't a match for our bad coding." – Jaron Lanier

Lesson. Need linear (or linearithmic) algorithm to keep pace with Moore's law.

Mergesort





Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

input
```
was had him and you his the but
```
sort left
```
and had him was you his the but
```
sort right
```
and had him was but his the you
```
merge
```
and but had him his the was you
```

Mergesort: Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Top-down mergesort

Merging

Merging. Combine two pre-sorted lists into a sorted whole.

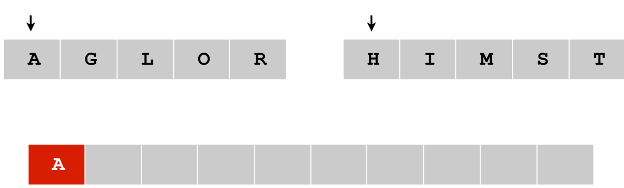
How to merge efficiently? Use an auxiliary array.

i	j	k	aux[k]	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
				and	had	him	was	but	his	the	you
0	4	0	and	and	had	him	was	but	his	the	you
1	4	1	but	and	had	him	was	but	his	the	you
1	5	2	had	and	had	him	was	but	his	the	you
2	5	3	him	and	had	him	was	but	his	the	you
3	5	4	his	and	had	him	was	but	his	the	you
3	6	5	the	and	had	him	was	but	his	the	you
3	6	6	was	and	had	him	was	but	his	the	you
4	7	7	you	and	had	him	was	but	his	the	you

Trace of the merge of the sorted left half with the sorted right half

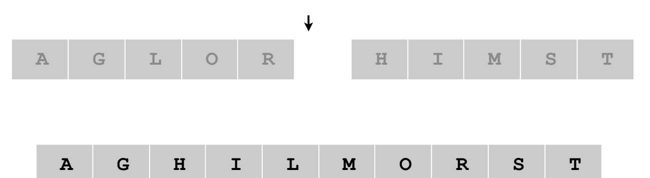
Merging

Merge.
- Keep track of smallest element in each sorted half.
- Choose smaller of two elements.
- Repeat until done.



Merging

Merge.
- Keep track of smallest element in each sorted half.
- Choose smaller of two elements.
- Repeat until done.



Merging

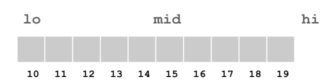
Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

```
String[] aux = new String[N];
// Merge into auxiliary array.
int i = lo, j = mid;
for (int k = 0; k < N; k++)
{
   if       (i == mid) aux[k] = a[j++];
   else if (j == hi)  aux[k] = a[i++];
   else if (a[j].compareTo(a[i]) < 0) aux[k] = a[j++];
   else                               aux[k] = a[i++];
}

// Copy back.
for (int k = 0; k < N; k++)
   a[lo + k] = aux[k];
```

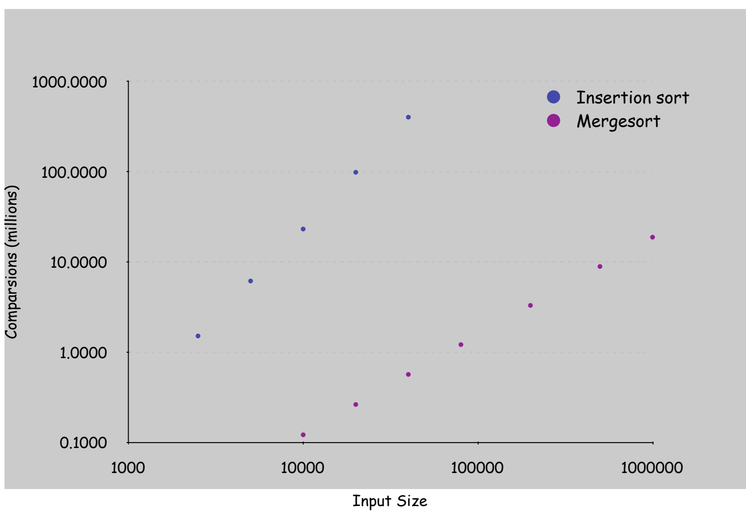

Mergesort: Java Implementation

```
public class Merge
{
   public static void sort(String[] a)
   {  sort(a, 0, a.length);  }

   // Sort a[lo, hi).
   public static void sort(String[] a, int lo, int hi)
   {
      int N = hi - lo;
      if (N <= 1) return;

      // Recursively sort left and right halves.
      int mid = lo + N/2;
      sort(a, lo, mid);
      sort(a, mid, hi);

      // Merge sorted halves (see previous slide).
   }

}
```



Mergesort: Empirical Analysis

Experimental hypothesis. Number of comparisons ≈ 20N.



Mergesort: Prediction and Verification

Experimental hypothesis. Number of comparisons ≈ 20N.

Prediction. 80 million comparisons for N = 4 million.

Observations.

N	Comparisons	Time
4 million	82.7 million	3.13 sec
4 million	82.7 million	3.25 sec
4 million	82.7 million	3.22 sec

Agrees.

Prediction. 400 million comparisons for N = 20 million.

Observations.

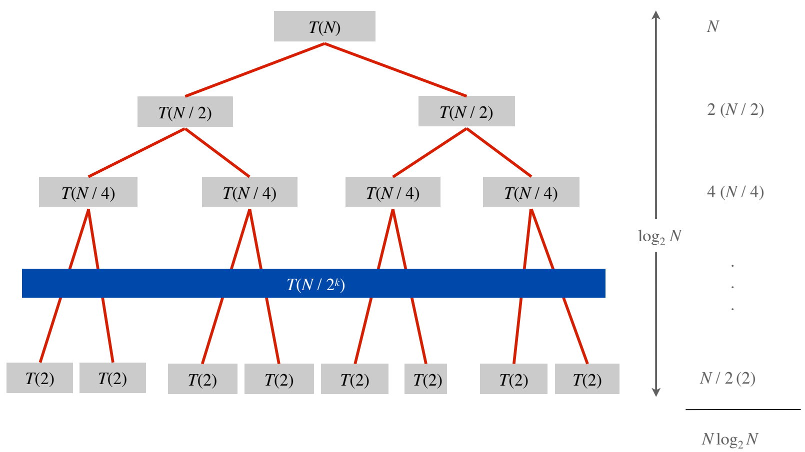
N	Comparisons	Time
20 million	460 million	17.5 sec
50 million	1216 million	45.9 sec

Not quite.

Mergesort: Mathematical Analysis

Analysis. To mergesort array of size N, mergesort two subarrays of size $N / 2$, and merge them together using $\leq N$ comparisons.

we assume N is a power of 2



Mergesort: Mathematical Analysis

Mathematical analysis.

analysis	comparisons
worst	$N \log_2 N$
average	$N \log_2 N$
best	$1/2\ N \log_2 N$

Validation. Theory agrees with observations.

N	actual	predicted
10,000	120 thousand	133 thousand
20 million	460 million	485 million
50 million	1,216 million	1,279 million

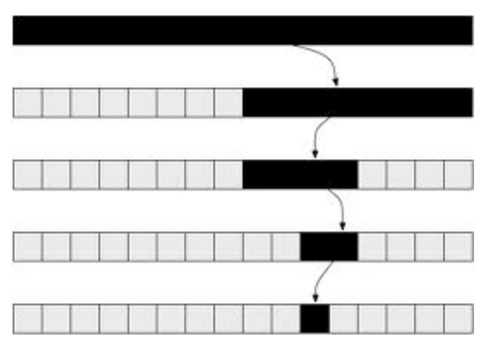
Mergesort: Lesson

Lesson. Great algorithms can be more powerful than supercomputers.

Computer	Comparisons Per Second	Insertion	Mergesort
laptop	10^7	3 centuries	3 hours
super	10^{12}	2 weeks	instant

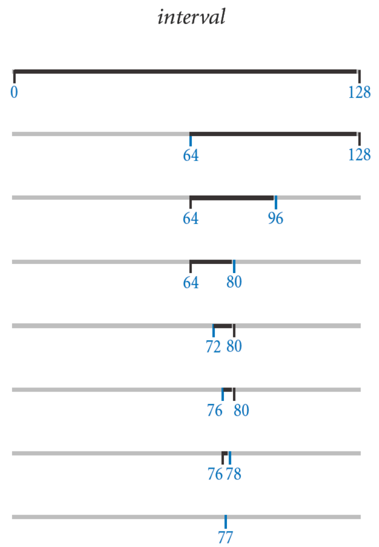
N = 1 billion

Binary Search



Twenty Questions

Intuition. Find a hidden integer.



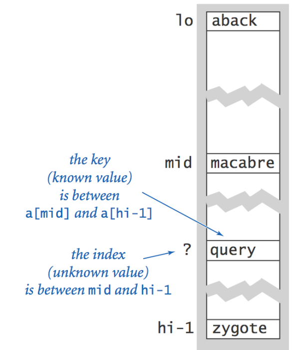
size	Q	A
128	< 64 ?	no
64	< 96 ?	yes
32	< 80 ?	yes
16	< 72 ?	no
8	< 76 ?	no
4	< 78 ?	yes
2	< 77 ?	no
1	= 77	

Binary Search

Idea:
- Sort the array
- Play "20 questions" to determine the index associated with a given key.

Ex. Dictionary, phone book, book index, credit card numbers, …

Binary search.
- Examine the middle key.
- If it matches, return its index.
- Otherwise, search either the left or right half.



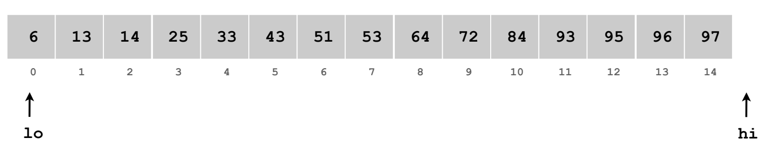
Binary search in a sorted array (one step)

Binary Search

Binary search. Given a `key` and sorted array `a[]`, find index `i` such that `a[i] = key`, or report that no such index exists.

Invariant. Algorithm maintains `a[lo] ≤ key ≤ a[hi-1]`.

Ex. Binary search for 33.



Binary Search

Binary search. Given a `key` and sorted array `a[]`, find index `i` such that `a[i] = key`, or report that no such index exists.

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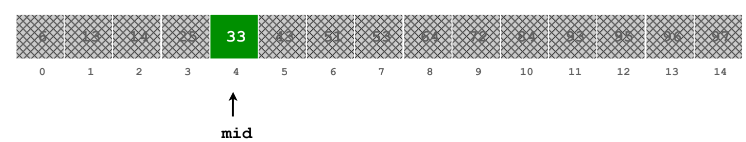
Ex. Binary search for 33.



Binary Search: Java Implementation

Invariant. Algorithm maintains `a[lo]` ≤ `key` ≤ `a[hi-1]`.

```
public static int search(String key, String[] a)
{
   return search(key, a, 0, a.length);
}

public static int search(String key, String[] a, int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);     // String compare: text p. 523
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```

Java library implementation: `Arrays.binarySearch()`

Binary Search: Mathematical Analysis

Analysis. To binary search in an array of size N: do one comparison, then binary search in an array of size $N / 2$.

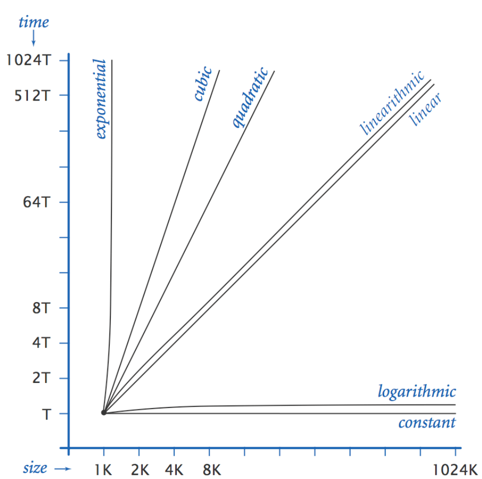
$$N \to N/2 \to N/4 \to N/8 \to \ldots \to 1$$

Q. How many times can you divide a number by 2 until you reach 1?

A. $\log_2 N$.

$$1$$
$$2 \to 1$$
$$4 \to 2 \to 1$$
$$8 \to 4 \to 2 \to 1$$
$$16 \to 8 \to 4 \to 2 \to 1$$
$$32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$1024 \to 512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$

Order of Growth Classifications



Orders of growth (log-log plot)

order of growth description	function	factor for doubling hypothesis
constant	1	1
logarithmic	$\log N$	1
linear	N	2
linearithmic	$N \log N$	2
quadratic	N^2	4
cubic	N^3	8
exponential	2^N	2^N

Commonly encountered growth functions

Order of Growth Classifications

Observation. A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```
while (N > 1) {
   N = N / 2;
   ...
}
```

$\lg N$

$\lg N = \log_2 N$

```
for (int i = 0; i < N; i++)
   ...
```

N

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

N^2

```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

$N \lg N$

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

2^N

Summary

Q. How can I evaluate the performance of my program?
A. Computational experiments, mathematical analysis

Q. What if it's not fast enough? Not enough memory?

- Understand why.
- Buy a faster computer.
- Learn a better algorithm (COS 226, COS 423).
- Discover a new algorithm.

attribute	better machine	better algorithm
cost	$$$ or more.	$ or less.
applicability	makes "everything" run faster	does not apply to some problems
improvement	incremental quantitative improvements expected	dramatic qualitative improvements possible