You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that. Also, limit your answers to one page or less—you just need to give enough detail to convince the grader. If you suspect a problem is open, just say so and give reasons for your suspicion.

§1 Write a linear program for finding the largest sphere contained inside a given polyhedron \( \{ x : Ax \leq b \} \) and explain why it is correct. What does it mean if this program is infeasible or unbounded?

§2 Suppose we are given points \((x_1, y_1), \ldots, (x_n, y_n)\) in the plane and want to fit a line \( y = ax + b \) to them. There are various notions of what a good line is; the most common one (linear regression) seeks to minimize the \( \ell_2 \) error:

\[
\varepsilon_2(a, b) = \sum_i (ax_i - b - y_i)^2
\]

and can be solved using linear algebra. Write linear programs which compute lines minimizing the \( \ell_1 \) and \( \ell_\infty \) error, namely

\[
\varepsilon_1(a, b) = \sum_i |ax_i - b - y_i|,
\]

\[
\varepsilon_\infty(a, b) = \max_i |ax_i - b - y_i|.
\]

What is the interpretation of dual feasible solutions for these programs?

§3 Helly’s theorem states that for any finite set of closed convex sets \( K_1, \ldots, K_m \) in \( \mathbb{R}^n \), if every \( n + 1 \) of them have a nonempty intersection then they all have a nonempty intersection. Use duality to prove Helly’s theorem for the special case of halfspaces \( K_j = \{ x : \langle x, a_j \rangle \leq b_j \} \).

§4 (Reducing to full-dimensionality in the Ellipsoid method.) Consider the feasibility linear program

\[
P = \exists! x : Ax \leq b,
\]

and the ‘thickened’ program

\[
P_\epsilon = \exists! x : Ax \leq b + \epsilon 1.
\]

Clearly if \( P \) is feasible then \( P_\epsilon \) is also feasible. Show that the converse is true for sufficiently small \( \epsilon \), and derive a bound on \( \epsilon \) in terms of the bit complexity of the inputs (assume \( a_{ij}, b_j \) are integers of size at most \( 2^L \).)
(von Neumann’s Minmax Theorem) A two-player zero sum game consists of two players min and max, who play strategies from finite sets $[n]$ and $[m]$ respectively, and a payoff matrix $A = (a_{ij})_{i \in [n], j \in [m]}$ which defines the outcome of the game for every pair $(i, j)$. If max reveals her strategy before min and each player tries to optimize her outcome, then that outcome will be exactly:

$$\max_{j \in [m]} \min_{i \in [n]} a_{ij}.$$  

On the other hand, if min goes first, then the outcome will be

$$\min_{i \in [n]} \max_{j \in [m]} a_{ij},$$

which may be different.

Now consider the setting where min and max are allowed to play mixed strategies, which are probability distributions over $[n]$ and $[m]$ respectively. A mixed strategy $x \in \Delta_n = \{x \in \mathbb{R}_+^n : \sum_i x_i = 1\}$ may be interpreted as a randomized rule in which min chooses $i \in [n]$ with probability $x_i$. The expected outcome for a pair of mixed strategies $(x, y)$ may be written as

$$E_{i \sim x, j \sim y} a_{ij} = y^T Ax.$$  

An equilibrium is a pair of mixed strategies for which the order in which they are announced does not matter as long as the random choice is made simultaneously for both players, i.e.:

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} y^T Ax = \max_{y \in \Delta_m} \min_{x \in \Delta_n} y^T Ax.$$  

Show that an equilibrium always exists and give a polynomial time algorithm for computing it in the case where $A$ has integer entries.

§6 Given a directed graph $G = (V, E)$ with edge capacities $c_e \geq 0$, and two distinguished vertices $s$ and $t$, the minimum cut problem seeks to find a subset of $E$ of minimum total weight which disconnects $s$ and $t$. Show that the following linear programming relaxation of this problem always has an integral optimum with $x_e \in \{0, 1\}$.

$$\min \sum_e x_e c_e$$

$$\forall e \in E \quad x_e \geq 0$$

$$\forall s-t paths p \quad \sum_{e \in p} x_e \geq 1.$$  

(hint: You need to show that for every cost function $c_e$ and feasible solution $x$, there is an integral feasible solution $x'$ with $\sum_e c_e x'_e \leq \sum_e c_e x_e$. Use the edge lengths $x_e$ to find a useful embedding of the vertices of the graph.)