Using LP rounding to design approximation algorithms.

Typically used in context of 0-1 optimization problem.
Example: Min Vertex Cover. Given $G = (V, E)$, find the smallest set $S \subseteq V$ such that for every edge (i, j) either $i \in S$ or $j \in S$

Integer programming formulation:

$$
\begin{aligned}
\text{Minimize } & \textstyle\sum_i x_i \\
& x_i \in \{0, 1\} \ \forall\, i \\
& x_i + x_j \geq 1 \qquad \forall\, (i, j) \in E
\end{aligned}
$$

LP formulation: relax to $0 \leq x_i \leq 1 \ \forall\, i$.

Observe: Optimum value of LP relaxation is a lowerbound on the integer optimum.

Rounding: going from fractional solution to 0-1 soln.

Example: Given fractional solution for Vertex Cover, consider set of all nodes i such that $x_i \geq \frac{1}{2}$.

- Is a vertex cover since $x_i + x_j \geq \frac{1}{2} \Rightarrow$ one of them is $\geq \frac{1}{2}$
- Has size at most $2 \times \sum_i x_i = 2 \times$ fractional opt
- This simple 2-approx is essentially the best we know of for VC !!.

Randomized rounding: make y_i =1 with prob. x_i and 0 with prob. $1-x_i$

Observations (Raghavan-Thompson): 1) For any coefficient vector a,
$E[a \cdot y] = a \cdot x$.(Linearity of Expectation)

(2) The y_i's are independent random variables, so one can use Chernoff bounds to upperbound the chance that $a \cdot y$ deviates much from the expectation.

3/4 –approx. for MAX-2SAT

Problem: Given 2-CNF formula, find assignment that maximizes number of satisfied clauses.

First we write the LP. Have a variable x_i for each boolean variable y_i and a variable z_c for each clause c. Require $0 \leq x_i, z_c \leq 1$

Objective is to maximize $\sum_c z_c$.
If clause c is $y_i \vee y_j$ represent by $x_i + x_j \geq z_c$. (Thus "1" represents "True" and "0" represents "False.")
If clause is $y_i \vee \neg y_j$ then represent by $x_i + (1-x_j) \geq z_c$, and so on.

Randomized rounding: make y_i =True with prob. x_i

$$
\begin{aligned}
\Pr[\text{clause } c \text{ satisfied}] = 1 - (1-x_i)(1-x_j) &= x_i + x_j - x_i x_j \\
&\geq z_c - x_i x_j \\
&\geq z_c - z_c^2/4 \text{ (by AM} \geq \text{GM)} \\
&\geq \tfrac{3}{4} z_c
\end{aligned}
$$

So E[# of clauses satisfied] \geq $\frac{3}{4} \sum_c z_c$

Running time?

- Method 1: Repeat poly(n, $1/\varepsilon$) times; take the best assignment.

  Averaging shows that at each repetition:
  Pr[assignment satisfies > $\frac{3}{4}-\varepsilon$ fraction of clauses] $\geq 4\varepsilon$

- Method 2: Observe that we only use pairwise independence.

Can do the rounding using pairwise indep. Variables.
Can exhaustively search through the probability space (recall HW1);
takes poly(n) time.

Method 2 gives deterministic algorithm!

Next example: $O(\log n)$-approximation for Set Cover.
(prototype of $O(\log n)$-approx for other problems, eg VLSI wiring)

Problem: Given sets $S_1, S_2, .., S_m$ of $\{1, .., n\}$, find smallest subset C such that $C \cap S_k \neq \emptyset \ \forall k$.

LP: $\min \sum_i x_i$

$$\sum_{i \in Sk} x_i \geq 1 \ \forall k$$
$$0 \leq x_i \leq 1$$

Solve LP. Do randomized rounding.

$\forall k$, $\Pr[S_k \text{ gets covered}] = 1 - \prod_{i \in Sk} (1-x_i) \geq 1 - (1-1/S)\hat{\ }S \geq 1-1/e$
where $S = |S_k|$.

Now repeat randomized rounding t times and take union of all the sets produced.
$\Pr[S_k$ still uncovered after t reps$] \leq (1/e)^t$.
Making $t = \log_e m + 1$ we see that this prob. is $\leq 1/em$.

E[size of final set] = $t \times$ fractional opt. $= O(\log m) \times$ fractional Opt.