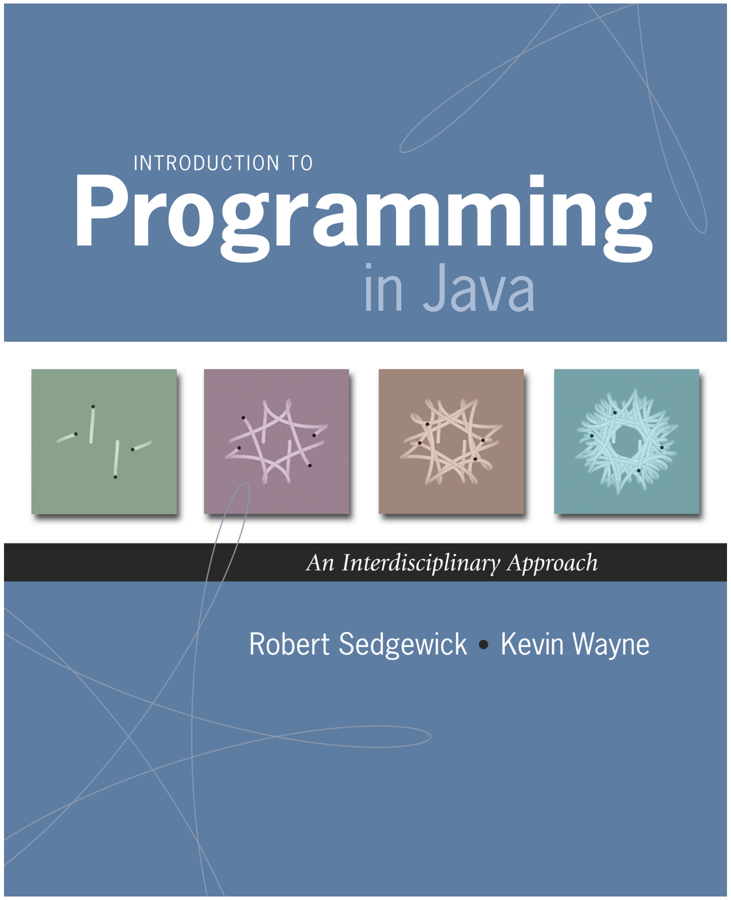
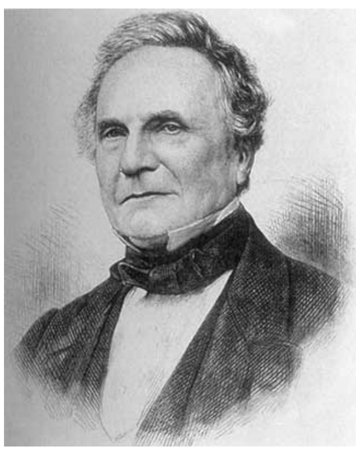
# 4.1, 4.2 Performance and Sorting

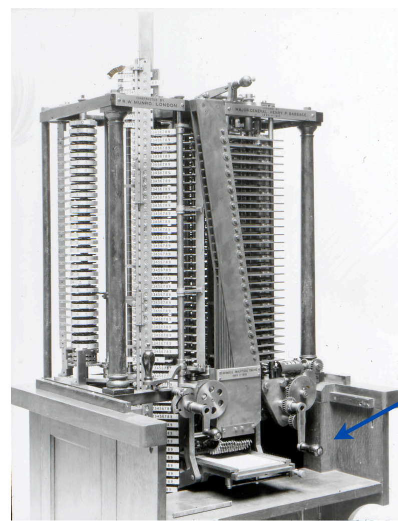


# Running Time

*"As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time?"* – Charles Babbage



Charles Babbage (1864)



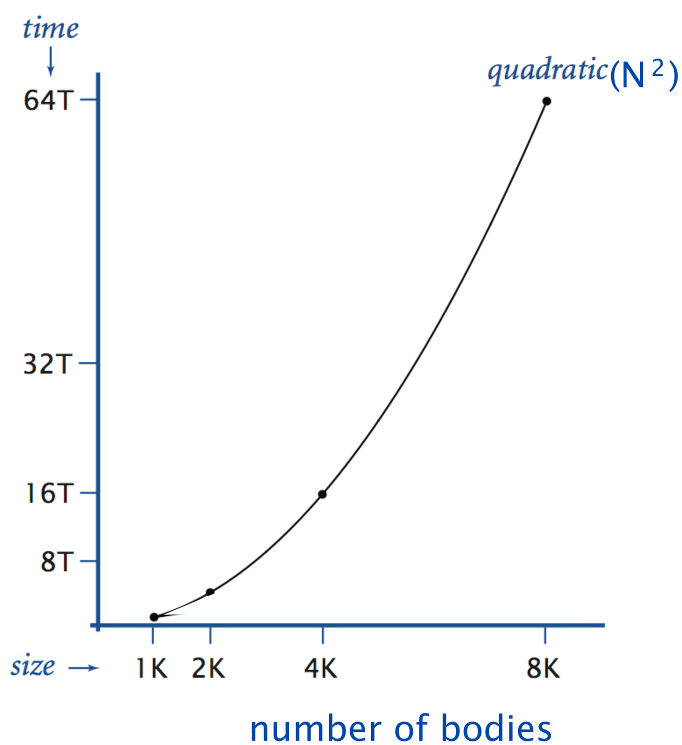
Analytic Engine

how many times do you have to turn the crank?

# Algorithmic Successes

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: $N^2$ steps.

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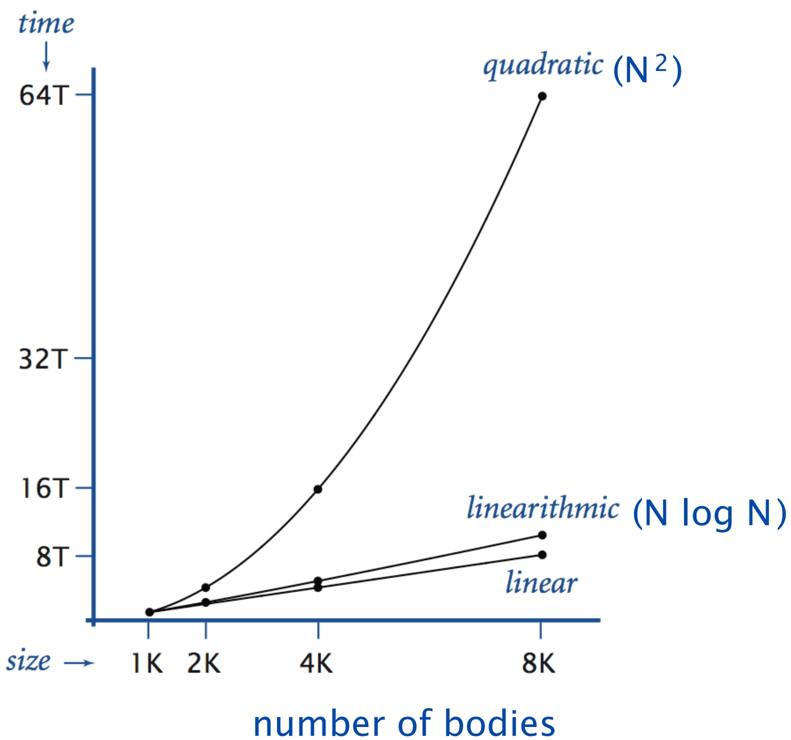
# Algorithmic Successes

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: $N^2$ steps.
- Barnes-Hut: N log N steps, enables new research.

Andrew Appel
PU '81

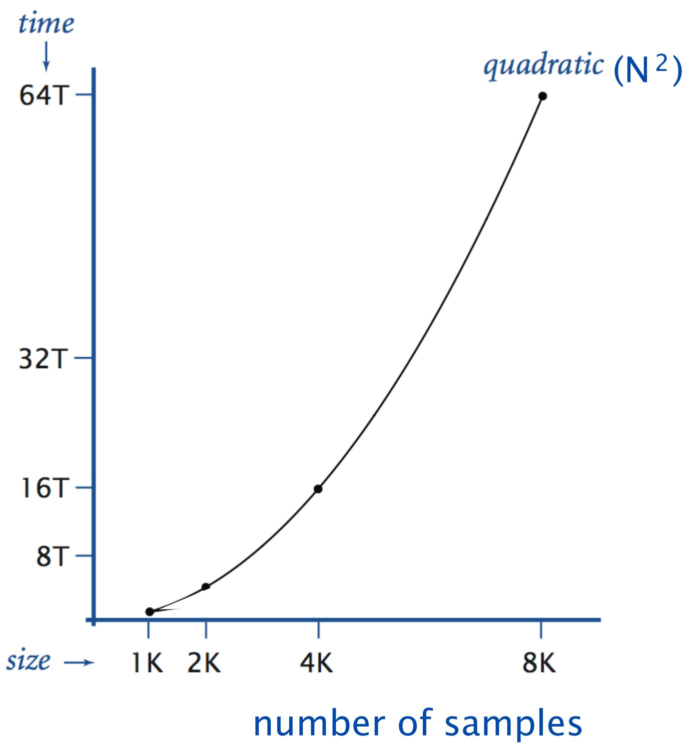



# Algorithmic Successes

Freidrich Gauss
1805

## Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: $N^2$ steps.

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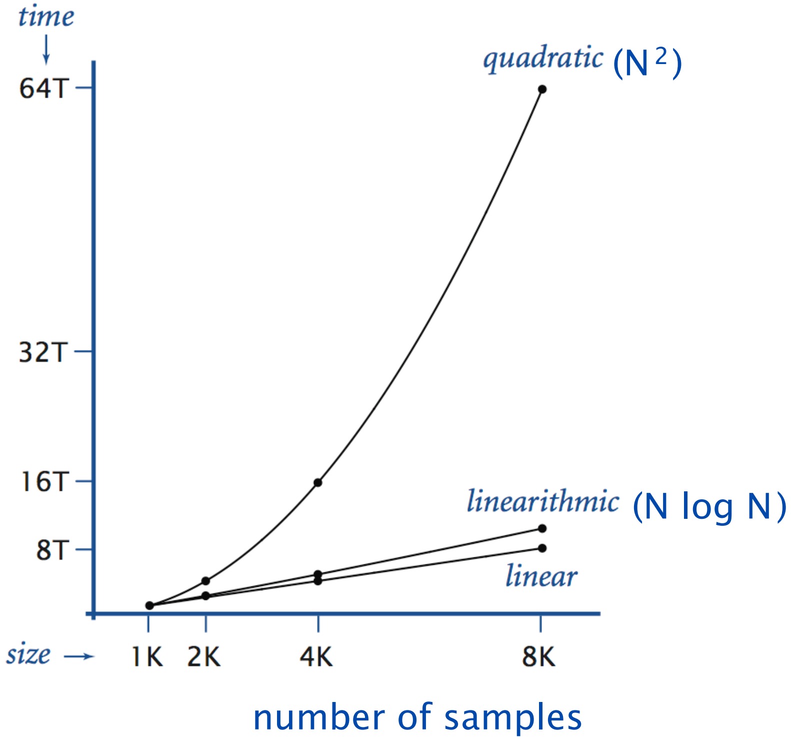
# Algorithmic Successes

John Tukey
1965

**Discrete Fourier transform.**

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ....
- Brute force: $N^2$ steps.
- FFT algorithm: N log N steps, enables new technology.

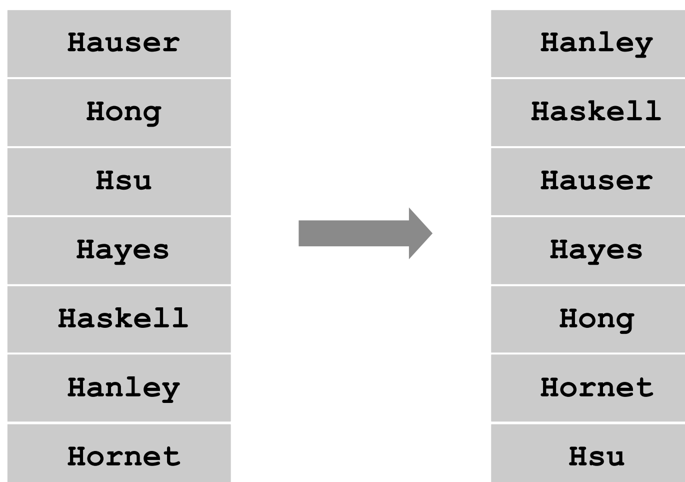



# Sorting

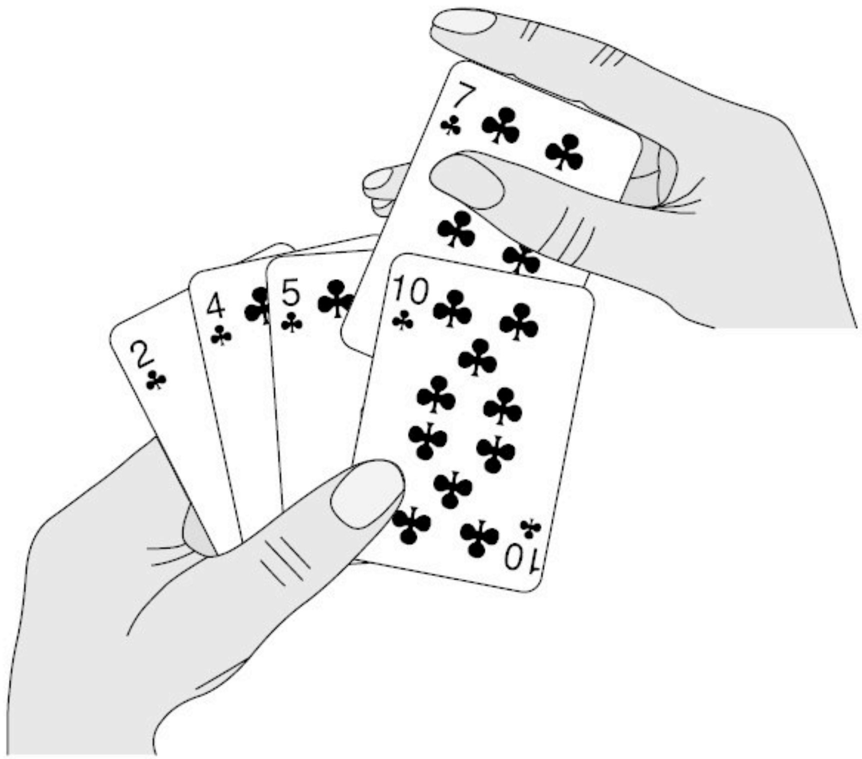
# Sorting

Sorting problem. Rearrange $N$ items in ascending order.

Applications. Binary search, statistics, databases, data compression, bioinformatics, computer graphics, scientific computing, (too numerous to list) ...



# Insertion Sort



# Insertion Sort

## Insertion sort.

- Brute-force sorting solution. ← *insertion sort is simpler and faster than bubble sort, so we don't teach bubble sort anymore*
- Move left-to-right through array.
- Insert each element into final position by exchanging it with larger elements to its left, one-by-one.

| i | j | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 6 | and | had | him | his | was | you | the | but |
| 6 | 5 | and | had | him | his | was | the | you | but |
| 6 | 4 | and | had | him | his | the | was | you | but |
| | | and | had | him | his | the | was | you | but |

*Inserting* `a[6]` *into position by exchanging with larger entries to its left*

# Insertion Sort

**Insertion sort.**

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

| i | j | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|
| | | was | had | him | and | you | his | the | but |
| 1 | 0 | had | was | him | and | you | his | the | but |
| 2 | 1 | had | him | was | and | you | his | the | but |
| 3 | 0 | and | had | him | was | you | his | the | but |
| 4 | 4 | and | had | him | was | you | his | the | but |
| 5 | 3 | and | had | him | his | was | you | the | but |
| 6 | 4 | and | had | him | his | the | was | you | but |
| 7 | 1 | and | but | had | him | his | the | was | you |
| | | and | but | had | him | his | the | was | you |

*Inserting* a[1] *through* a[N-1] *into position (insertion sort)*

# Insertion Sort: Java Implementation

```
public class Insertion
{

   public static void sort(String[] a)
   {
      int N = a.length;
      for (int i = 1; i < N; i++)
         for (int j = i; j > 0; j--)
            if (a[j-1] > a[j])
               exch(a, j-1, j);
            else break;
   }


   private static void exch(String[] a, int i, int j)
   {
      String swap = a[i];
      a[i] = a[j];
      a[j] = swap;
   }
}
```

# Insertion Sort: Observation

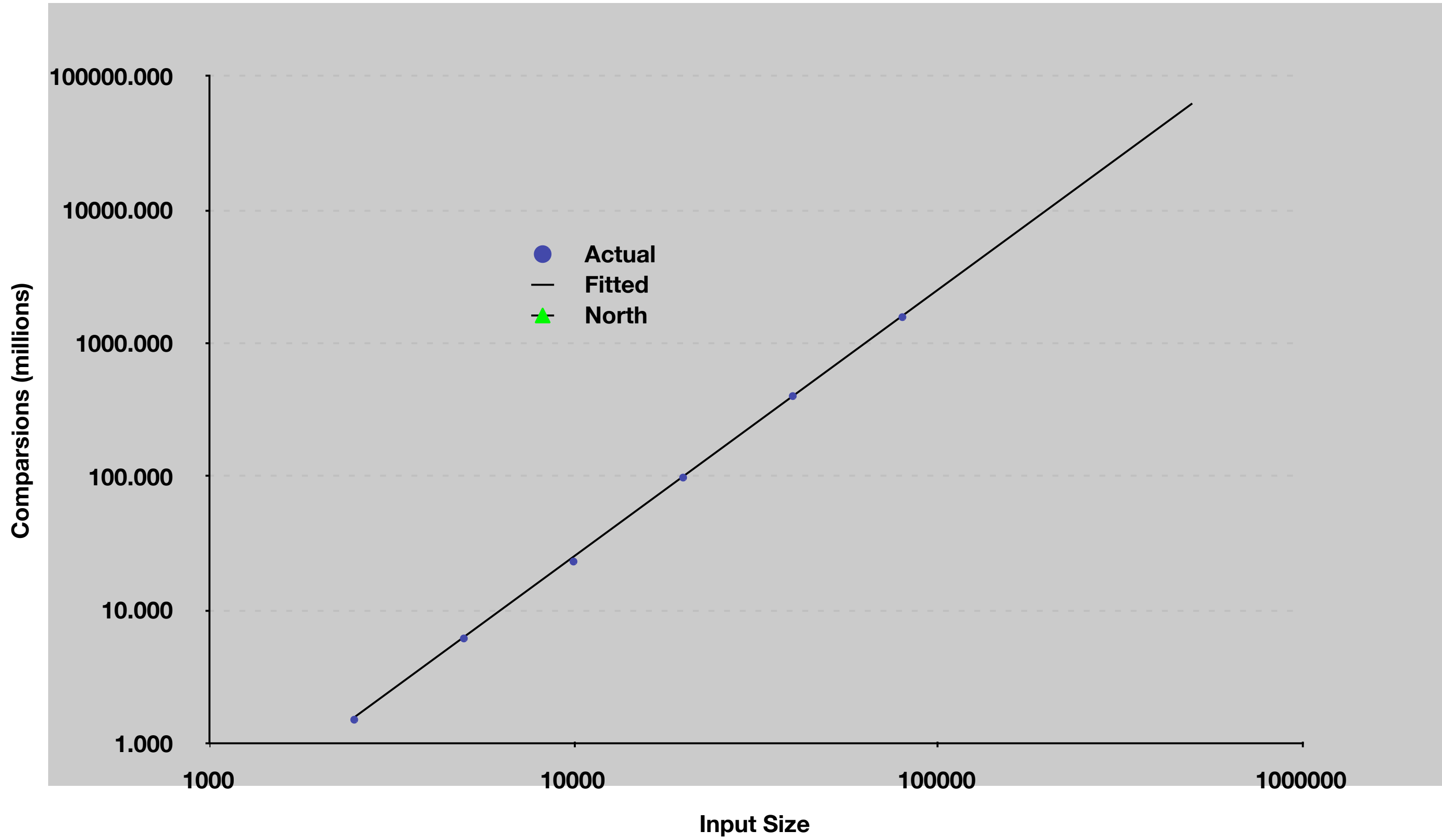
Observe and tabulate running time for various values of N.

- Data source: N random numbers between 0 and 1.
- Machine: Apple G5 1.8GHz with 1.5GB memory running OS X.
- Timing: Skagen wristwatch.

| N | Comparisons | Time |
|---|---|---|
| 5,000 | 6.2 million | 0.13 seconds |
| 10,000 | 25 million | 0.43 seconds |
| 20,000 | 99 million | 1.5 seconds |
| 40,000 | 400 million | 5.6 seconds |
| 80,000 | 1600 million | 23 seconds |

# Insertion Sort: Empirical Analysis

Data analysis. Plot # comparisons vs. input size on log-log scale.



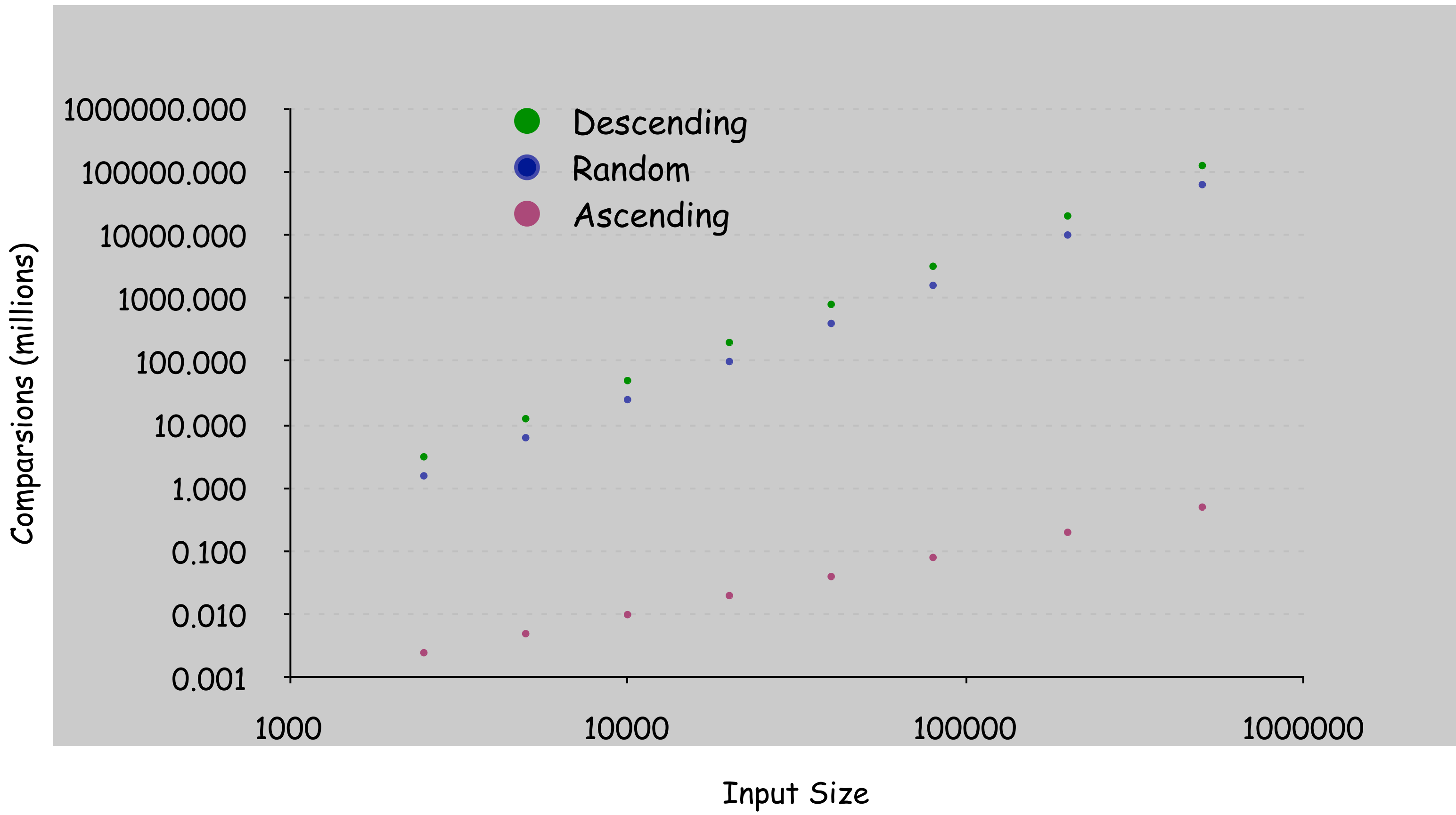

slope

Hypothesis. # comparisons grows quadratically with input size $\sim N^2 / 4$.

# Insertion Sort: Empirical Analysis

Observation. Number of compares depends on input family.

- Descending: $\sim N^2 / 2$.
- Random: $\sim N^2 / 4$.
- Ascending: $\sim N$.



# Analysis: Empirical vs. Mathematical

Empirical analysis.

- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

Mathematical analysis.

- Analyze algorithm to estimate # ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

Critical difference. Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.

# Insertion Sort: Mathematical Analysis

Worst case. [descending]

- Iteration $i$ requires $i$ comparisons.
- Total = $(0 + 1 + 2 + ... + N\text{-}1) \sim N^2 / 2$ compares.


Average case. [random]

- Iteration $i$ requires $i / 2$ comparisons on average.
- Total = $(0 + 1 + 2 + ... + N\text{-}1) / 2 \sim N^2 / 4$ compares

# Insertion Sort: Lesson

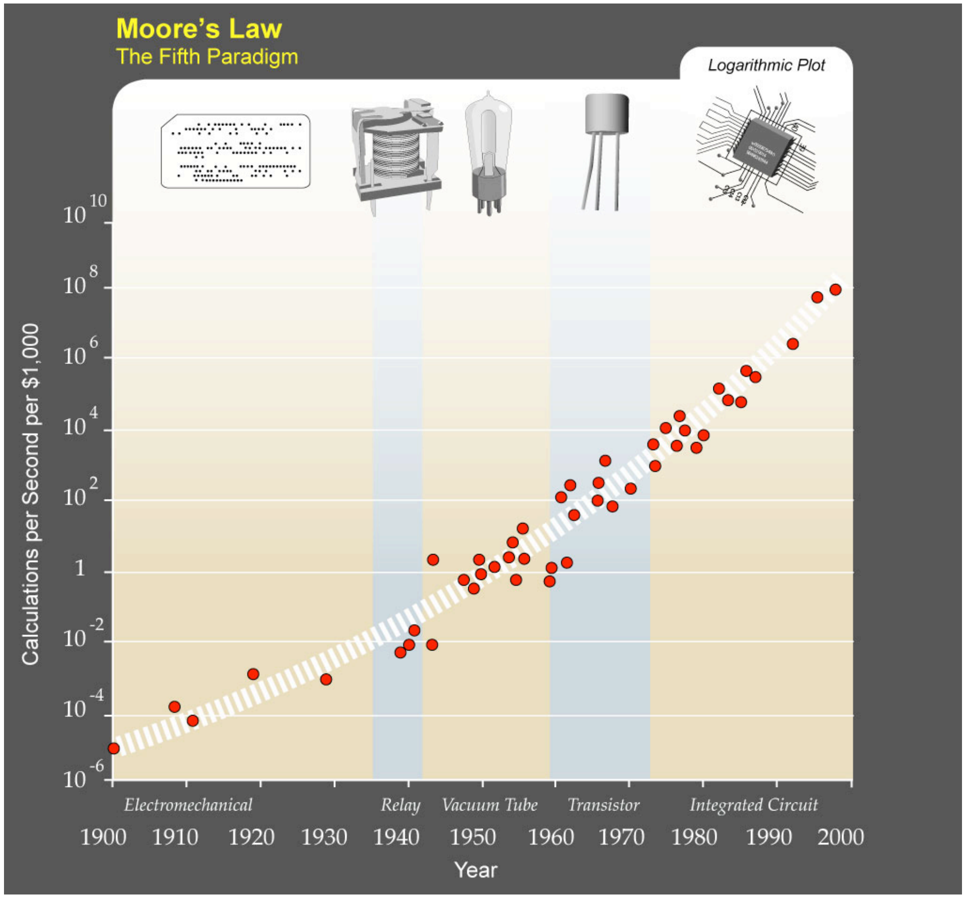
**Lesson.** Supercomputer can't rescue a bad algorithm.

| Computer | Comparisons Per Second | Thousand | Million | Billion |
|---|---|---|---|---|
| laptop | $10^7$ | instant | 1 day | 3 centuries |
| super | $10^{12}$ | instant | 1 second | 2 weeks |

# Moore's Law

Moore's law. Transistor density on a chip doubles every 2 years.

Variants. Memory, disk space, bandwidth, computing power per $.




http://en.wikipedia.org/wiki/Moore's_law

# Moore's Law and Algorithms

Quadratic algorithms do not scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

*"Software inefficiency can always outpace Moore's Law. Moore's Law isn't a match for our bad coding."* – *Jaron Lanier*

Lesson. Need linear (or linearithmic) algorithm to keep pace with Moore's law.

# Announcements

Exam 1 looms.

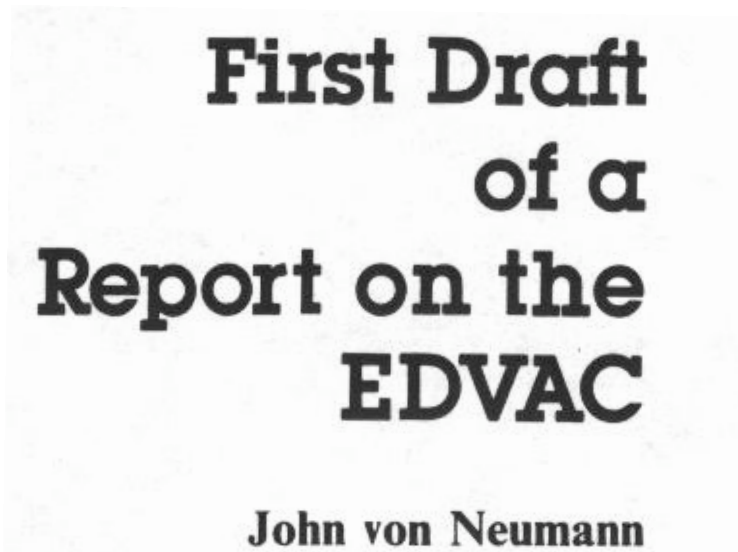
Written exam Tuesday 3/13 during your lecture time.  Room TBD.

Programming exam Tuesday 3/13 or Wednesday 3/14 in your precept.

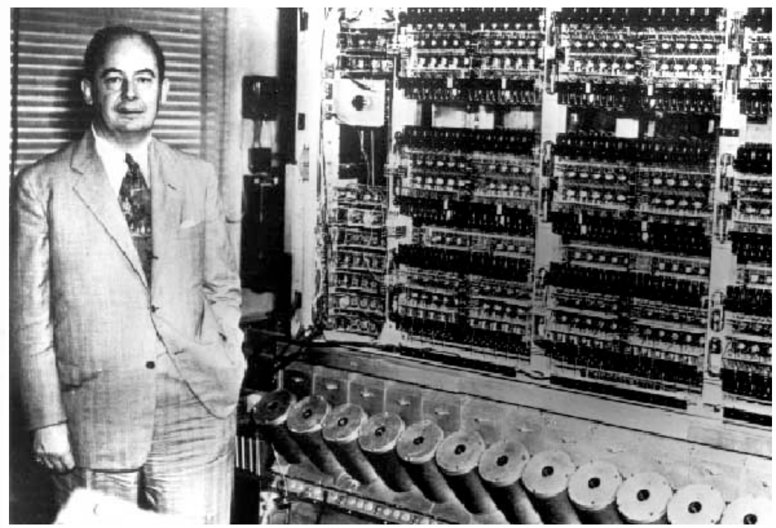
Review session will be held.

Rooms, rules, details on Exams page of website.

# Mergesort


First Draft
of a
Report on the
EDVAC

John von Neumann



# Mergesort

**Mergesort.**

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

*input*

```
was had him and you his the but
```

*sort left*

```
and had him was you his the but
```

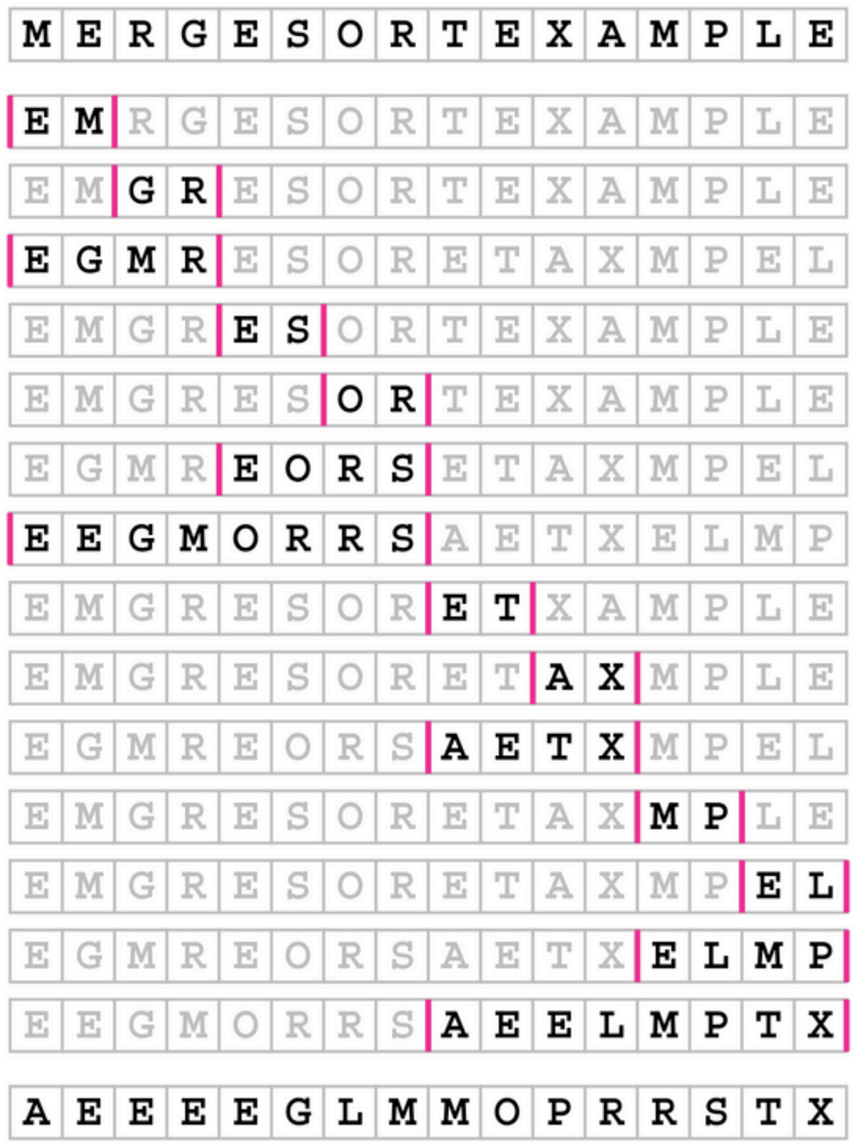
*sort right*

```
and had him was but his the you
```

*merge*

```
and but had him his the was you
```

# Mergesort: Example




*Top-down mergesort*

# Merging

**Merging.** Combine two pre-sorted lists into a sorted whole.

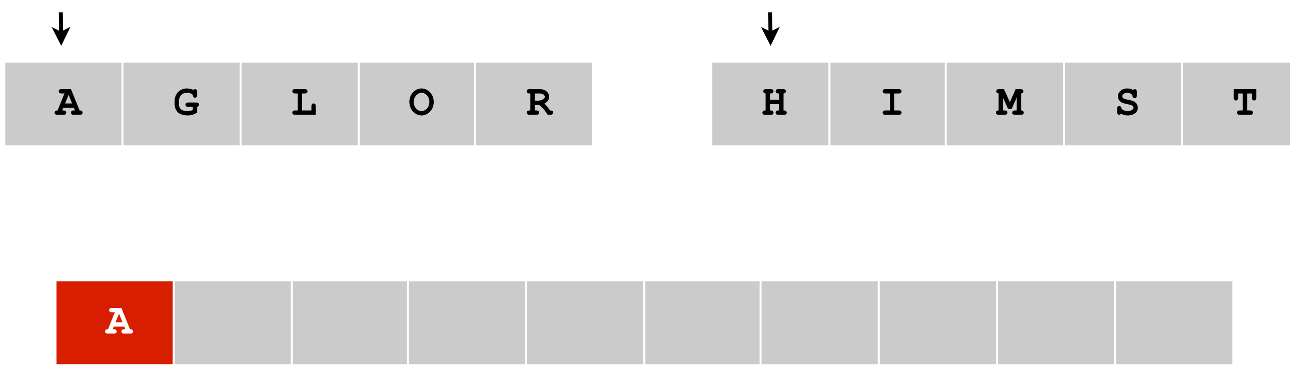
**How to merge efficiently?** Use an auxiliary array.

| i | j | k | aux[k] | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | and | had | him | was | but | his | the | you |
| 0 | 4 | 0 | and | and | had | him | was | but | his | the | you |
| 1 | 4 | 1 | but | and | had | him | was | but | his | the | you |
| 1 | 5 | 2 | had | and | had | him | was | but | his | the | you |
| 2 | 5 | 3 | him | and | had | him | was | but | his | the | you |
| 3 | 5 | 4 | his | and | had | him | was | but | his | the | you |
| 3 | 6 | 5 | the | and | had | him | was | but | his | the | you |
| 3 | 6 | 6 | was | and | had | him | was | but | his | the | you |
| 4 | 7 | 7 | you | and | had | him | was | but | his | the | you |

*Trace of the merge of the sorted left half with the sorted right half*

# Merging

Merge.
- Keep track of smallest element in each sorted half.
- Choose smaller of two elements.
- Repeat until done.



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**Merge.**

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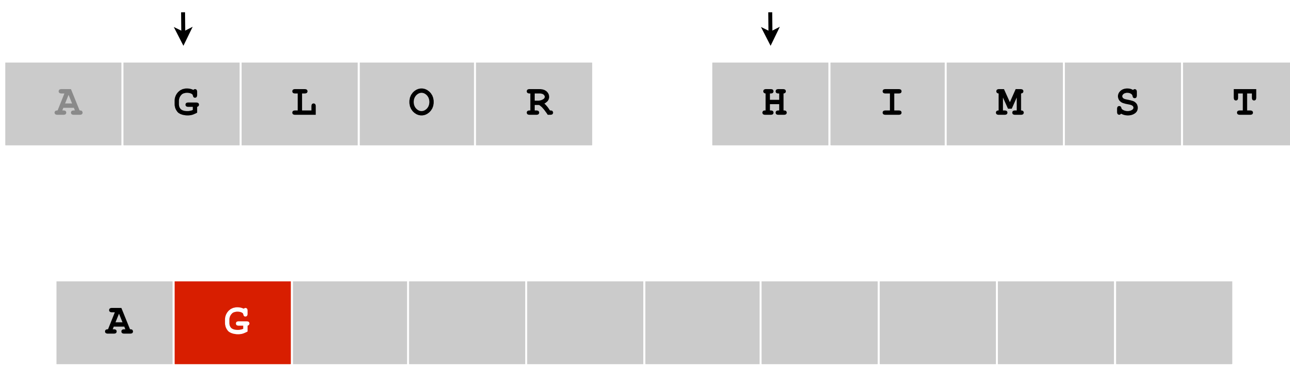

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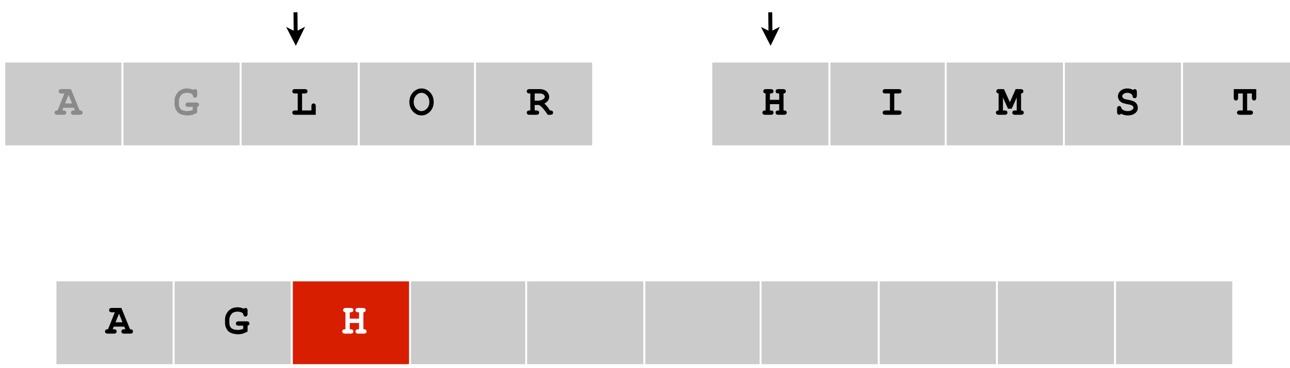

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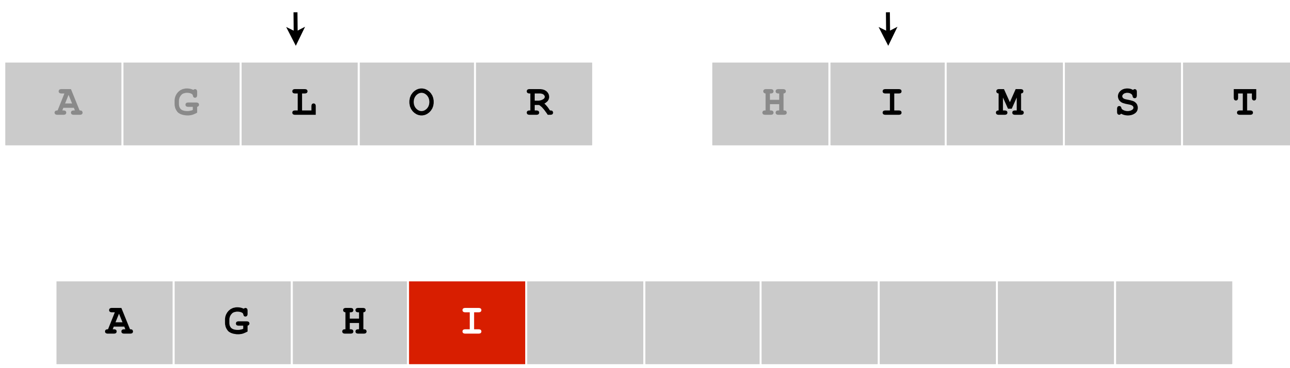

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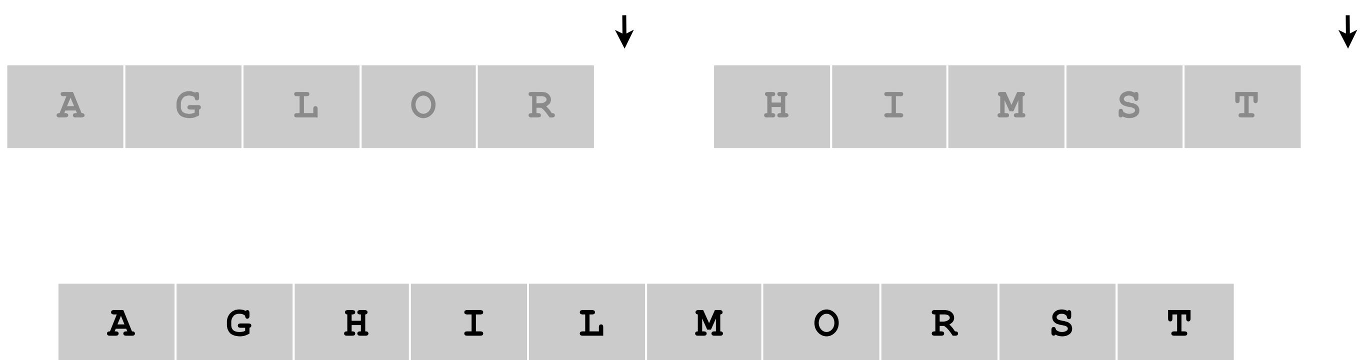


# Merging

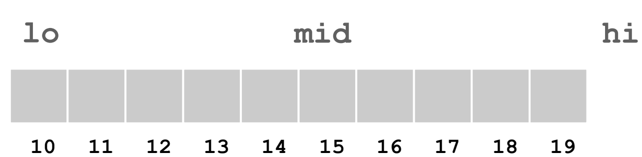
Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

```
String[] aux = new String[N];
// Merge into auxiliary array.
int i = lo, j = mid;
for (int k = 0; k < N; k++)
{
   if      (i == mid) aux[k] = a[j++];
   else if (j == hi)  aux[k] = a[i++];
   else if (a[j].compareTo(a[i]) < 0) aux[k] = a[j++];
   else                               aux[k] = a[i++];
}

// Copy back.
for (int k = 0; k < N; k++)
   a[lo + k] = aux[k];
```

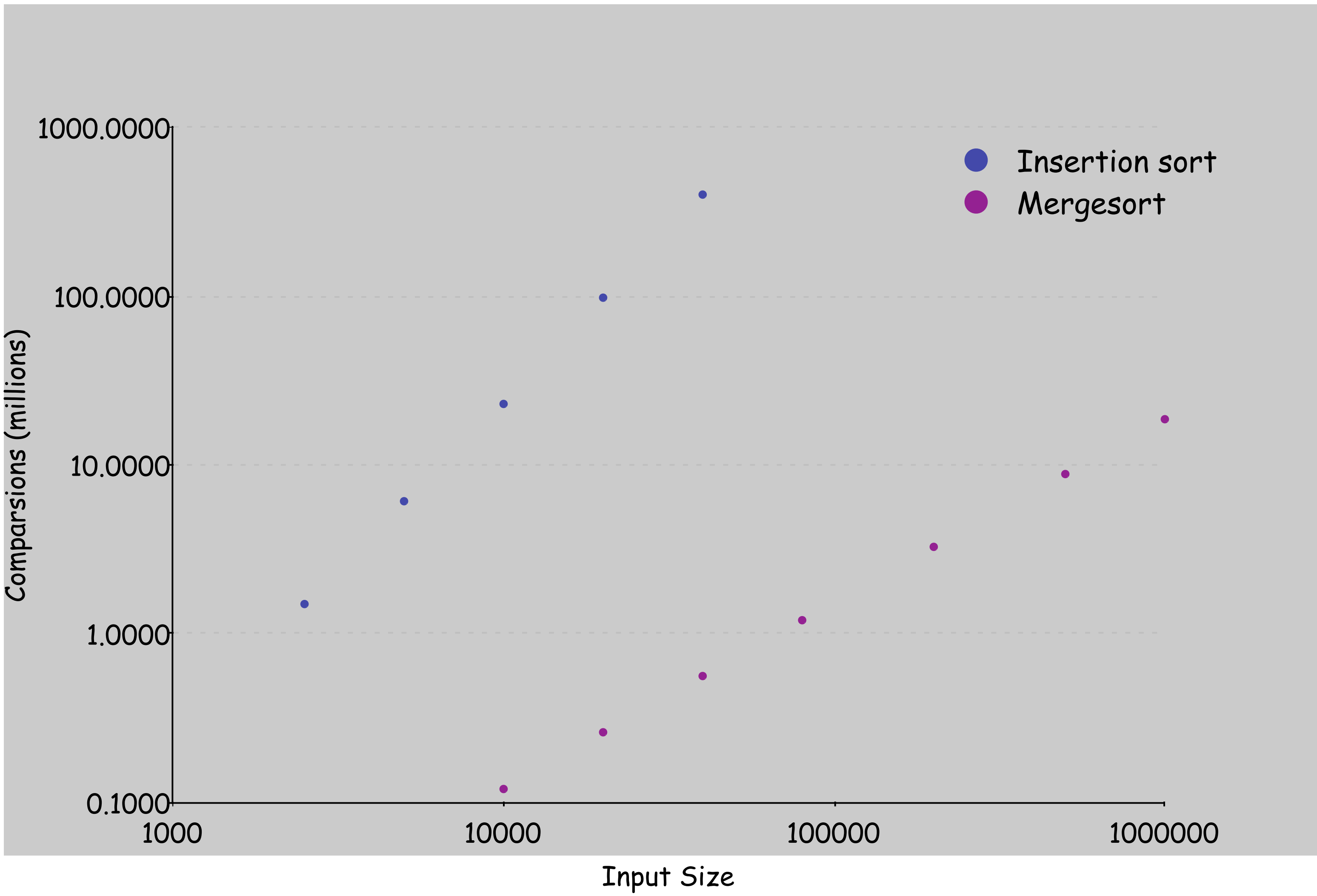
# Mergesort: Java Implementation

```
public class Merge
{
   public static void sort(String[] a)
   {  sort(a, 0, a.length);  }

   // Sort a[lo, hi).
   public static void sort(String[] a, int lo, int hi)
   {
      int N = hi - lo;
      if (N <= 1) return;

      // Recursively sort left and right halves.
      int mid = lo + N/2;
      sort(a, lo, mid);
      sort(a, mid, hi);

      // Merge sorted halves (see previous slide).
   }

}
```



# Mergesort: Empirical Analysis

Experimental hypothesis. Number of comparisons ≈ 20N.



# Mergesort: Prediction and Verification

Experimental hypothesis. Number of comparisons $\approx$ 20N.

Prediction. 80 million comparisons for N = 4 million.

Observations.

| N | Comparisons | Time |
|---|---|---|
| 4 million | 82.7 million | 3.13 sec |
| 4 million | 82.7 million | 3.25 sec |
| 4 million | 82.7 million | 3.22 sec |

Agrees.

Prediction. 400 million comparisons for N = 20 million.

Observations.

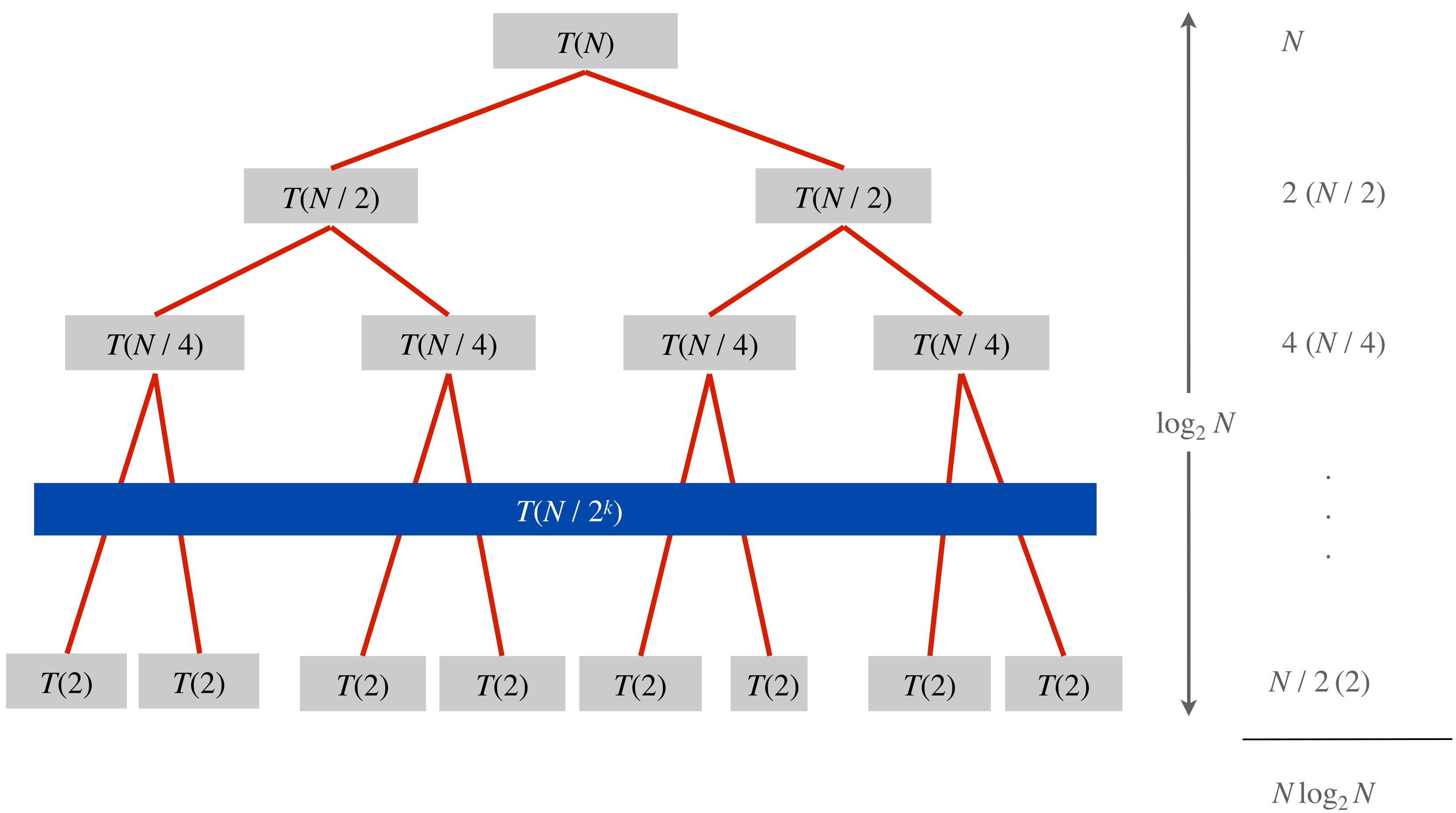
| N | Comparisons | Time |
|---|---|---|
| 20 million | 460 million | 17.5 sec |
| 50 million | 1216 million | 45.9 sec |

Not quite.

# Mergesort: Mathematical Analysis

**Analysis.** To mergesort array of size $N$, mergesort two subarrays of size $N / 2$, and merge them together using $\leq N$ comparisons.

we assume $N$ is a power of 2



# Mergesort: Mathematical Analysis

**Mathematical analysis.**

| analysis | comparisons |
|---|---|
| worst | $N \log_2 N$ |
| average | $N \log_2 N$ |
| best | $1/2\ N \log_2 N$ |

**Validation.** Theory agrees with observations.

| N | actual | predicted |
|---|---|---|
| 10,000 | 120 thousand | 133 thousand |
| 20 million | 460 million | 485 million |
| 50 million | 1,216 million | 1,279 million |

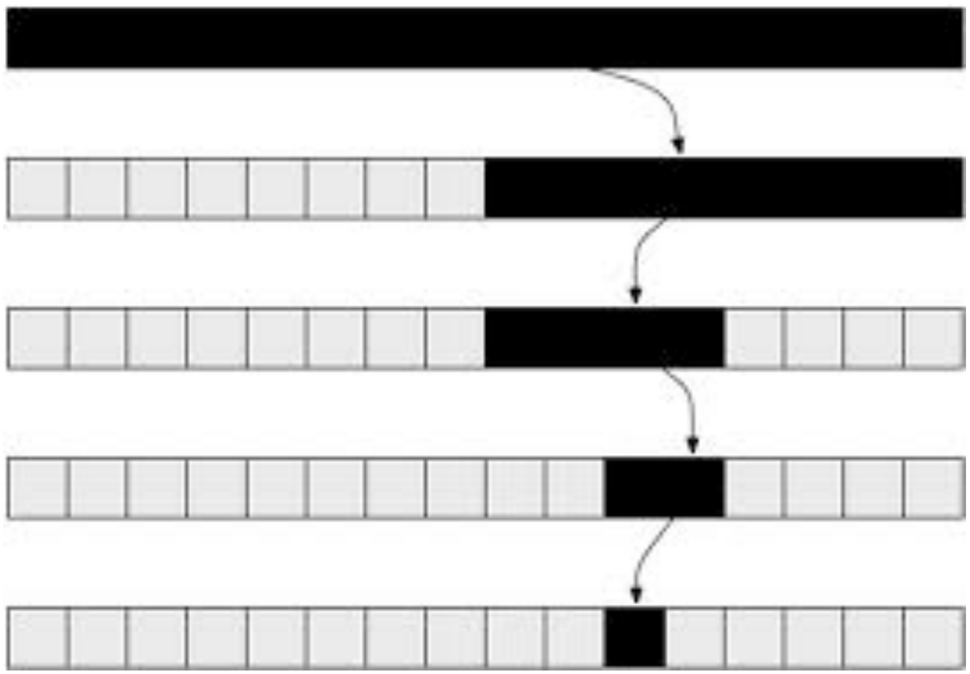
# Mergesort: Lesson

**Lesson.** Great algorithms can be more powerful than supercomputers.

| Computer | Comparisons Per Second | Insertion | Mergesort |
|---|---|---|---|
| laptop | $10^7$ | 3 centuries | 3 hours |
| super | $10^{12}$ | 2 weeks | instant |

N = 1 billion

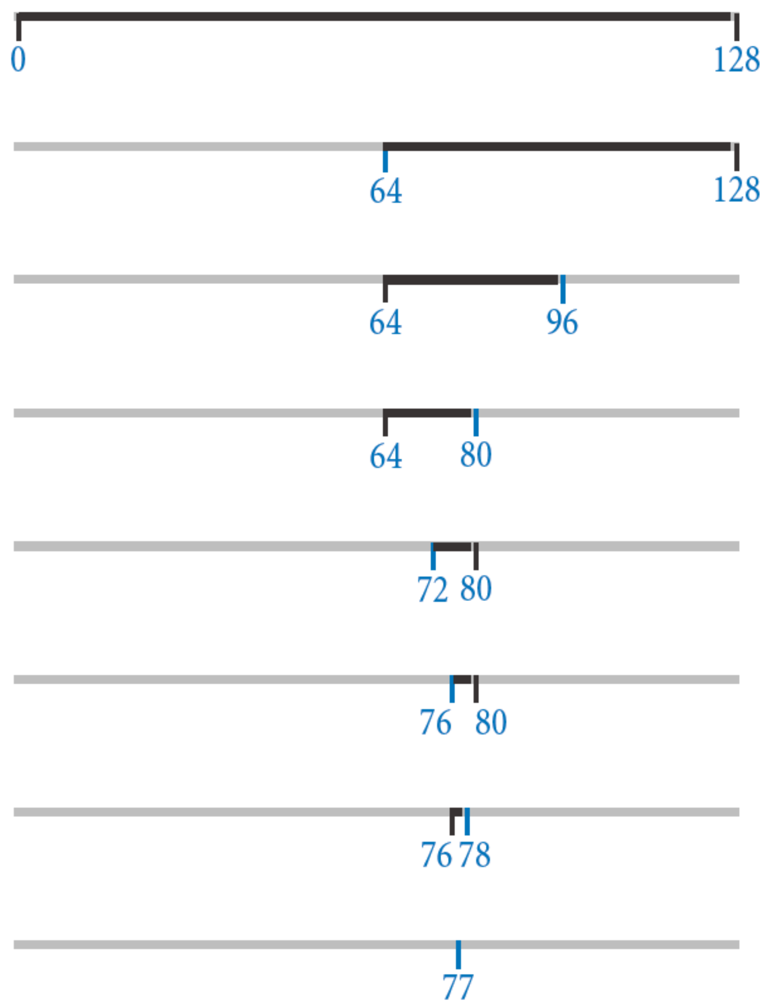
# Binary Search



# Twenty Questions

**Intuition.** Find a hidden integer.

*interval*




| *size* | *Q* | *A* |
|---|---|---|
| 128 | < 64 ? | *no* |
| 64 | < 96 ? | *yes* |
| 32 | < 80 ? | *yes* |
| 16 | < 72 ? | *no* |
| 8 | < 76 ? | *no* |
| 4 | < 78 ? | *yes* |
| 2 | < 77 ? | *no* |
| 1 | = 77 | |

# Binary Search

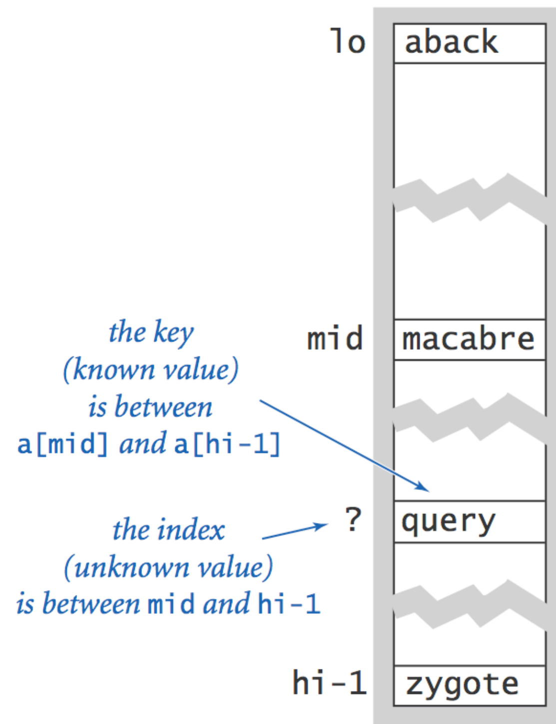
**Idea:**

- Sort the array (stay tuned)
- Play "20 questions" to determine the index associated with a given key.

**Ex.** Dictionary, phone book, book index, credit card numbers, …

**Binary search.**

- Examine the middle key.
- If it matches, return its index.
- Otherwise, search either the left or right half.




*Binary search in a sorted array (one step)*

# Binary Search

Binary search. Given a `key` and sorted array `a[]`, find index `i` such that `a[i] = key`, or report that no such index exists.

Invariant. Algorithm maintains `a[lo]` ≤ `key` ≤ `a[hi-1]`.

Ex. Binary search for 33.

| 6 | 13 | 14 | 25 | 33 | 43 | 51 | 53 | 64 | 72 | 84 | 93 | 95 | 96 | 97 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

↑ **lo** (at index 0) ↑ **hi** (after index 14)

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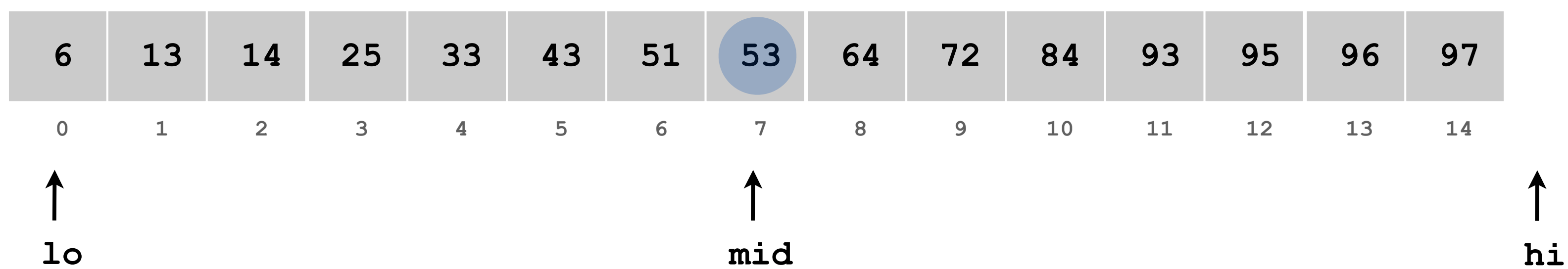

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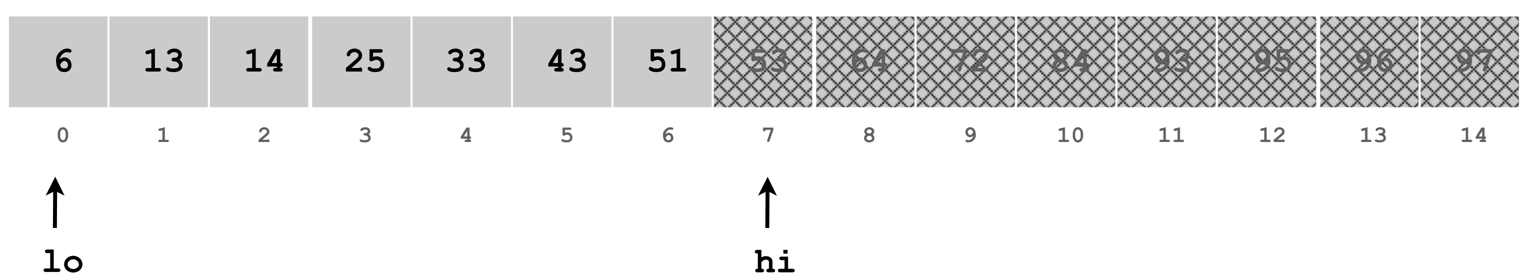

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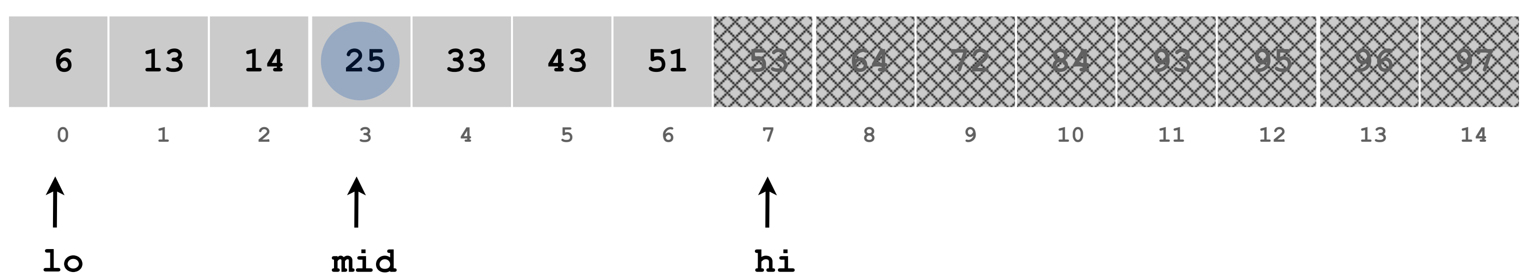

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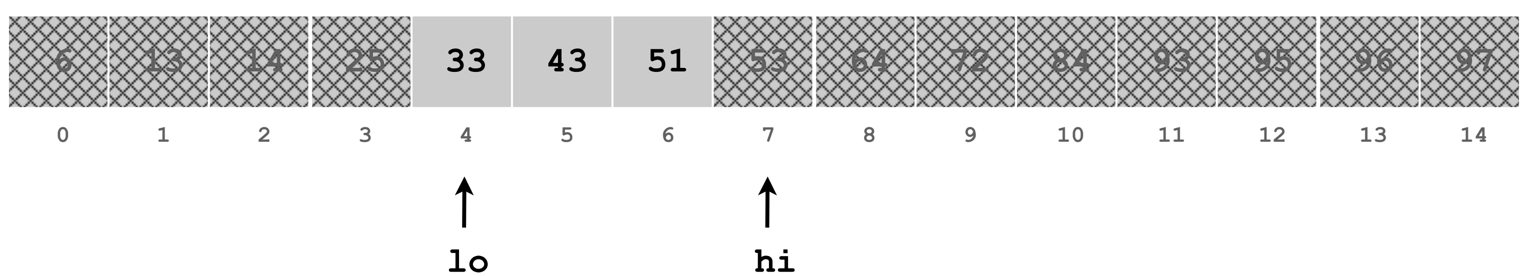

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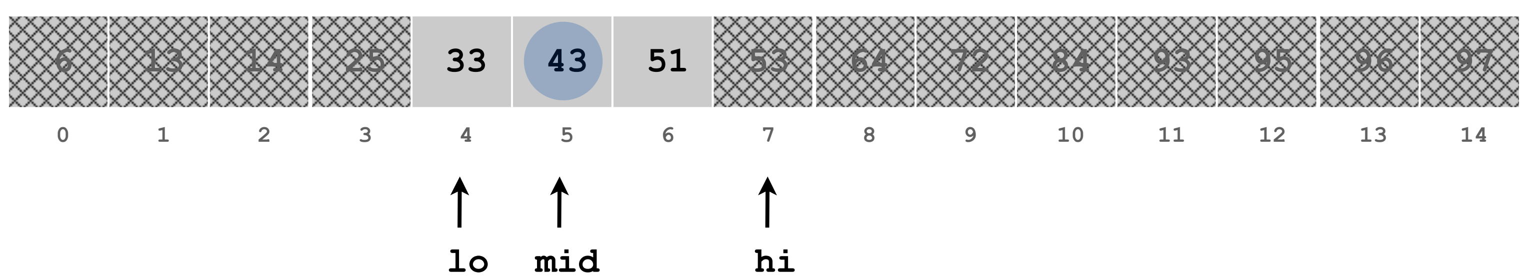

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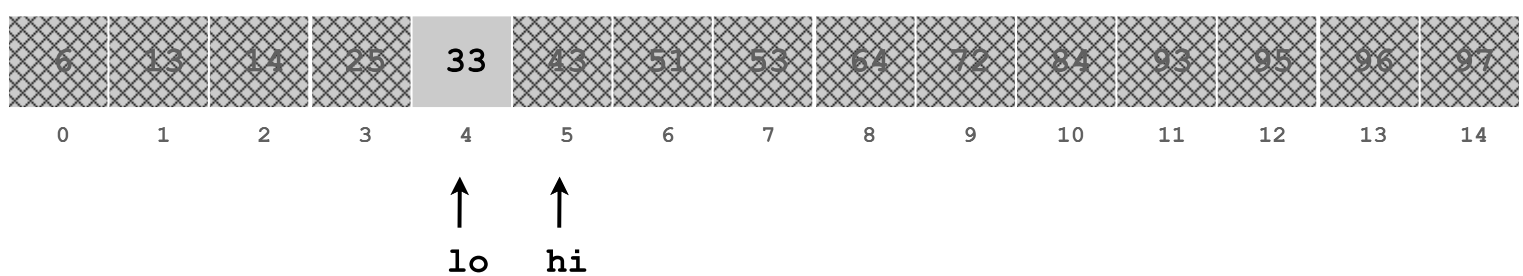

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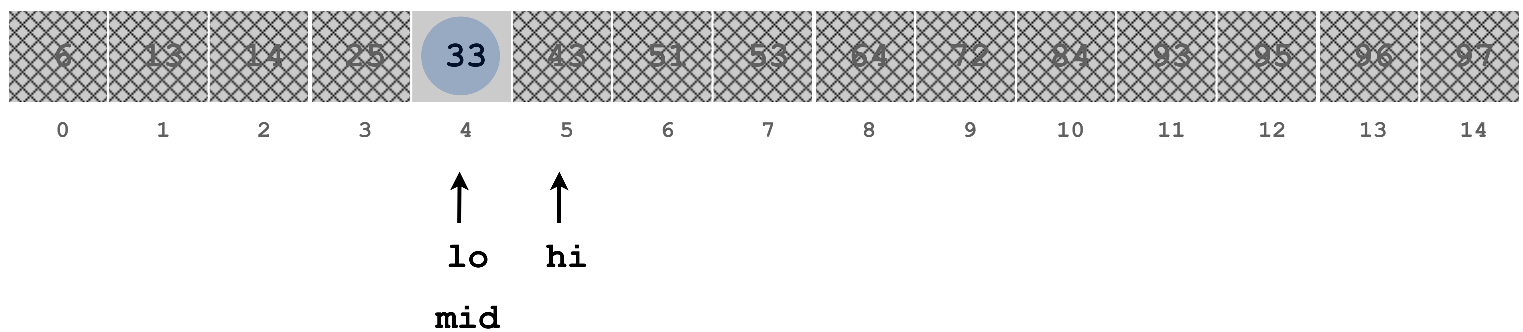
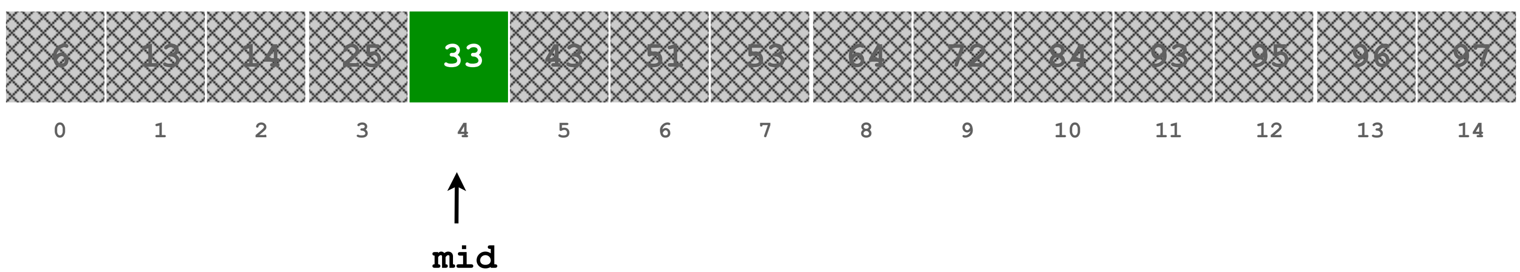


# Binary Search

Binary search. Given a `key` and sorted array `a[]`, find index `i` such that `a[i] = key`, or report that no such index exists.

Invariant. Algorithm maintains `a[lo]` ≤ `key` ≤ `a[hi-1]`.

Ex. Binary search for 33.



# Binary Search: Java Implementation

Invariant. Algorithm maintains `a[lo]` ≤ `key` ≤ `a[hi-1]`.

```
public static int search(String key, String[] a)
{
   return search(key, a, 0, a.length);
}

public static int search(String key, String[] a, int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```

Java library implementation: `Arrays.binarySearch()`

# Binary Search: Mathematical Analysis

**Analysis.** To binary search in an array of size $N$: do one comparison, then binary search in an array of size $N / 2$.

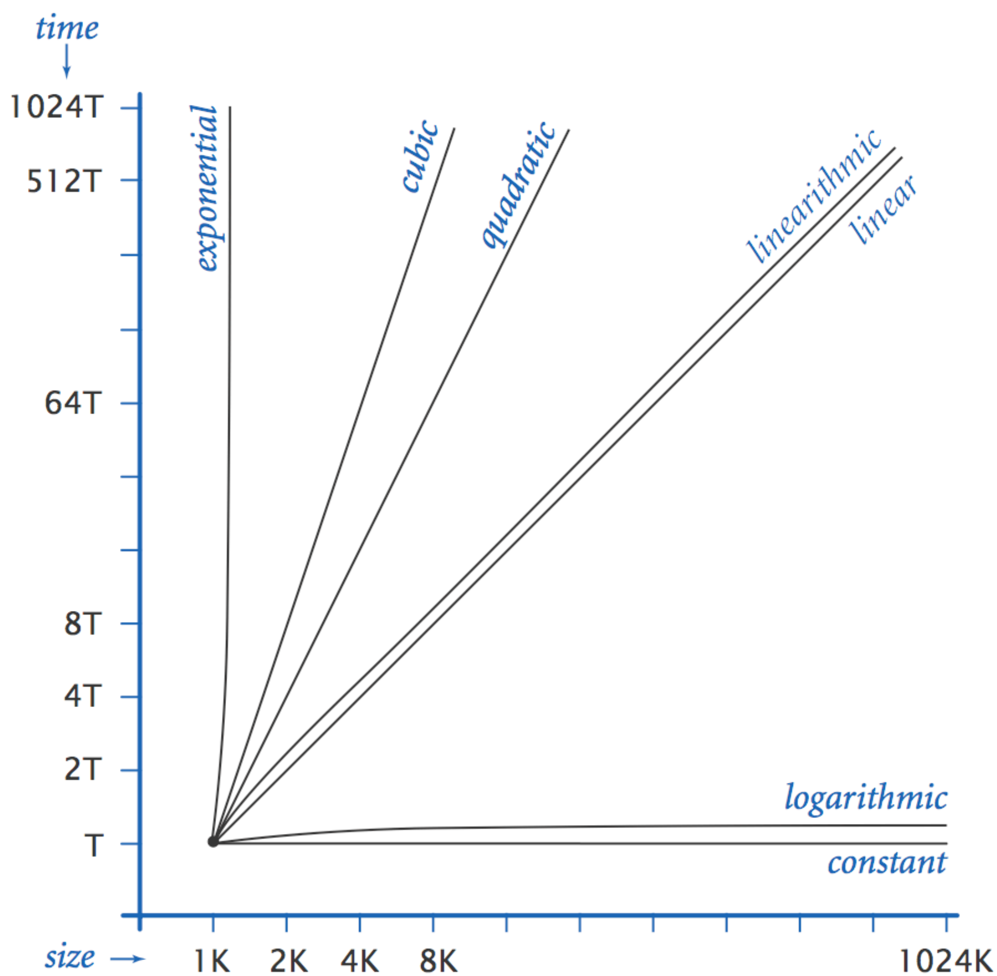
$$N \rightarrow N/2 \rightarrow N/4 \rightarrow N/8 \rightarrow \ldots \rightarrow 1$$

**Q.** How many times can you divide a number by 2 until you reach 1?

**A.** $\log_2 N$.

$$1$$
$$2 \rightarrow 1$$
$$4 \rightarrow 2 \rightarrow 1$$
$$8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$256 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$512 \rightarrow 256 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$1024 \rightarrow 512 \rightarrow 256 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

# Order of Growth Classifications




*Orders of growth (log-log plot)*

| order of growth | | factor for doubling hypothesis |
|---|---|---|
| *description* | *function* | |
| constant | $1$ | 1 |
| logarithmic | $\log N$ | 1 |
| linear | $N$ | 2 |
| linearithmic | $N \log N$ | 2 |
| quadratic | $N^2$ | 4 |
| cubic | $N^3$ | 8 |
| exponential | $2^N$ | $2^N$ |

*Commonly encountered growth functions*

# Order of Growth Classifications

Observation. A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```
while (N > 1) {
   N = N / 2;
   ...
}
```

lg N

lg N = $\log_2 N$

```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

N lg N

```
for (int i = 0; i < N; i++)
   ...
```

N

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

$2^N$

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

$N^2$

# Summary

Q. How can I evaluate the performance of my program?

A. Computational experiments, mathematical analysis

Q. What if it's not fast enough? Not enough memory?

- Understand why.
- Buy a faster computer.
- Learn a better algorithm (COS 226, COS 423).
- Discover a new algorithm.

| attribute | better machine | better algorithm |
| --- | --- | --- |
| cost | $$$ or more. | $ or less. |
| applicability | makes "everything" run faster | does not apply to some problems |
| improvement | incremental quantitative improvements expected | dramatic qualitative improvements possible |