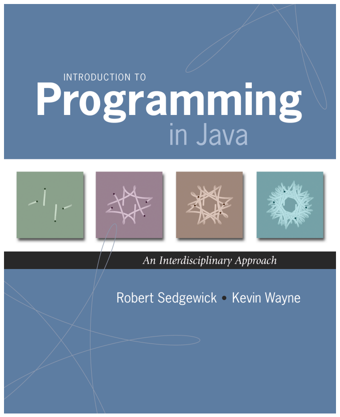
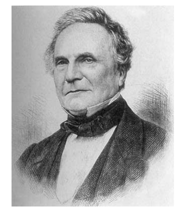
# 4.1, 4.2 Performance and Sorting

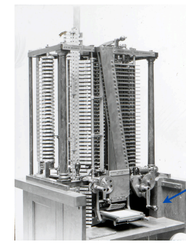



## Running Time

> *"As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time?"* – Charles Babbage



Charles Babbage (1864)



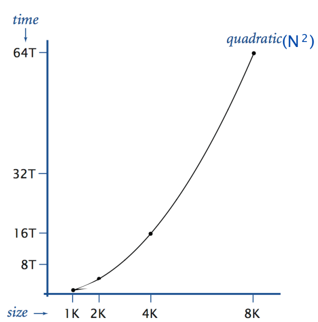
Analytic Engine

how many times do you have to turn the crank?

## Algorithmic Successes

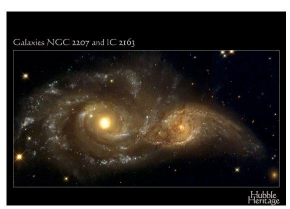
N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: $N^2$ steps.




number of bodies

)

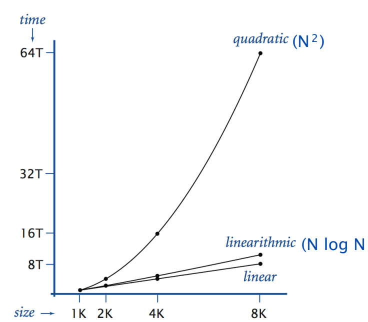



## Algorithmic Successes

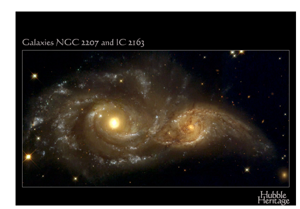
N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: $N^2$ steps.
- Barnes-Hut: N log N steps, enables new research.

Andrew Appel
PU '81




number of bodies

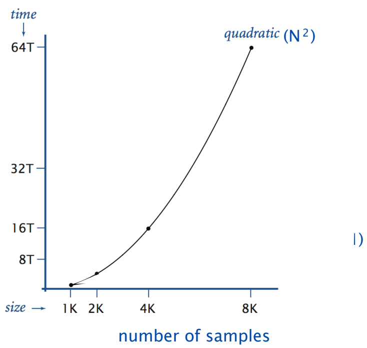


## Algorithmic Successes

Freidrich Gauss
1805

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: $N^2$ steps.

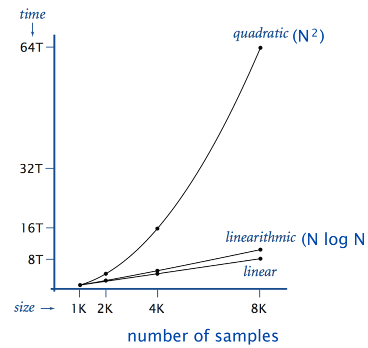



## Algorithmic Successes

John Tukey
1965

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: $N^2$ steps.
- FFT algorithm: N log N steps, enables new technology.




# Sorting

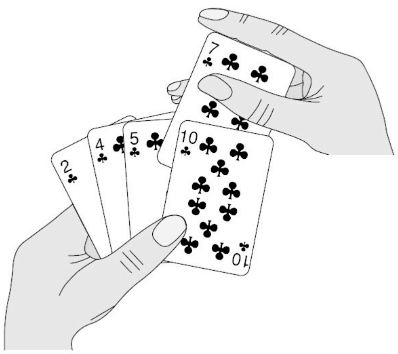
## Sorting

Sorting problem. Rearrange $N$ items in ascending order.

Applications. Binary search, statistics, databases, data compression, bioinformatics, computer graphics, scientific computing, (too numerous to list) ...

| | | |
|---|---|---|
| Hauser | | Hanley |
| Hong | | Haskell |
| Hsu | | Hauser |
| Hayes | → | Hayes |
| Haskell | | Hong |
| Hanley | | Hornet |
| Hornet | | Hsu |

# Insertion Sort



## Insertion Sort

**Insertion sort.**

- Brute-force sorting solution. ← insertion sort is simpler and faster than bubble sort, so we don't teach bubble sort anymore
- Move left-to-right through array.
- Insert each element into final position by exchanging it with larger elements to its left, one-by-one.

| i | j | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 6 | and | had | him | his | was | you | the | but |
| 6 | 5 | and | had | him | his | was | the | you | but |
| 6 | 4 | and | had | him | his | the | was | you | but |
| | | and | had | him | his | the | was | you | but |

*Inserting* `a[6]` *into position by exchanging with larger entries to its left*

## Insertion Sort

**Insertion sort.**

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

| i | j | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|
| | | was | had | him | and | you | his | the | but |
| 1 | 0 | had | was | him | and | you | his | the | but |
| 2 | 1 | had | him | was | and | you | his | the | but |
| 3 | 0 | and | had | him | was | you | his | the | but |
| 4 | 4 | and | had | him | was | you | his | the | but |
| 5 | 3 | and | had | him | his | was | you | the | but |
| 6 | 4 | and | had | him | his | the | was | you | but |
| 7 | 1 | and | but | had | him | his | the | was | you |
| | | and | but | had | him | his | the | was | you |

*Inserting* `a[1]` *through* `a[N-1]` *into position (insertion sort)*

## Insertion Sort: Java Implementation

```
public class Insertion
{

   public static void sort(String[] a)
   {
      int N = a.length;
      for (int i = 1; i < N; i++)
         for (int j = i; j > 0; j--)
            if (a[j-1] > a[j])
               exch(a, j-1, j);
            else break;
   }


   private static void exch(String[] a, int i, int j)
   {
      String swap = a[i];
      a[i] = a[j];
      a[j] = swap;
   }
}
```

## Insertion Sort: Observation

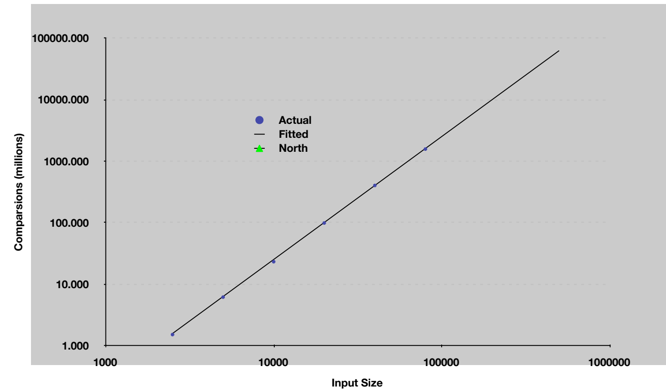
Observe and tabulate running time for various values of N.

- Data source: N random numbers between 0 and 1.
- Machine: Apple G5 1.8GHz with 1.5GB memory running OS X.
- Timing: Skagen wristwatch.

| N | Comparisons | Time |
|---|---|---|
| 5,000 | 6.2 million | 0.13 seconds |
| 10,000 | 25 million | 0.43 seconds |
| 20,000 | 99 million | 1.5 seconds |
| 40,000 | 400 million | 5.6 seconds |
| 80,000 | 1600 million | 23 seconds |

## Insertion Sort: Empirical Analysis

Data analysis. Plot # comparisons vs. input size on log-log scale.



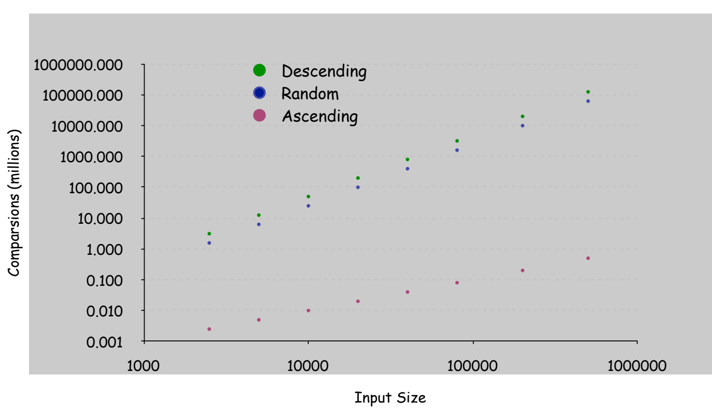

slope

Hypothesis. # comparisons grows quadratically with input size $\sim N^2/4$.

## Insertion Sort: Empirical Analysis

Observation. Number of compares depends on input family.

- Descending: $\sim N^2/2$.
- Random: $\sim N^2/4$.
- Ascending: $\sim N$.




## Analysis: Empirical vs. Mathematical

Empirical analysis.

- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

Mathematical analysis.

- Analyze algorithm to estimate # ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

Critical difference. Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.

## Insertion Sort: Mathematical Analysis

Worst case. [descending]

- Iteration $i$ requires $i$ comparisons.
- Total = $(0 + 1 + 2 + ... + N\text{-}1) \sim N^2 / 2$ compares.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| E | F | G | H | I | J | D | C | B | A |
| | | | | | | $i$ | | | |

Average case. [random]

- Iteration $i$ requires $i / 2$ comparisons on average.
- Total = $(0 + 1 + 2 + ... + N\text{-}1) / 2 \sim N^2 / 4$ compares

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | C | D | F | H | J | E | B | I | G |
| | | | | | | $i$ | | | |

## Insertion Sort: Lesson

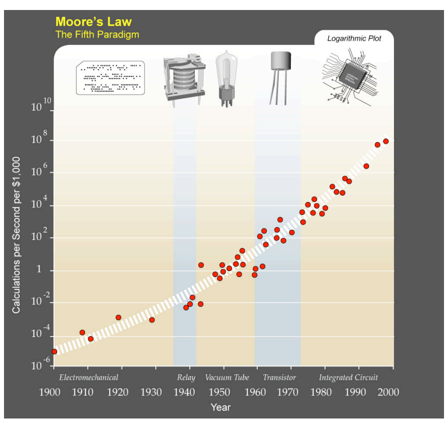
Lesson. Supercomputer can't rescue a bad algorithm.

| Computer | Comparisons Per Second | Thousand | Million | Billion |
|---|---|---|---|---|
| laptop | $10^7$ | instant | 1 day | 3 centuries |
| super | $10^{12}$ | instant | 1 second | 2 weeks |

## Moore's Law

Moore's law. Transistor density on a chip doubles every 2 years.

Variants. Memory, disk space, bandwidth, computing power per $.




http://en.wikipedia.org/wiki/Moore's_law

## Moore's Law and Algorithms

Quadratic algorithms do not scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

*"Software inefficiency can always outpace Moore's Law. Moore's Law isn't a match for our bad coding."* – *Jaron Lanier*

Lesson. Need linear (or linearithmic) algorithm to keep pace with Moore's law.

## Announcements

Exam 1 looms.

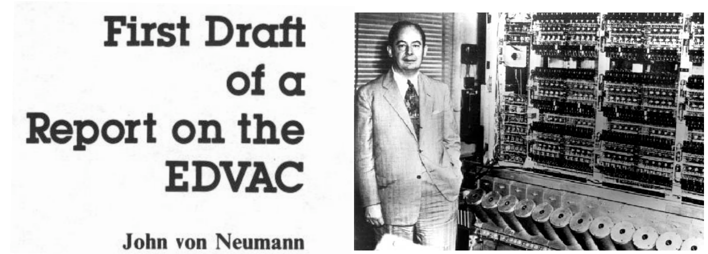
Written exam Tuesday 3/13 during your lecture time. Room TBD.

Programming exam Tuesday 3/13 or Wednesday 3/14 in your precept.

Review session will be held.

Rooms, rules, details on Exams page of website.

# Mergesort




## Mergesort

Mergesort.
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

*input*
```
was had him and you his the but
```
*sort left*
```
and had him was you his the but
```
*sort right*
```
and had him was but his the you
```
*merge*
```
and but had him his the was you
```

## Mergesort: Example

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **M** | **E** | **R** | **G** | **E** | **S** | **O** | **R** | **T** | **E** | **X** | **A** | **M** | **P** | **L** | **E** |
| **E** | **M** | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| E | M | **G** | **R** | E | S | O | R | T | E | X | A | M | P | L | E |
| **E** | **G** | **M** | **R** | E | S | O | R | E | T | A | X | M | P | E | L |
| E | M | G | R | **E** | **S** | O | R | T | E | X | A | M | P | L | E |
| E | M | G | R | E | S | **O** | **R** | T | E | X | A | M | P | L | E |
| E | G | M | R | **E** | **O** | **R** | **S** | E | T | A | X | M | P | E | L |
| **E** | **E** | **G** | **M** | **O** | **R** | **R** | **S** | A | E | T | X | E | L | M | P |
| E | M | G | R | E | S | O | R | **E** | **T** | X | A | M | P | L | E |
| E | M | G | R | E | S | O | R | E | T | **A** | **X** | M | P | L | E |
| E | G | M | R | E | O | R | S | **A** | **E** | **T** | **X** | M | P | E | L |
| E | M | G | R | E | S | O | R | E | T | A | X | **M** | **P** | L | E |
| E | M | G | R | E | S | O | R | E | T | A | X | M | P | **E** | **L** |
| E | G | M | R | E | O | R | S | A | E | T | X | **E** | **L** | **M** | **P** |
| E | E | G | M | O | R | R | S | **A** | **E** | **E** | **L** | **M** | **P** | **T** | **X** |
| **A** | **E** | **E** | **E** | **E** | **G** | **L** | **M** | **M** | **O** | **P** | **R** | **R** | **S** | **T** | **X** |

*Top-down mergesort*

## Merging

Merging. Combine two pre-sorted lists into a sorted whole.

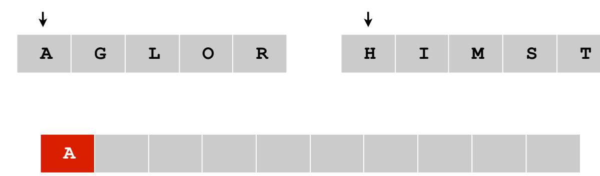
How to merge efficiently? Use an auxiliary array.

| i | j | k | aux[k] | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | and | had | him | was | but | his | the | you |
| 0 | 4 | 0 | and | and | had | him | was | but | his | the | you |
| 1 | 4 | 1 | but | and | had | him | was | but | his | the | you |
| 1 | 5 | 2 | had | and | had | him | was | but | his | the | you |
| 2 | 5 | 3 | him | and | had | him | was | but | his | the | you |
| 3 | 5 | 4 | his | and | had | him | was | but | his | the | you |
| 3 | 6 | 5 | the | and | had | him | was | but | his | the | you |
| 3 | 6 | 6 | was | and | had | him | was | but | his | the | you |
| 4 | 7 | 7 | you | and | had | him | was | but | his | the | you |

*Trace of the merge of the sorted left half with the sorted right half*

## Merging

Merge.
- Keep track of smallest element in each sorted half.
- Choose smaller of two elements.
- Repeat until done.

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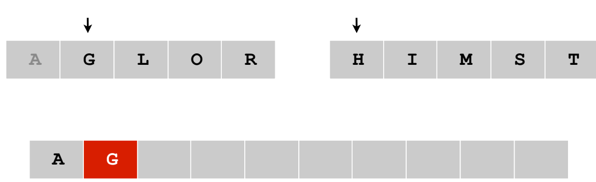


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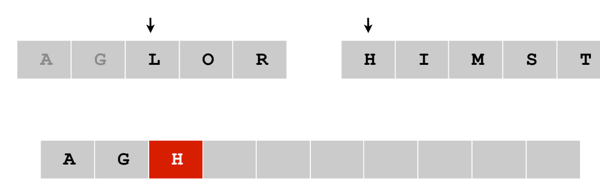

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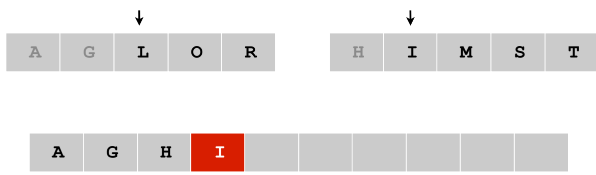


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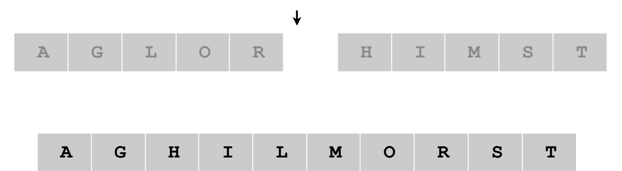



## Merging

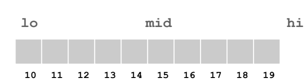
Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

```
String[] aux = new String[N];
// Merge into auxiliary array.
int i = lo, j = mid;
for (int k = 0; k < N; k++)
{
   if      (i == mid)  aux[k] = a[j++];
   else if (j == hi)   aux[k] = a[i++];
   else if (a[j].compareTo(a[i]) < 0) aux[k] = a[j++];
   else                               aux[k] = a[i++];
}

// Copy back.
for (int k = 0; k < N; k++)
   a[lo + k] = aux[k];
```

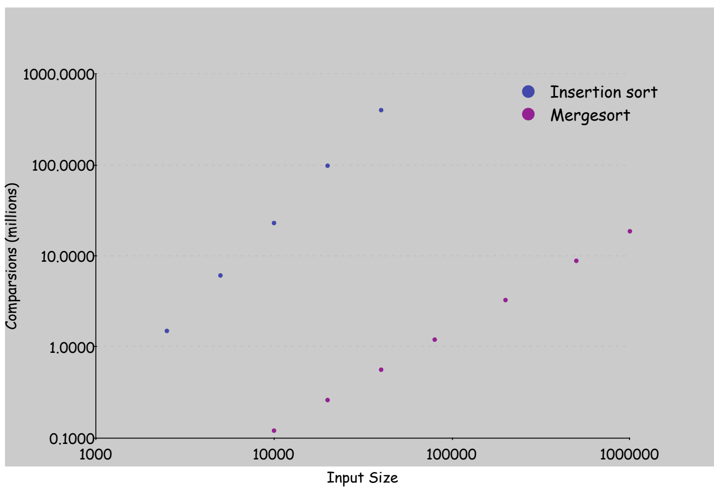
## Mergesort: Java Implementation

```
public class Merge
{
   public static void sort(String[] a)
   {  sort(a, 0, a.length);  }

   // Sort a[lo, hi).
   public static void sort(String[] a, int lo, int hi)
   {
      int N = hi - lo;
      if (N <= 1) return;

      // Recursively sort left and right halves.
      int mid = lo + N/2;
      sort(a, lo, mid);
      sort(a, mid, hi);

      // Merge sorted halves (see previous slide).
   }

}
```



## Mergesort: Empirical Analysis

Experimental hypothesis. Number of comparisons ≈ 20N.




## Mergesort: Prediction and Verification

Experimental hypothesis. Number of comparisons ≈ 20N.

Prediction. 80 million comparisons for N = 4 million.

Observations.

| N | Comparisons | Time |
|---|---|---|
| 4 million | 82.7 million | 3.13 sec |
| 4 million | 82.7 million | 3.25 sec |
| 4 million | 82.7 million | 3.22 sec |

Agrees.

Prediction. 400 million comparisons for N = 20 million.

Observations.

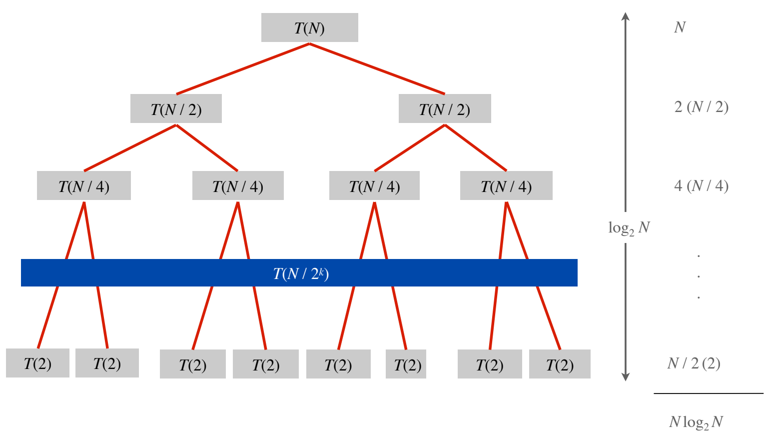
| N | Comparisons | Time |
|---|---|---|
| 20 million | 460 million | 17.5 sec |
| 50 million | 1216 million | 45.9 sec |

Not quite.

## Mergesort: Mathematical Analysis

Analysis. To mergesort array of size $N$, mergesort two subarrays of size $N / 2$, and merge them together using ≤ $N$ comparisons.

we assume $N$ is a power of 2




## Mergesort: Mathematical Analysis

Mathematical analysis.

| analysis | comparisons |
|---|---|
| worst | $N \log_2 N$ |
| average | $N \log_2 N$ |
| best | $1/2\ N \log_2 N$ |

Validation. Theory agrees with observations.

| N | actual | predicted |
|---|---|---|
| 10,000 | 120 thousand | 133 thousand |
| 20 million | 460 million | 485 million |
| 50 million | 1,216 million | 1,279 million |

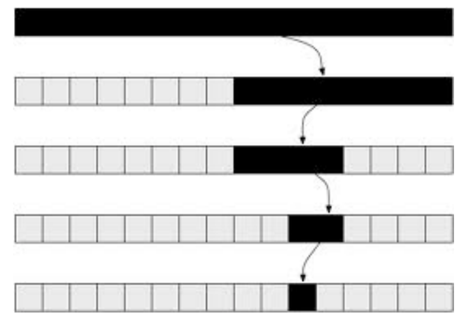
## Mergesort: Lesson

Lesson. Great algorithms can be more powerful than supercomputers.

| Computer | Comparisons Per Second | Insertion | Mergesort |
|---|---|---|---|
| laptop | $10^7$ | 3 centuries | 3 hours |
| super | $10^{12}$ | 2 weeks | instant |

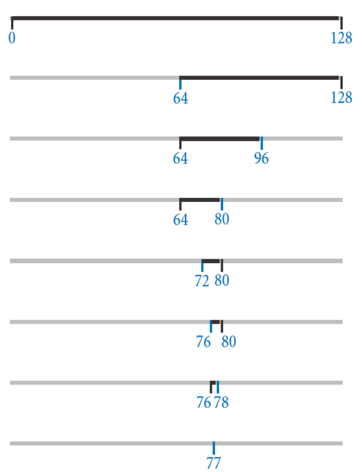
N = 1 billion

# Binary Search



## Twenty Questions

Intuition. Find a hidden integer.



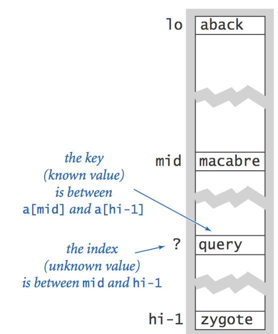

| interval | size | Q | A |
|---|---|---|---|
| 0–128 | 128 | < 64 ? | *no* |
| 64–128 | 64 | < 96 ? | *yes* |
| 64–96 | 32 | < 80 ? | *yes* |
| 64–80 | 16 | < 72 ? | *no* |
| 72–80 | 8 | < 76 ? | *no* |
| 76–80 | 4 | < 78 ? | *yes* |
| 76–78 | 2 | < 77 ? | *no* |
| 77 | 1 | = 77 | |

## Binary Search

Idea:
- Sort the array (stay tuned)
- Play "20 questions" to determine the index associated with a given key.

Ex. Dictionary, phone book, book index, credit card numbers, …

Binary search.
- Examine the middle key.
- If it matches, return its index.
- Otherwise, search either the left or right half.



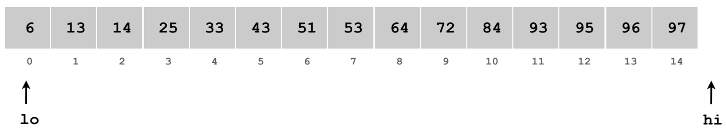

*Binary search in a sorted array (one step)*

## Binary Search

Binary search. Given a `key` and sorted array `a[]`, find index `i` such that `a[i] = key`, or report that no such index exists.

Invariant. Algorithm maintains `a[lo] ≤ key ≤ a[hi-1]`.

Ex. Binary search for 33.

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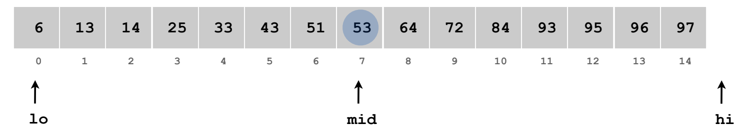


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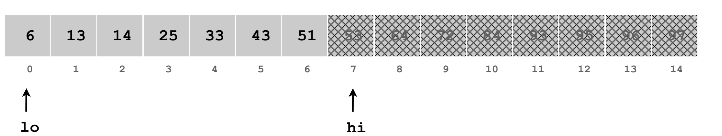


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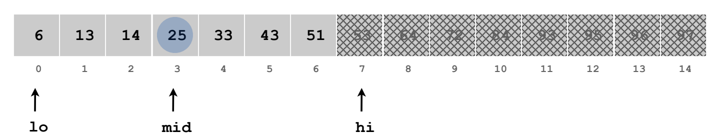
Ex. Binary search for 33.



## Binary Search

Binary search. Given a `key` and sorted array `a[]`, find index `i` such that `a[i] = key`, or report that no such index exists.

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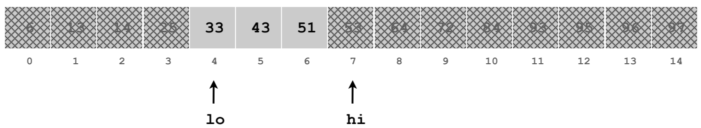


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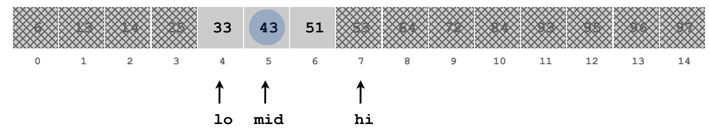


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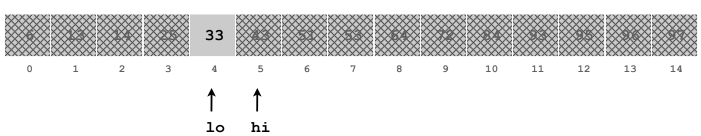


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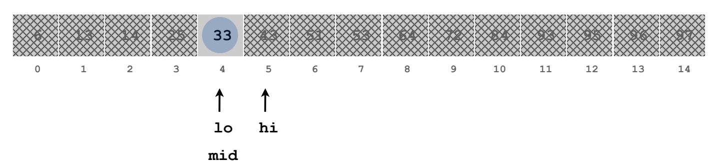

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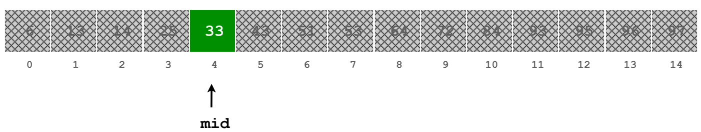



## Binary Search: Java Implementation

Invariant. Algorithm maintains **`a[lo]`** ≤ **`key`** ≤ **`a[hi-1]`**.

```
public static int search(String key, String[] a)
{
   return search(key, a, 0, a.length);
}

public static int search(String key, String[] a, int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```

Java library implementation: **`Arrays.binarySearch()`**

## Binary Search: Mathematical Analysis

Analysis. To binary search in an array of size $N$: do one comparison, then binary search in an array of size $N / 2$.

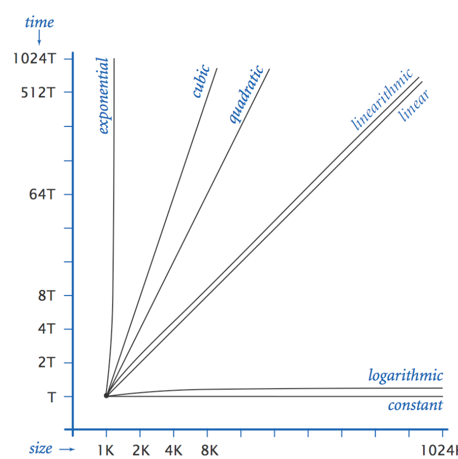
$$N \to N/2 \to N/4 \to N/8 \to \ldots \to 1$$

Q. How many times can you divide a number by 2 until you reach 1?

A. $\log_2 N$.

$$1$$
$$2 \to 1$$
$$4 \to 2 \to 1$$
$$8 \to 4 \to 2 \to 1$$
$$16 \to 8 \to 4 \to 2 \to 1$$
$$32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$1024 \to 512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$

## Order of Growth Classifications




*Orders of growth (log-log plot)*

| order of growth: description | order of growth: function | factor for doubling hypothesis |
|---|---|---|
| constant | $1$ | 1 |
| logarithmic | $\log N$ | 1 |
| linear | $N$ | 2 |
| linearithmic | $N \log N$ | 2 |
| quadratic | $N^2$ | 4 |
| cubic | $N^3$ | 8 |
| exponential | $2^N$ | $2^N$ |

*Commonly encountered growth functions*

## Order of Growth Classifications

Observation. A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```
while (N > 1) {
   N = N / 2;
   ...
}
```

lg N

lg N = $\log_2$ N

```
for (int i = 0; i < N; i++)
   ...
```

N

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

$N^2$

```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

N lg N

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

$2^N$

## Summary

Q. How can I evaluate the performance of my program?
A. Computational experiments, mathematical analysis

Q. What if it's not fast enough? Not enough memory?

- Understand why.
- Buy a faster computer.
- Learn a better algorithm (COS 226, COS 423).
- Discover a new algorithm.

| attribute | better machine | better algorithm |
|---|---|---|
| cost | $$$ or more. | $ or less. |
| applicability | makes "everything" run faster | does not apply to some problems |
| improvement | incremental quantitative improvements expected | dramatic qualitative improvements possible |