Collaboration Policy: You may collaborate with other students on these problems. Collaboration is limited to discussion of ideas only, and you should write up the solutions entirely on your own and list your collaborators as well as cite any references you may have used.

1. Problem 6.5 in Williamson-Shmoys.

2. Problem 7.5 in Williamson-Shmoys.

3. Consider the following optimization problem with robust conditions:

\[ \min \{ c^T x : x \in \mathbb{R}^n; Ax \geq b \text{ for any } A \in F \}, \]

where \( b \in \mathbb{R}^m \) and \( F \) is a set of \( m \times n \) matrices:

\[ F = \{ A; \forall i, j; a_{ij}^{\min} \leq a_{ij} \leq a_{ij}^{\max} \}. \]

(a) Considering \( F \) as a polytope in \( \mathbb{R}^{m \times n} \), what are the vertices of \( F \)?

(b) Show that instead of conditions for all \( A \in F \), it is enough to consider the vertices of \( F \). Write the resulting linear program. What is its size? Is this polynomial in the size of the input namely \( m, n \) and the sizes of \( b, c, a_{ij}^{\min} \) and \( a_{ij}^{\max} \)?

(c) Derive a more efficient description of the linear program: Write the conditions on \( x \) given by one row of \( A \), for all choices of \( A \). Formulate the condition as a linear program. Use duality and formulate the original problem as a linear program. What is the size of this one? Is this polynomial in the size of the input?

4. The goal of this exercise is to prove von Neumann’s minimax theorem via LP duality. Given an \( n \times m \) payoff matrix \( A \) for a zero-sum game, let \( \Delta_n \) be the set of mixed strategies for the row player and \( \Delta_m \) be the set of mixed strategies for the column player, i.e.

\[ \Delta_n = \{ x \in \mathbb{R}^n | x_i \geq 0, \sum x_i = 1 \} \]

\[ \Delta_m = \{ y \in \mathbb{R}^m | y_i \geq 0, \sum y_i = 1 \} \]

(A mixed strategy is a probability distribution over the choices of the player). We want to prove von Neumann’s minimax theorem,

\[ \min_{x \in \Delta_n} \max_{y \in \Delta_m} y^T A x = \max_{y \in \Delta_m} \min_{x \in \Delta_n} y^T A x \]
(a) First show that the optimal response for the column player is to pick a single column (i.e. a pure strategy) and similarly for the row player. In other words, show that:

\[ \min_{x \in \Delta_n} \max_{y \in \Delta_m} y^T Ax = \min_{x \in \Delta_n} \max_{i \in \{1 \ldots m\}} A_i x \]

\[ \max_{y \in \Delta_m} \min_{x \in \Delta_n} y^T Ax = \max_{y \in \Delta_m} \min_{j \in \{1 \ldots n\}} y^T A_j \]

Hence we have reduced the problem to proving

\[ \min_{x \in \Delta_n} \max_{i \in \{1 \ldots m\}} A_i x = \max_{y \in \Delta_m} \min_{j \in \{1 \ldots n\}} y^T A_j \]

(b) Prove the preceding equality via LP duality.

5. Suppose we wish to solve the following flow problem: There are \( n \) nodes in an undirected graph, and the edges should be thought of as pipes of a certain capacity. For each pair \( \{i, j\} \) we wish to send 1 unit of flow between them. All these flows must be routed through the pipes and should not violate any capacity. Let \( z \) be the minimum number such that if all edge pipes have capacity \( z \) then the flows can be routed in the network.

(a) Express the problem of finding \( z \) as a linear program, and argue that it can be solved in polynomial time.

(b) Use the multiplicative weights method to design a algorithm that solves the above linear program approximately. How long does your algorithm take to find \( z \) correctly up to an additive error \( \epsilon > 0 \) ?