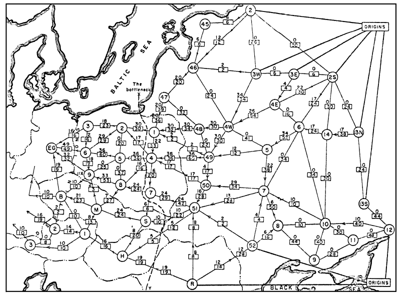
6.4 Maximum Flow



- overview
- Ford-Fulkerson
- implementations
- applications

Mincut problem

Given. A weighted digraph with identified source s and target t.

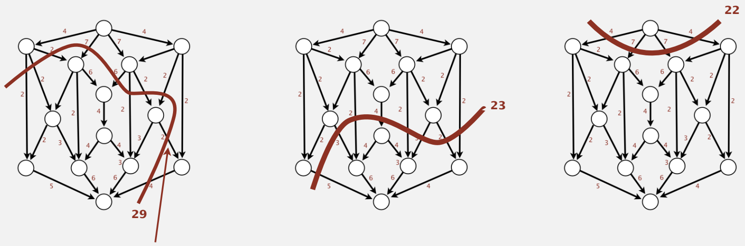
Def. A cut is a partition of the vertices into two disjoint sets.

Def. An st-cut is a cut that places s in one of its sets (C_s) and t in the other (C_t).

Def. An st-cut's weight is the sum of the weights of its st-crossing edges.

edges from C_s to C_t

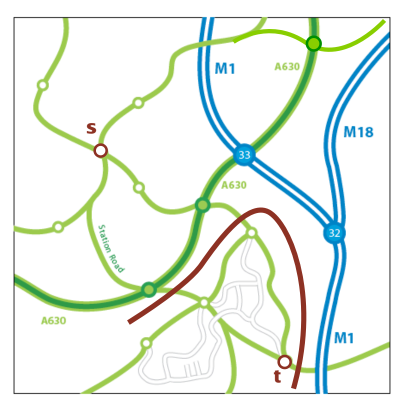
Minimum st-cut (mincut) problem. Find an st-cut of minimal weight.



Note: don't count edges from C_t to C_s

Typical mincut application

Find cheapest way to cut connection between s and t.



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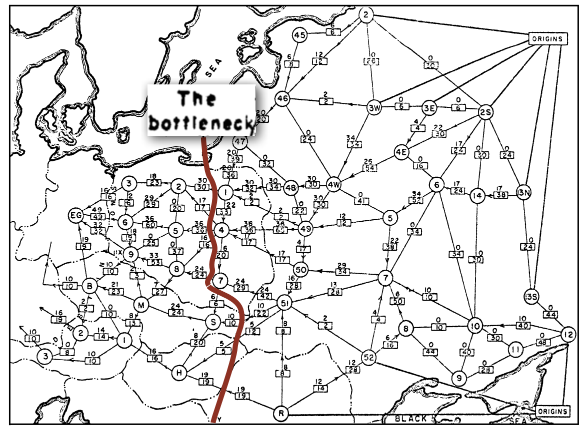
Mincut application (1950s)

Rail network connecting Soviet Union with Eastern European countries



"Free world" goal: Know how to cut supplies if cold war turns into real war (map declassified by Pentagon in 1999).

Potential mincut application (2010s)

Facebook graph

Government-in-power's goal: Cut off communication to specified set of people.

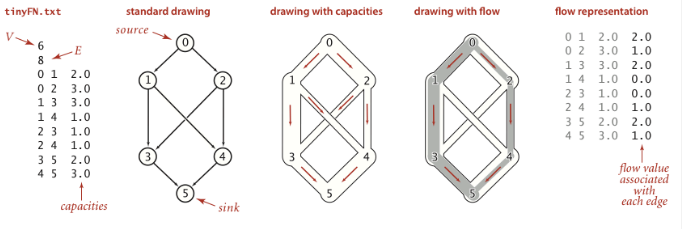
Maxflow problem

Flow network.

- Weighted digraph with a source s (indegree 0) and sink t (outdegree 0)
- An edge's weight is its capacity (positive)
- Add additional flow variable to each edge(no greater than its capacity)

Maximum st-flow (maxflow) problem: Assign flows to edges that

- Maintain local equilibrium: inflow = outflow at every vertex (except s and t).
- Maximize total flow into t.



tinyFN.txt

V
E

```
6
8
0 1  2.0
0 2  3.0
1 3  3.0
1 4  1.0
2 3  1.0
2 4  1.0
3 5  2.0
4 5  3.0
```

capacities

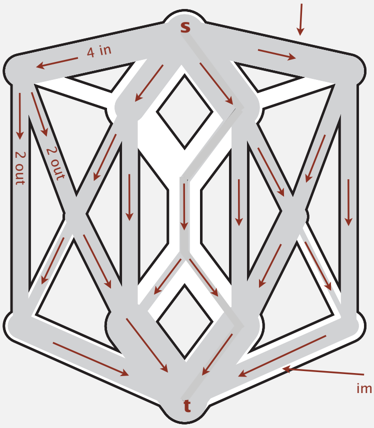
flow representation

```
0 1  2.0  2.0
0 2  3.0  1.0
1 3  3.0  2.0
1 4  1.0  0.0
2 3  1.0  0.0
2 4  1.0  1.0
3 5  2.0  2.0
4 5  3.0  1.0
```

flow value associated with each edge

A physical model

Oil flowing in pipes



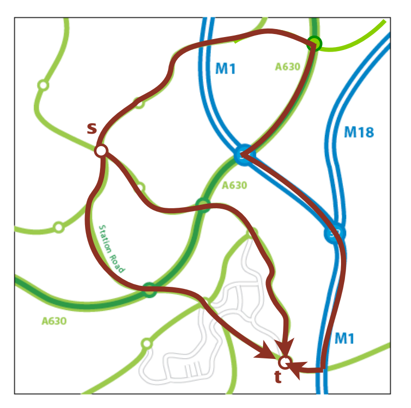
all flow goes down
(just to simplify diagrams)

local equilibrium
at each vertex

room for
improvement?

Typical maxflow application

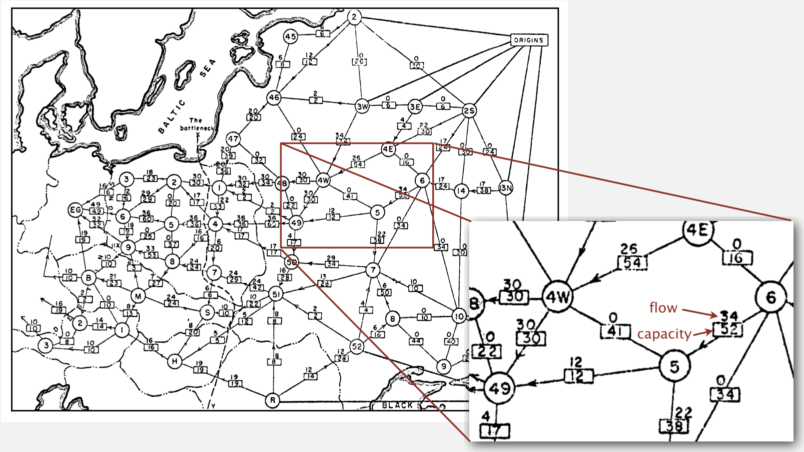
Find best way to distribute goods from s to t.



www.blog.spoongraphics.co.uk/tutorials/creating-road-maps-in-adobe-illustrator

Maxflow application (1950s)

Rail network connecting Soviet Union with Eastern European countries



Soviet Union goal: Know how to maximize flow of supplies to Eastern Europe.

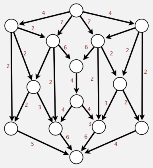
Potential mincut application (2010s)

Facebook graph

"Free world" goal: Maximize flow of information to specified set of people.

Overview (summary)

Given. A weighted digraph,
source s and target t.



Minimum st-cut (mincut) problem. Find an st-cut of minimal weight.

Maximum st-flow (maxflow) problem: Assign flows to edges that

- Maintain local equilibrium: inflow = outflow at every vertex (except s and t).
- Maximize total flow into t.

Remarkable fact. These two problems are equivalent!

- Two very rich algorithmic problems
- Cornerstone problems in combintorial optimization
- Beautiful mathematical duality

Maxflow / mincut applications

Maxflow/mincut is a widely applicable problem-solving model

- Data mining.
- Open-pit mining.
- Project selection.
- Image processing.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network connectivity/reliability.
- Many many more . . .

- overview
- **APIs**
- Ford Fulkerson
- implementations
- applications

APIs (cf. `EdgeWeightedDigraph`, `DirectedEdge`)

public class	FlowNetwork	
	`FlowNetwork(int V)`	*empty V-vertex flow network*
	`FlowNetwork(In in)`	*construct from input stream*
`int`	`V()`	*number of vertices*
`int`	`E()`	*number of edges*
`void`	`addEdge(FlowEdge e)`	*add e to this flow network*
`Iterable<FlowEdge>`	`adj(int v)`	*edges pointing from v*
`Iterable<FlowEdge>`	`edges()`	*all edges in this flow network*
`String`	`toString()`	*string representation*

Flow network API

public class	FlowEdge	
	`FlowEdge(int v, int w, double cap)`	
`int`	`from()`	*vertex this edge points from*
`int`	`to()`	*vertex this edge points to*
`int`	`other(int v)`	*other endpoint*
`double`	`capacity()`	*capacity of this edge*
`double`	`flow()`	*flow in this edge*
`double`	`residualCapacityTo(int v)`	*residual capicity toward v*
`double`	`addFlowTo(int v, double delta)`	*add delta flow toward v*
`String`	`toString()`	*string representation*

manipulate flow values
(stay tuned)

Flow edge: implementation in Java (cf. `DirectedEdge`)

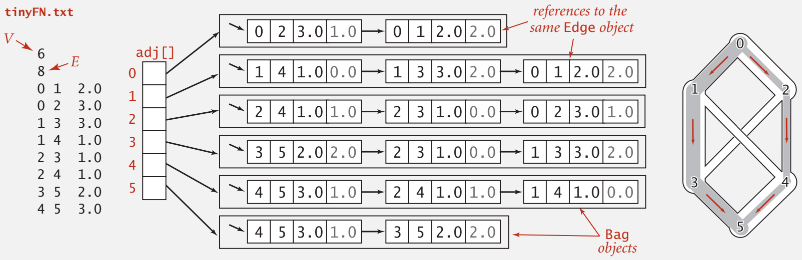
```
public class FlowEdge
{
    private final int v;                // from
    private final int w;                // to
    private final double capacity;      // capacity
    private double flow;                // flow

    public FlowEdge(int v, int w, double capacity, double flow)
    {
        this.v          = v;
        this.w          = w;
        this.capacity   = capacity;
        this.flow       = flow;
    }

    public int from()          { return v;         }
    public int to()            { return w;         }
    public double capacity()   { return capacity;  }
    public double flow()       { return flow;      }

    public int other(int vertex)
    {
        if      (vertex == v) return w;
        else if (vertex == w) return v;
        else throw new RuntimeException("Illegal endpoint");
    }

    public double residualCapacityTo(int vertex)          {...}
    public void addResidualFlowTo(int vertex, double delta) {...}
}
```

flow variable

stay tuned

Flow network representation

A flow network is an array of bags of flow edges.



Flow network representation

Flow network: implementation in Java (cf. `EdgeWeightedDigraph`)

```
public class FlowNetwork
{
    private final int V;
    private int E;
    private Bag<FlowEdge>[] adj;

    public FlowNetwork(int V)
    {
        this.V = V;
        this.E = 0;
        adj = (Bag<FlowEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<FlowEdge>();
    }

    public int V() { return V; }
    public int E() { return E; }

    public void addEdge(FlowEdge e)
    {
        E++;
        int v = e.from();
        int w = e.to();
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<FlowEdge> adj(int v)
    {  return adj[v]; }

}
```

array of bags of flow edges

constructor

add edge (to both adj lists)

iterator for adjacent edges

Typical client code: check that a flow is feasible

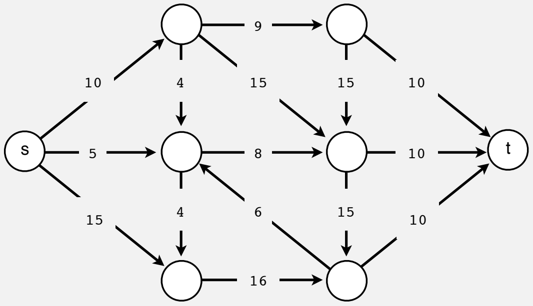
```
private boolean localEq(FlowNetwork G, int v)
{  // Check local equilibrium at v.
   double EPSILON = 1E-11;
   double netflow = 0.0;
   for (FlowEdge e : G.adj(v))
      if (v == e.from()) netflow -= e.flow();
      else               netflow += e.flow();
   return Math.abs(netflow) < EPSILON;
}
private boolean isFeasible(FlowNetwork G)
{
   for (int v = 0; v < G.V(); v++)
     for (FlowEdge e : G.adj(v))
        if (e.flow() < 0 || e.flow() > e.capacity())
            return false;

   for (int v = 0; v < G.V(); v++)
      if (v !=s && v != t && !localEq(G, v))
         return false;

   return true;
}
```

check that each flow
is nonnegative
and no greater than capacity

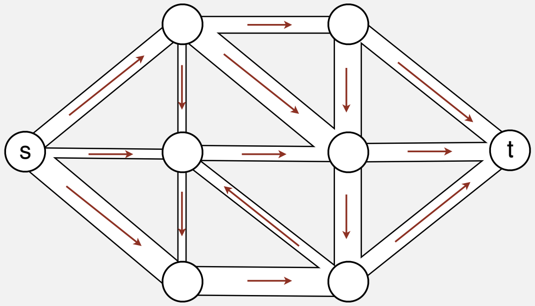
check local equilibrium
at each vertex

- APIs
- **Ford Fulkerson**
- implementations
- applications

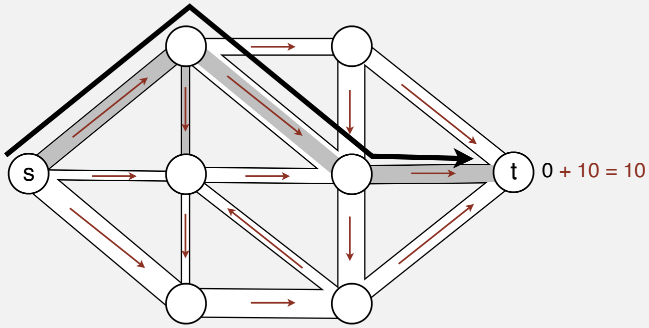
Idea: increase flow along augmenting paths



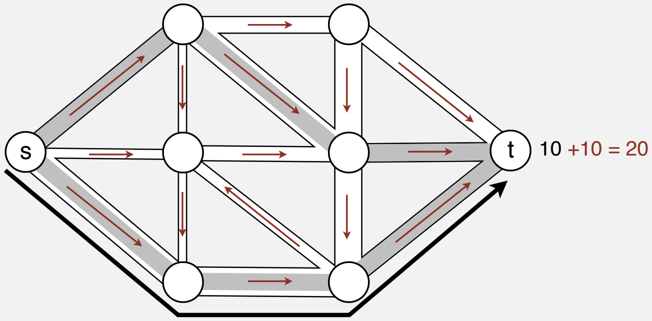
Idea: increase flow along augmenting paths



Idea: increase flow along augmenting paths

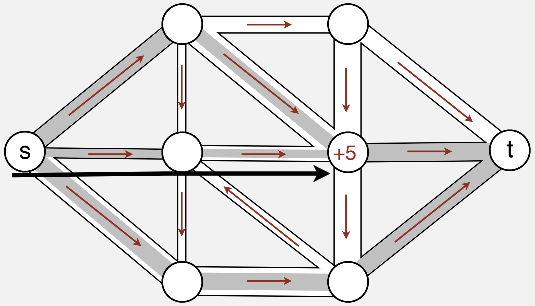


Idea: increase flow along augmenting paths



Idea: increase flow along augmenting paths

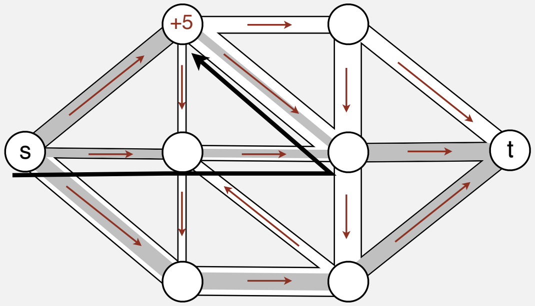
Problem: Can get stuck with no way to add more flow to t.



Idea: increase flow along augmenting paths

Problem: Can get stuck with no way to add more flow to t.

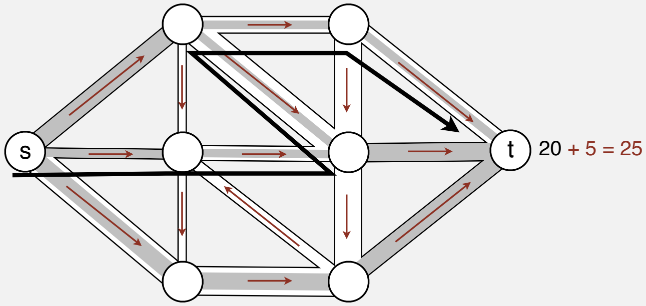
Solution: Go backwards along an edge with flow (removing some flow).



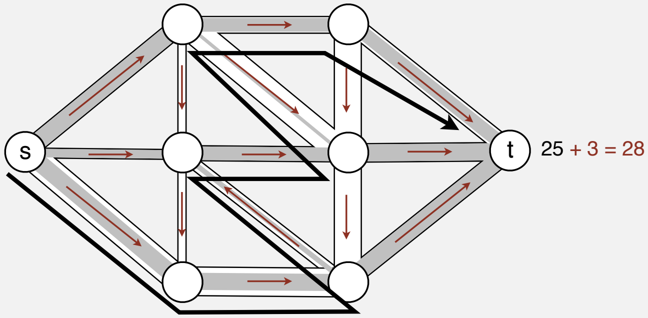
Idea: increase flow along augmenting paths

Augmenting paths in general

- increase flow on forward edge (if not full)
- decrease flow on backward edge (if not empty)



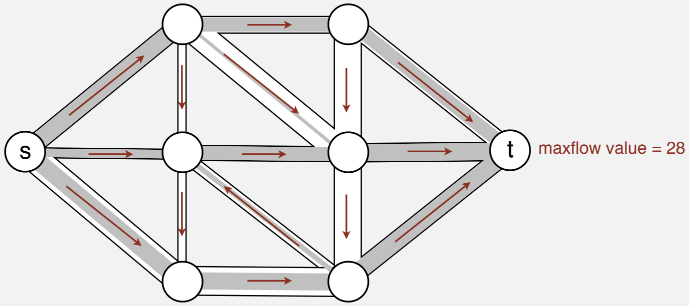
Idea: increase flow along augmenting paths



Idea: increase flow along augmenting paths

Eventually all paths from s to t are blocked by either a

- full forward edge
- empty backward edge



Ford-Fulkerson algorithm

Generic method for solving maxflow problem.
- Start with 0 flow everywhere.
- Find an augmenting path.
- Increase the flow on that path, by as much as possible.
- Repeat until no augmenting paths are left.

Questions.

Q. Does this process give a maximum flow?

A. Yes! It also finds a mincut (!!). [Classic result]

Q. How do we find an augmenting path?

A. Easy. Adapt standard graph-searching methods.

Q. How many augmenting paths (does the process even terminate)?

A. Difficult to know: depends on graph model, search method.

Mincut problem (revisited)

Given. A weighted digraph with identified source s and target t.

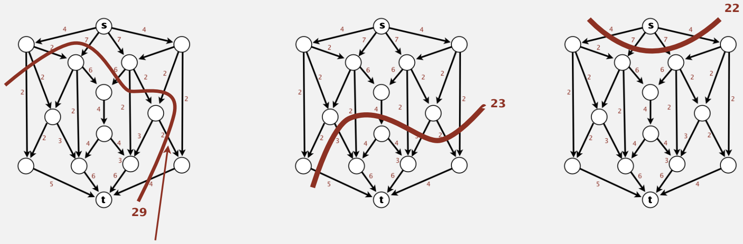
Def. A cut is a partition of the vertices into two disjoint sets.

Def. An st-cut is a cut that places s in one of its sets (C_s) and t in the other (C_t).

Def. An st-cut's weight is the sum of the weights of its st-crossing edges.

edges from C_s to C_t

Mincut problem. Find an st-cut of minimal weight.



Note: don't count edges from C_t to C_s

Mincut problem (revisited with slight change in terminology)

Given. A flow network with identified source s and target t.

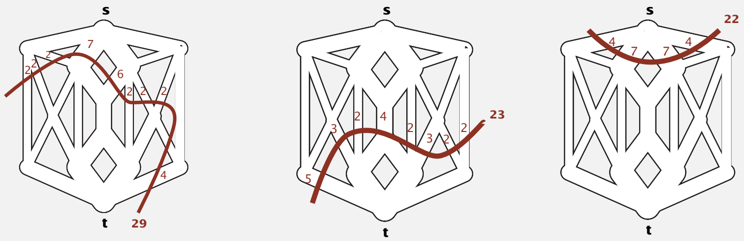
Def. A cut is a partition of the vertices into two disjoint sets.

Def. An st-cut is a cut that places s in one of its sets (C_s) and t in the other (C_t).

Def. An st-cut's capacity is the sum of the capacities of its st-crossing edges.

edges from C_s to C_t

Mincut problem. Find an st-cut of minimal capacity.



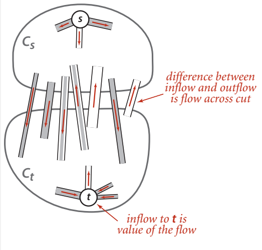
Amazing fact. Mincut and maxflow problems are equivalent.

Relationship between flows and cuts

Def. The flow across an st-cut is the sum of the flows on its st-crossing edges minus the sum of the flows of its ts-crossing edges.

Thm. For any st-flow, the flow across every st-cut equals the value of the flow.

Pf. By induction on the size of C_t.
- true when $C_t = \{t\}$.
- true by local equilibrium when moving a vertex from C_s to C_t



difference between inflow and outflow is flow across cut

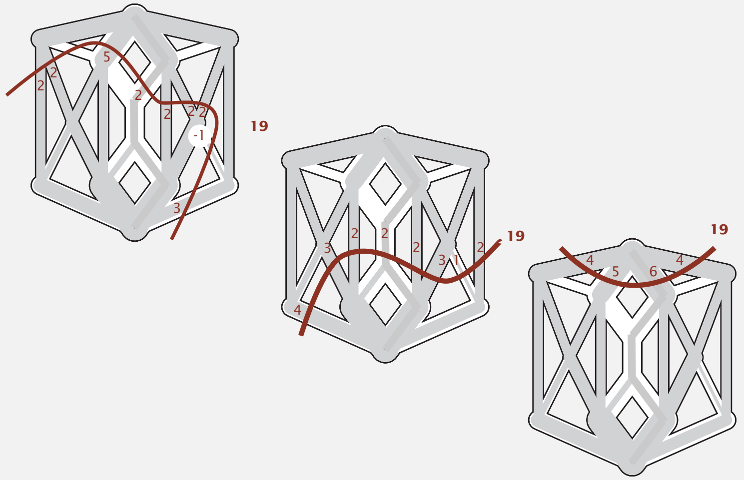
inflow to **t** is value of the flow

Corollary 1. Outflow from s = inflow to t = value.

Corollary 2. No st-flow's value can exceed the capacity of any st-cut.

Relationship between flows and cuts (example)

For any st-flow, the flow across every st-cut equals the value of the flow.



Maxflow-mincut theorem

Thm. The following three conditions are equivalent for any st-flow f:

i. There exists an st-cut whose capacity equals the value of the flow f.
ii. f is a maxflow.
iii. There is no augmenting path with respect to f.

Pf.

i. implies ii. [no flow's value can exceed any cut's capacity]

ii. implies iii. by contradiction [aug path would give higher-value flow, so f could not be maximal].

iii. implies i.

C_s: set of all vertices connected to s by an undirected path with no full forward or empty backward edges.

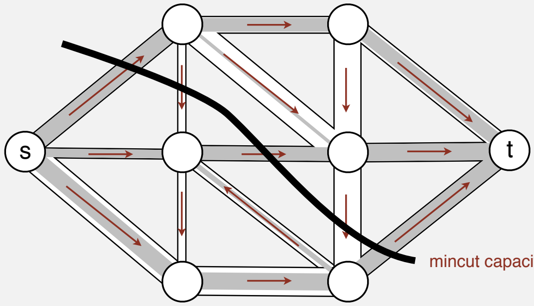
C_t: all other vertices.

capacity = flow across [st-crossing edges full, ts-crossing edges empty] .
= value of f [capacity of any cut = value of f].

FF termination

Eventually all paths from s to t are blocked by either a

- full forward edge
- empty backward edge

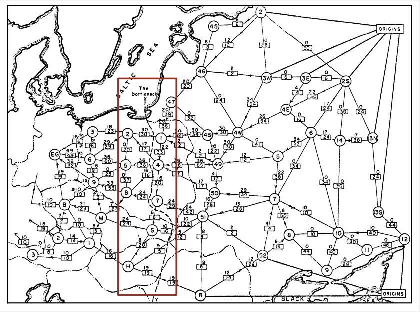


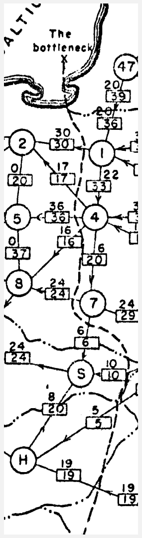
maxflow value = 28

mincut capacity = 28

Mincut:
Consider only paths with no full forward or empty backward edges.
C_s is the set of vertices reachable from s; C_t is the set of remaining vertices.

Maxflow/mincut application (1950s)





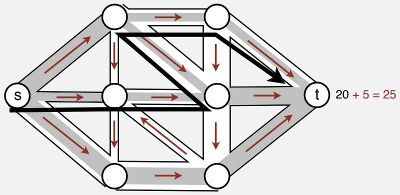
"bottleneck" is mincut (all forward edges full)

value of flow = 30+17+36+16+24+6+10+5+19 = 163,000 tons

Integrality property

Corollary to maxflow-mincut theorem. When capacities are integers, there exists an integer-valued maxflow, and the Ford Fulkerson algorithm finds it.

Pf. Flow increases by augmenting path value, which is either unused capacity in a forward edge or flow in a backwards edge [and always an integer].



Bottom line: Ford-Fulkerson always works when weights are integers.

Note: When weights are not integers, it could converge to the wrong value!

Possible strategies for augmenting paths

FF algorithm: any strategy for choosing augmenting paths will give a maxflow.
[Caveat: Can have convergence problems when weights are not integers.]

Shortest path?

aug path with fewest number of edges

This lecture (guaranteed to converge)

DFS path?

Random path?

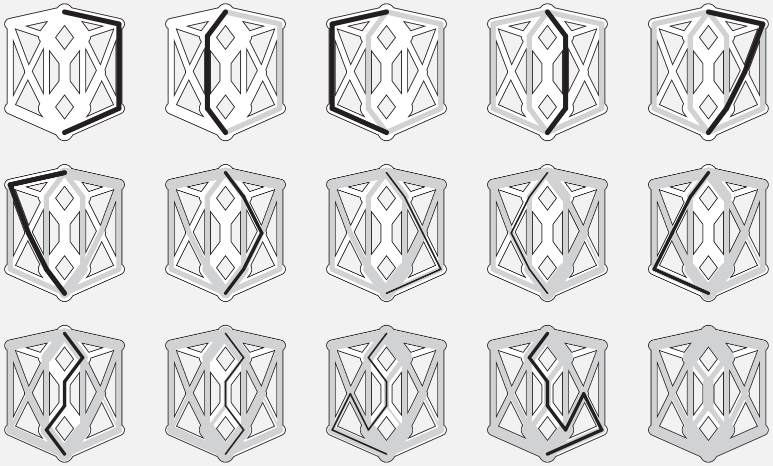
Fattest path?

max capacity aug path

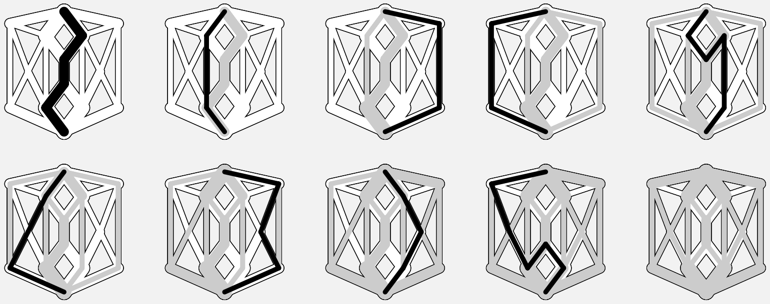
All easy to implement
- Define residual graph
- Find paths in residual graph.

Performance depends on network properties (stay tuned)

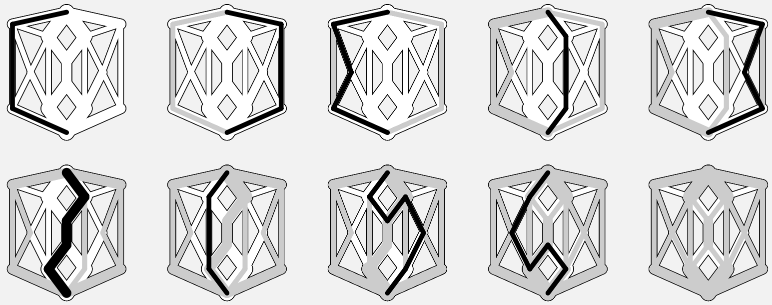
Shortest augmenting path



Fattest augmenting path

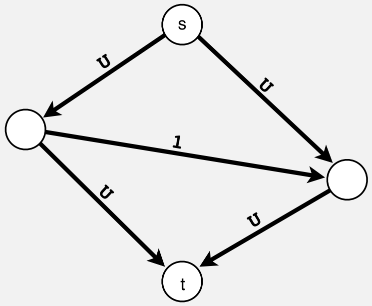


Random augmenting path

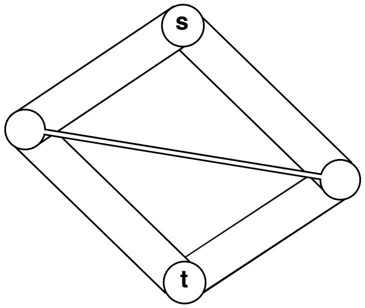


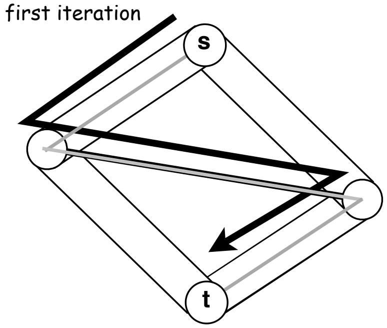
Bad case for Ford-Fulkerson

Bad news: Even when weights are integers, number of augmenting paths could be equal to the value of the maxflow.



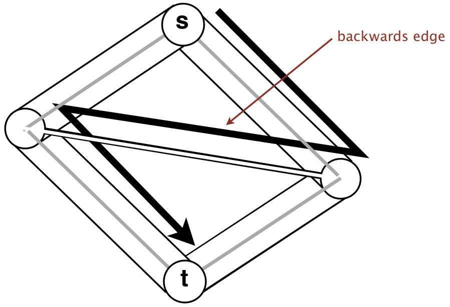
Good news: This case is easily avoided [use shortest augmenting path].





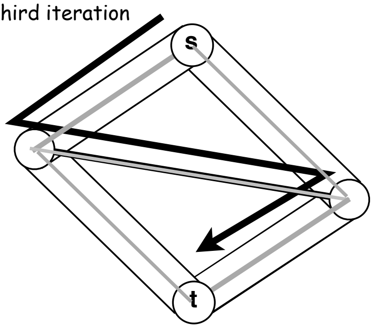
0 + 1 = 1

second iteration



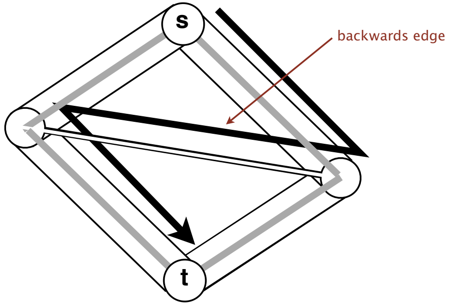
1 + 2 = 2

third iteration



2 + 1 = 3

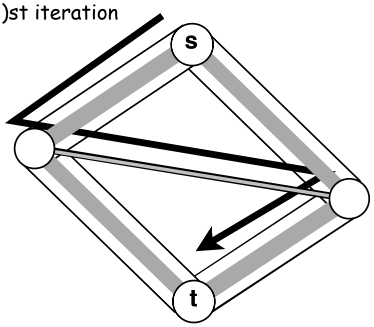
fourth iteration



3 + 4 = 4

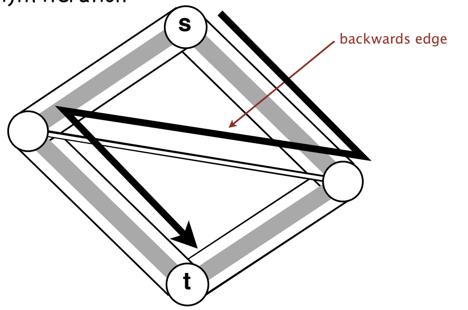
• • •

(2i-1)st iteration



2i – 2 + 1 = 2i – 1

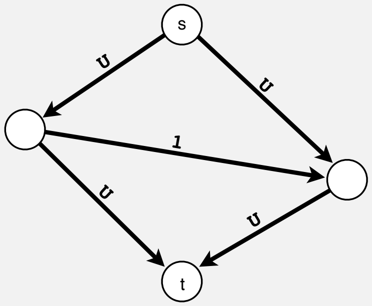
(2i)th iteration



2i – 1 + 1 = 2i

Bad case for Ford-Fulkerson

Bad news: Even when weights are integers, number of augmenting paths could be equal to the value of the maxflow.

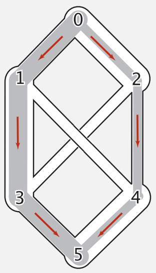


Good news: This case is easily avoided [use shortest augmenting path].

Flow network representation (revisited)

Residual digraph. Another view of a flow network

drawing with flow

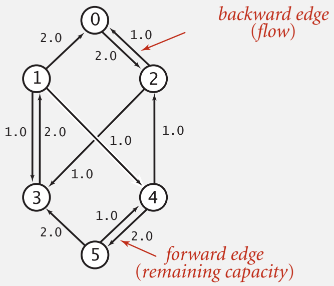


flow representation

```
0 1  2.0  2.0
0 2  3.0  1.0
1 3  3.0  2.0
1 4  1.0  0.0
2 3  1.0  0.0
2 4  1.0  1.0
3 5  2.0  2.0
4 5  3.0  1.0
```

capacity *flow*

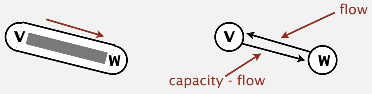
residual network



Finding an augmenting path is equivalent to finding a path in residual digraph.

Residual network implementation

```
public class FlowEdge
{
   private final int v;              // from
   private final int w;              // to
   private final double capacity;    // capacity
   private double flow;              // flow

   public double residualCapacityTo(int vertex)
   {
      if      (vertex == v) return flow;
      else if (vertex == w) return capacity - flow;
      else throw new RuntimeException("Illegal endpoint");
   }

   public void addResidualFlowTo(int vertex, double delta)
   {
      if      (vertex == v) flow -= delta;
      else if (vertex == w) flow += delta;
      else throw new RuntimeException("Illegal endpoint");
   }

}
```



flow

capacity - flow

Ford-Fulkerson: Java implementation

```
public class FordFulkerson
{
   private boolean[] marked;  // true if s->v path in residual digraph
   private FlowEdge[] edgeTo; // last edge on s->v path
   private double value;

   public FordFulkerson(FlowNetwork G, int s, int t)
   {
      value = 0;
      while (hasAugmentingPath(G, s, t))
      {
         double bottle = Double.POSITIVE_INFINITY;
         for (int v = t; v != s; v = edgeTo[v].other(v))
            bottle = Math.min(bottle, edgeTo[v].residualCapacityTo(v));

         for (int v = t; v != s; v = edgeTo[v].other(v))
            edgeTo[v].addResidualFlowTo(v, bottle);

         value += bottle;
      }
   }

   public double hasAugmentingPath(FlowNetwork G, int s, int t)
   {  /* See next slide. */  }

   public double value()
   {  return value;  }

   public boolean inCut(int v)
   {  return marked[v];  }
```

compute bottleneck capacity

augment flow

Finding a shortest augmenting path (cf. breadth-first search)

```
private boolean hasAugmentingPath(FlowNetwork G, int s, int t)
{
   edgeTo = new FlowEdge[G.V()];
   marked = new boolean[G.V()];

   Queue<Integer> q = new Queue<Integer>();
   q.enqueue(s);
   marked[s] = true;
   while (!q.isEmpty())
   {
      int v = q.dequeue();

      for (FlowEdge e : G.adj(v))
      {
         int w = e.other(v);
         if (e.residualCapacityTo(w) > 0 && !marked[w])
         {
            edgeTo[w] = e;
            marked[w] = true;
            q.enqueue(w);
         }
      }
   }

   return marked[t];
}
```

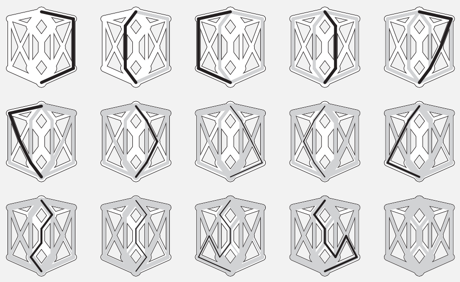
is there a path from s to w in the residual graph?

save last edge on path, mark w, and add w to the queue

Analysis of Ford-Fulkerson (shortest augmenting path)

Thm. Ford-Fulkerson (shortest augmenting path) uses at most EV/2 augmenting paths.

Pf. [see text]



Cor. Ford-Fulkerson (shortest augmenting path) examines at most $E^2V/2$ edges.

Pf. Each BFS examines at most E edges.

Summary: possible strategies for augmenting paths

All easy to implement

- Define residual graph
- Find paths in residual graph.

Shortest path: Use BFS.

DFS path: Use DFS.

Fattest path: Use a PQ, ala shortest paths.

Random path: Use a randomized queue.

Performance depends on network properties

- how many augmenting paths?
- how many edges examined to find each augmenting path?

Analysis of maxflow algorithms

(Yet another) holy grail for mathematicians/theoretical computer scientists.

For sparse graphs with E edges, integer capacities (max U).

year	method	worst case order of growth	discovered by
1951	simplex	O (E^3 U)	Dantzig
1955	augmenting paths	O (E^2 U)	Ford-Fulkerson
1970	shortest aug path	O (E^3)	Edmunds-Karp
1970	fattest aug path	O (E^2 log E log U)	Edmunds-Karp
1973	capacity scaling	O (E^2 log U)	Dinitz-Gabow
1983	preflow-push	O (E^2 log E)	Sleator-Tarjan
1997	length function	$\tilde{O}$ ($E^{3/2}$)	Goldberg-Rao
2011	electrical flow	$\tilde{O}$ ($E^{4/3}$) *	Christiano-Kelner-Madry-Spielman-Teng
?		O (E)	

~ ignore log factors

* approximation

Warning: Worst-case order-of-growth analysis is generally not useful for predicting or comparing algorithm performance in practice.

O-notation considered harmful (Lecture 2 revisited)

Facebook and Google: Huge sparse graphs are of interest (10^{10} – 10^{11} edges).

Time to solve maxflow:

Algorithm A: $\tilde{O}(E^{3/2})$.

Algorithm B: $\tilde{O}(E^{4/3})$.*

~ ignore log factors

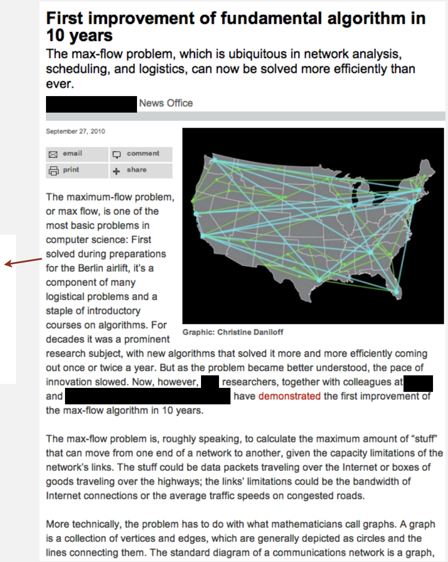
* approximation algorithm

Q. Which algorithm should Facebook and Google be interested in?

A. Who knows? These mathematical results are not relevant!

- Upper bound on worst case [may never take stated time].
- Unknown constants [most published maxflow algs never are implemented].
- $E^{1/6}$ savings likely offset by ignored log factors [40-50 vs. 30-40+].
- Performance for practical graph models likely unknown [and not studied].
- Approximation algorithm [cost of accuracy may be too high].

O-notation considered harmful



First improvement of fundamental algorithm in 10 years

The max-flow problem, which is ubiquitous in network analysis, scheduling, and logistics, can now be solved more efficiently than ever.

[redacted] News Office

September 27, 2010

Graphic: Christine Daniloff

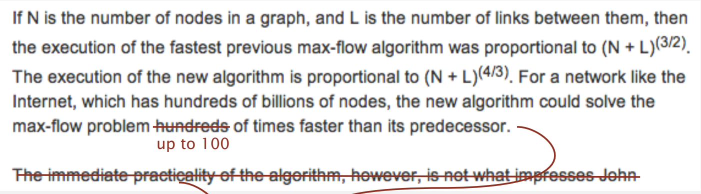
The maximum-flow problem, or max flow, is one of the most basic problems in computer science: First solved during preparations for the Berlin airlift, it's a component of many logistical problems and a staple of introductory courses on algorithms. For decades it was a prominent research subject, with new algorithms that solved it more and more efficiently coming out once or twice a year. But as the problem became better understood, the pace of innovation slowed. Now, however, [redacted] researchers, together with colleagues at [redacted] and [redacted] have demonstrated the first improvement of the max-flow algorithm in 10 years.

The max-flow problem is, roughly speaking, to calculate the maximum amount of "stuff" that can move from one end of a network to another, given the capacity limitations of the network's links. The stuff could be data packets traveling over the Internet or boxes of goods traveling over the highways; the links' limitations could be the bandwidth of Internet connections or the average traffic speeds on congested roads.

More technically, the problem has to do with what mathematicians call graphs. A graph is a collection of vertices and edges, which are generally depicted as circles and the lines connecting them. The standard diagram of a communications network is a graph,

Source? Schrijver's authoritative survey attributes T. E. Harris (author of the Soviet rail network report) as the first to formulate the problem in 1954.

O-notation considered harmful



If N is the number of nodes in a graph, and L is the number of links between them, then the execution of the fastest previous max-flow algorithm was proportional to $(N + L)^{(3/2)}$. The execution of the new algorithm is proportional to $(N + L)^{(4/3)}$. For a network like the Internet, which has hundreds of billions of nodes, the new algorithm could solve the max-flow problem ~~hundreds~~ up to 100 of times faster than its predecessor.

~~The immediate practicality of the algorithm, however, is not what impresses John~~

if the constant-factor costs were the same for both algorithms and if the internet were the worst case for both algorithms, which there is no reason to believe.

Moreover, these mathematical results are approximate, ignoring factors that could run into the hundreds for the internet graph.

The algorithm also computes an approximation to the maxflow, not the actual maxflow, and slows down as the approximation improves.

The algorithm has not been implemented or tested on graphs the size of the internet (or at all, for that matter). The algorithm would have to be implemented and tested before any claim to immediate practicality could be assessed.

It is likely that simpler approaches involving parallelism will be used in practice.

Summary

Minimum st-cut (mincut) problem. Find an st-cut of minimal weight.

Maximum st-flow (maxflow) problem: Assign flows to edges that
- Maintain local equilibrium: inflow = outflow at every vertex (except s and t).
- Maximize total flow into t.

Proven successful approaches.
- Ford-Fulkerson (various augmenting-path strategies).
- Preflow-push (various versions).

Open research challenges.
- Practice: Solve maxflow/mincut problems for real networks in linear time.
- Theory: Prove it for worst-case networks.

- APIs
- Ford Fulkerson
- implementations
- **applications**

Maxflow / mincut applications

Maxflow/mincut is a widely applicable problem-solving model

- Data mining.
- Open-pit mining.
- Project selection.
- Image processing.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network connectivity/reliability.
- Many many more . . .

Bipartite matching problem

N students apply for N jobs

Each get several offers

Is there a way to match all student to jobs?

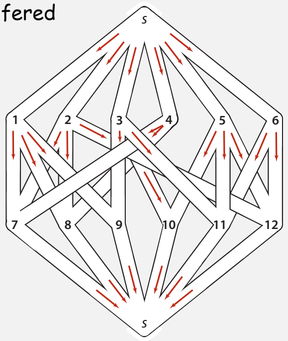
1 Alice
  Adobe
  Amazon
  Facebook
2 Bob
  Adobe
  Amazon
  Yahoo
3 Carol
  Facebook
  Google
  IBM
4 Dave
  Adobe
  Amazon
5 Eliza
  Google
  IBM
  Yahoo
6 Frank
  IBM
  Yahoo

7 Adobe
  Alice
  Bob
  Dave
8 Amazon
  Alice
  Bob
  Dave
9 Facebook
  Alice
  Carol
10 Google
  Carol
  Eliza
11 IBM
  Carol
  Eliza
  Frank
12 Yahoo
  Bob
  Eliza
  Frank

Network flow formulation of bipartite matching

To formulate a bipartite matching problem as a network flow problem

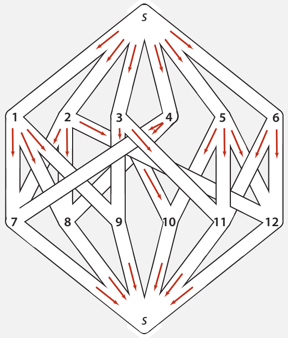
- create s, t, one vertex for each student, and one vertex for each job
- add edge from s to each student
- add edge from each job to t
- add edge from student to each job offered
- give all edges capacity 1



Bipartite matching problem formulated as a network flow problem

1 Alice
  Adobe
  Amazon
  Facebook
2 Bob
  Adobe
  Amazon
  Yahoo
3 Carol
  Facebook
  Google
  IBM
4 Dave
  Adobe
  Amazon
5 Eliza
  Google
  IBM
  Yahoo
6 Frank
  IBM
  Yahoo

7 Adobe
  Alice
  Bob
  Dave
8 Amazon
  Alice
  Bob
  Dave
9 Facebook
  Alice
  Carol
10 Google
  Carol
  Eliza
11 IBM
  Carol
  Eliza
  Frank
12 Yahoo
  Bob
  Eliza
  Frank



1-1 correspondence between maxflow solution and bipartite matching solution

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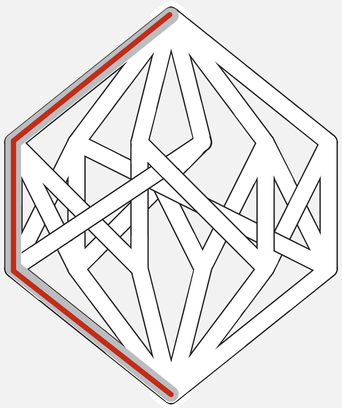
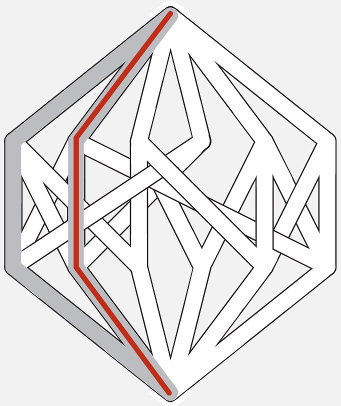
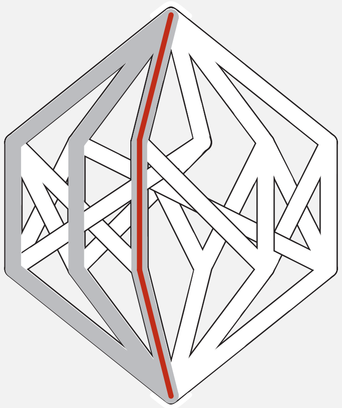
Maxflow solution (FF shortest augmenting path)



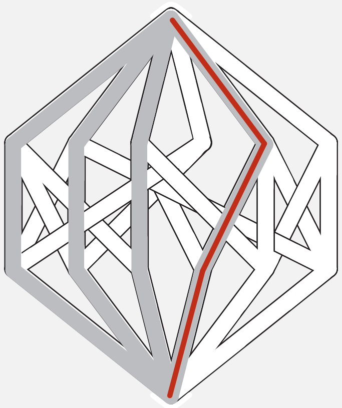
Maxflow solution (FF shortest augmenting path)



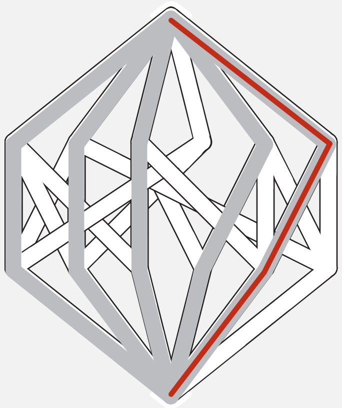
Maxflow solution (FF shortest augmenting path)



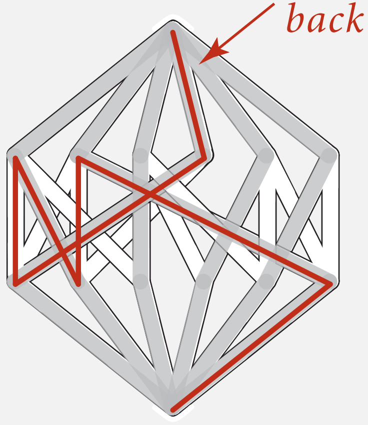
Maxflow solution (FF shortest augmenting path)



Maxflow solution (FF shortest augmenting path)

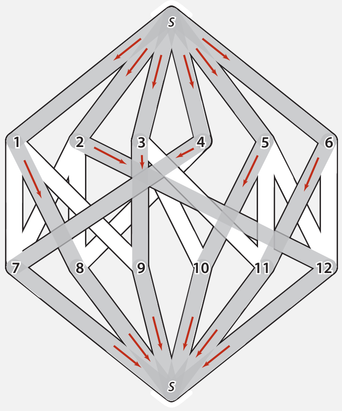


Maxflow solution (FF shortest augmenting path)



path with back edges

Maxflow solution



Maxflow solution corresponds directly to matching solution

1 Alice
- Adobe
- Amazon
- Facebook

2 Bob
- Adobe
- Amazon
- Yahoo

3 Carol
- Facebook
- Google
- IBM

4 Dave
- Adobe
- Amazon

5 Eliza
- Google
- IBM
- Yahoo

6 Frank
- IBM
- Yahoo

7 Adobe
- Alice
- Bob
- Dave

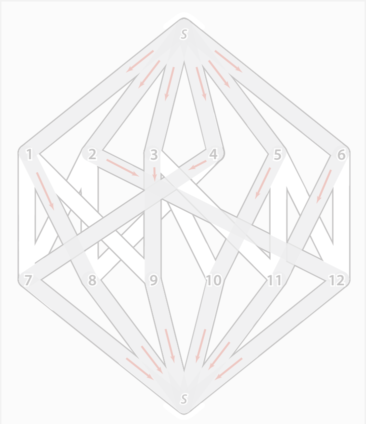
8 Amazon
- Alice
- Bob
- Dave

9 Facebook
- Alice
- Carol

10 Google
- Carol
- Eliza

11 IBM
- Carol
- Eliza
- Frank

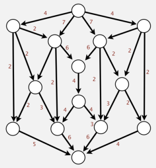
12 Yahoo
- Bob
- Eliza
- Frank



Alice — Amazon
Bob — Yahoo
Carol — Facebook
Dave — Adobe
Eliza — Google
Frank — IBM

Overview (summary)

Given. A weighted digraph, source s and target t.



Minimum st-cut (mincut) problem. Find an st-cut of minimal weight.

Maximum st-flow (maxflow) problem: Assign flows to edges that
- Maintain local equilibrium: inflow = outflow at every vertex (except s and t).
- Maximize total flow into t.

Remarkable fact. These two problems are equivalent!
- Two very rich algorithmic problems
- Cornerstone problems in combinatorial optimisation
- Beautiful mathematical duality
- Still much to be learned!