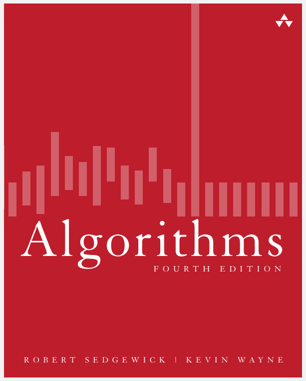
5. Strings



- 5.1 Strings Sorts
- 5.2 Tries
- 5.3 Substring Search
- 5.4 Regular Expressions
- 5.5 Data Compression

String processing

String. Sequence of characters.

Important fundamental abstraction.

- Information processing.
- Genomic sequences.
- Communication systems (e.g., email).
- Programming systems (e.g., Java programs).
- …

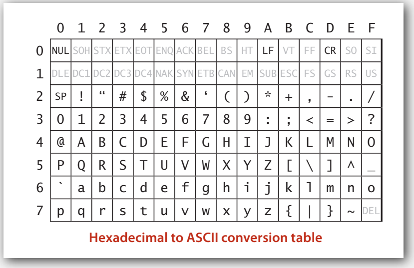
> *" The digital information that underlies biochemistry, cell biology, and development can be represented by a simple string of G's, A's, T's and C's. This string is the root data structure of an organism's biology. "* — M. V. Olson

---

The char data type

C char data type. Typically an 8-bit integer.

- Supports 7-bit ASCII.
- Need more bits to represent certain characters.



	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	HT	LF	VT	FF	CR	SO	SI
1	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
2	SP	!	“	#	$	%	&	‘	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	DEL

Hexadecimal to ASCII conversion table

Java char data type. A 16-bit unsigned integer.

- Supports original 16-bit Unicode.
- Supports 21-bit Unicode 3.0 (awkwardly).

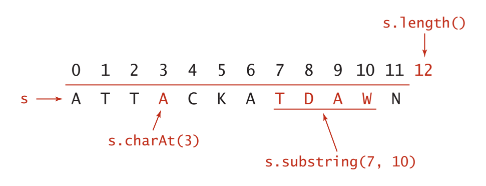
The String data type

String data type. Sequence of characters (immutable).

Indexing. Get the *ith* character.

Substring extraction. Get a contiguous sequence of characters from a string.

String concatenation. Append one character to end of another string.



The String data type: Java implementation

```
public final class String implements Comparable<String>
{
   private char[] value;   // characters
   private int offset;     // index of first char in array
   private int count;      // length of string
   private int hash;       // cache of hashCode()

   private String(int offset, int count, char[] value)
   {
      this.offset = offset;
      this.count  = count;
      this.value  = value;
   }

   public String substring(int from, int to)
   {  return new String(offset + from, to - from, value);  }

   public char charAt(int index)
   {  return value[index + offset];  }

   public String concat(String that)
   {
      char[] val = new char[this.length() + that.length()];
      ...
      return new String(0, this.length() + that.length(), val);
   }
}
```

strings share underlying `char[]` array

The String data type: performance

String data type. Sequence of characters (immutable).

Underlying implementation. Immutable `char[]` array, offset, and length.

	String	
operation	guarantee	extra space
`charAt()`	1	1
`substring()`	1	1
`concat()`	N	N

Memory. $40 + 2N$ bytes for a virgin `String` of length N.

use `byte[]` or `char[]` instead of `String` to save space

The StringBuilder data type

StringBuilder data type. Sequence of characters (mutable).

Underlying implementation. Doubling `char[]` array and length.

	String		**StringBuilder**	
operation	guarantee	extra space	guarantee	extra space
`charAt()`	1	1	1	1
`substring()`	1	1	N	N
`concat()`	N	N	1 *	1 *

* amortized

Remark. `StringBuffer` data type is similar, but thread safe (and slower).

String vs. StringBuilder

Challenge. How to reverse a string?

A.

```
public static String reverse(String s)
{
   String rev = "";
   for (int i = s.length() - 1; i >= 0; i--)
      rev += s.charAt(i);
   return rev;
}
```

quadratic time

B.

```
public static String reverse(String s)
{
   StringBuilder rev = new StringBuilder();
   for (int i = s.length() - 1; i >= 0; i--)
      rev.append(s.charAt(i));
   return rev.toString();
}
```

linear time

String challenge: array of suffixes

Challenge. How to form array of suffixes?

input string

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

suffixes

0	a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
1	a	c	a	a	g	t	t	t	a	c	a	a	g	c	
2	c	a	a	g	t	t	t	a	c	a	a	g	c		
3	a	a	g	t	t	t	a	c	a	a	g	c			
4	a	g	t	t	t	a	c	a	a	g	c				
5	g	t	t	t	a	c	a	a	g	c					
6	t	t	t	a	c	a	a	g	c						
7	t	t	a	c	a	a	g	c							
8	t	a	c	a	a	g	c								
9	a	c	a	a	g	c									
10	c	a	a	g	c										
11	a	a	g	c											
12	a	g	c												
13	g	c													
14	c														

String vs. StringBuilder

Challenge. How to form array of suffixes?

A.

```
public static String[] suffixes(String s)
{
   int N = s.length();
   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = s.substring(i, N);
   return suffixes;
}
```

linear time and space

B.

```
public static String[] suffixes(String s)
{
   int N = s.length();
   StringBuilder sb = new StringBuilder(s);
   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = sb.substring(i, N);
   return suffixes;
}
```

quadratic time and space

Alphabets

Digital key. Sequence of digits over fixed alphabet.

Radix. Number of digits R in alphabet.

name	R()	lgR()	characters
BINARY	2	1	01
OCTAL	8	3	01234567
DECIMAL	10	4	0123456789
HEXADECIMAL	16	4	0123456789ABCDEF
DNA	4	2	ACTG
LOWERCASE	26	5	abcdefghijklmnopqrstuvwxyz
UPPERCASE	26	5	ABCDEFGHIJKLMNOPQRSTUVWXYZ
PROTEIN	20	5	ACDEFGHIKLMNPQRSTVWY
BASE64	64	6	ABCDEFGHIJKLMNOPQRSTUVWXYZabcdef ghijklmnopqrstuvwxyz0123456789+/
ASCII	128	7	*ASCII characters*
EXTENDED_ASCII	256	8	*extended ASCII characters*
UNICODE16	65536	16	*Unicode characters*

Standard alphabets

5.1 String Sorts

- key-indexed counting
- LSD string sort
- MSD string sort
- 3-way string quicksort
- suffix arrays

Review: summary of the performance of sorting algorithms

Frequency of operations = key compares.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2/2$	$N^2/4$	no	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	no	no	`compareTo()`

* probabilistic

Lower bound. ~ $N \lg N$ compares are required by any compare-based algorithm.

Q. Can we do better (despite the lower bound)?

A. Yes, if we don't depend on compares.

- key-indexed counting
- LSD string sort
- MSD string sort
- 3-way string quicksort
- longest repeated substring

Key-indexed counting: assumptions about keys

Assumption. Keys are integers between 0 and $R - 1$.
Implication. Can use key as an array index.

Applications.

- Sort string by first letter.
- Sort class roster by section.
- Sort phone numbers by area code.
- Subroutine in a sorting algorithm.

Remark. Keys may have associated data ⇒ can't just count up number of keys of each value.

input		sorted result	
name	*section*	*(by section)*	
`Anderson`	2	`Harris`	1
`Brown`	3	`Martin`	1
`Davis`	3	`Moore`	1
`Garcia`	4	`Anderson`	2
`Harris`	1	`Martinez`	2
`Jackson`	3	`Miller`	2
`Johnson`	4	`Robinson`	2
`Jones`	3	`White`	2
`Martin`	1	`Brown`	3
`Martinez`	2	`Davis`	3
`Miller`	2	`Jackson`	3
`Moore`	1	`Jones`	3
`Robinson`	2	`Taylor`	3
`Smith`	4	`Williams`	3
`Taylor`	3	`Garcia`	4
`Thomas`	4	`Johnson`	4
`Thompson`	4	`Smith`	4
`White`	2	`Thomas`	4
`Williams`	3	`Thompson`	4
`Wilson`	4	`Wilson`	4

↑ *keys are small integers*

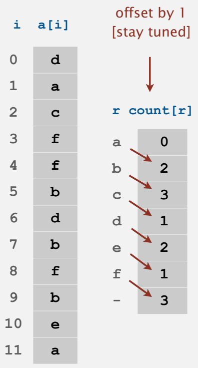
Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
-
-
-

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

count frequencies →



Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
-
-

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

compute cumulates

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	0
b	2
c	5
d	6
e	8
f	9
-	12

6 keys < d, 8 keys < e
so d's go in `a[6]` and `a[7]`

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move records.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

copy back

i	a[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

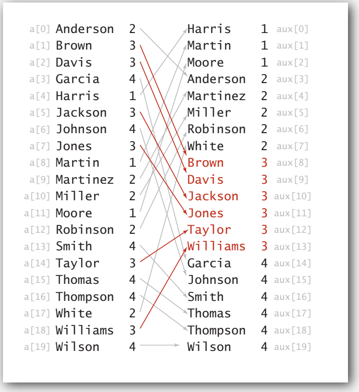
Key-indexed counting: analysis

Proposition. Key-indexed counting uses $8N + 3R$ array accesses to sort N records whose keys are integers between 0 and $R - 1$.

Proposition. Key-indexed counting uses extra space proportional to $N + R$.

Stable? Yes!

In-place? No.



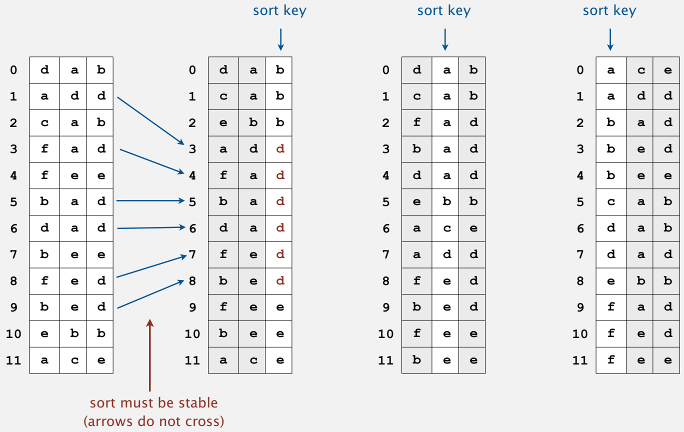
a[0]	Anderson	2	Harris	1	aux[0]
a[1]	Brown	3	Martin	1	aux[1]
a[2]	Davis	3	Moore	1	aux[2]
a[3]	Garcia	4	Anderson	2	aux[3]
a[4]	Harris	1	Martinez	2	aux[4]
a[5]	Jackson	3	Miller	2	aux[5]
a[6]	Johnson	4	Robinson	2	aux[6]
a[7]	Jones	3	White	2	aux[7]
a[8]	Martin	1	Brown	3	aux[8]
a[9]	Martinez	2	Davis	3	aux[9]
a[10]	Miller	2	Jackson	3	aux[10]
a[11]	Moore	1	Jones	3	aux[11]
a[12]	Robinson	2	Taylor	3	aux[12]
a[13]	Smith	4	Williams	3	aux[13]
a[14]	Taylor	3	Garcia	4	aux[14]
a[15]	Thomas	4	Johnson	4	aux[15]
a[16]	Thompson	4	Smith	4	aux[16]
a[17]	White	2	Thomas	4	aux[17]
a[18]	Williams	3	Thompson	4	aux[18]
a[19]	Wilson	4	Wilson	4	aux[19]

- key-indexed counting
- **LSD string sort**
- MSD string sort
- 3-way string quicksort
- suffix arrays

Least-significant-digit-first string sort

LSD string sort.

- Consider characters from right to left.
- Stably sort using d^{th} character as the key (using key-indexed counting).



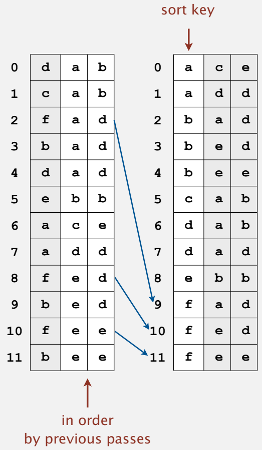
0	d a b	d a b	d a b	a c e
1	a d d	c a b	c a b	a d d
2	c a b	e b b	f a d	b a d
3	f a d	a d d	b a d	b e d
4	f e e	f a d	d a d	b e e
5	b a d	b a d	e b b	c a b
6	d a d	d a d	a c e	d a b
7	b e e	f e d	a d d	d a d
8	f e d	b e d	f e d	e b b
9	b e d	f e e	b e d	f a d
10	e b b	b e e	f e e	f e d
11	a c e	a c e	b e e	f e e

LSD string sort: correctness proof

Proposition. LSD sorts fixed-length strings in ascending order.

Pf. [thinking about the future]

- If the characters not yet examined differ, it doesn't matter what we do now.
- If the characters not yet examined agree, stability ensures later pass won't affect order.



0	d a b	0	a c e
1	c a b	1	a d d
2	f a d	2	b a d
3	b a d	3	b e d
4	d a d	4	b e e
5	e b b	5	c a b
6	a c e	6	d a b
7	a d d	7	d a d
8	f e d	8	e b b
9	b e d	9	f a d
10	f e e	10	f e d
11	b e e	11	f e e

LSD string sort: Java implementation

```
public class LSD
{
   public static void sort(String[] a, int W)
   {
      int R = 256;
      int N = a.length;
      String[] aux = new String[N];

      for (int d = W-1; d >= 0; d--)
      {
         int[] count = new int[R+1];
         for (int i = 0; i < N; i++)
            count[a[i].charAt(d) + 1]++;
         for (int r = 0; r < R; r++)
            count[r+1] += count[r];
         for (int i = 0; i < N; i++)
            aux[count[a[i].charAt(d)]++] = a[i];
         for (int i = 0; i < N; i++)
            a[i] = aux[i];
      }
   }
}
```

fixed-length W strings

radix R

do key-indexed counting
for each digit from right to left

key-indexed
counting

LSD string sort: example

Input	d = 6	d = 5	d = 4	d = 3	d= 2	d= 1	d = 0	Output
4PGC938	2IYE230	3CIO720	2IYE230	2RLA629	1ICK750	3ATW723	1ICK750	1ICK750
2IYE230	3CIO720	3CIO720	4JZY524	2RLA629	1ICK750	3CIO720	1ICK750	1ICK750
3CIO720	1ICK750	3ATW723	2RLA629	4PGC938	4PGC938	3CIO720	1OHV845	1OHV845
1ICK750	1ICK750	4JZY524	2RLA629	2IYE230	1OHV845	1ICK750	1OHV845	1OHV845
1OHV845	3CIO720	2RLA629	3CIO720	1ICK750	1OHV845	1ICK750	1OHV845	1OHV845
4JZY524	3ATW723	2RLA629	3CIO720	1ICK750	1OHV845	2IYE230	2IYE230	2IYE230
1ICK750	4JZY524	2IYE230	3ATW723	3CIO720	3CIO720	4JZY524	2RLA629	2RLA629
3CIO720	1OHV845	4PGC938	1ICK750	3CIO720	3CIO720	1OHV845	2RLA629	2RLA629
1OHV845	1OHV845	1OHV845	1ICK750	1OHV845	2RLA629	1OHV845	3ATW723	3ATW723
1OHV845	1OHV845	1OHV845	1OHV845	1OHV845	2RLA629	1OHV845	3CIO720	3CIO720
2RLA629	4PGC938	1OHV845	1OHV845	1OHV845	3ATW723	4PGC938	3CIO720	3CIO720
2RLA629	2RLA629	1ICK750	1OHV845	3ATW723	2IYE230	2RLA629	4JZY524	4JZY524
3ATW723	2RLA629	1ICK750	4PGC938	4JZY524	4JZY524	2RLA629	4PGC938	4PGC938

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2 /2$	$N^2 /4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 W N	2 W N	N + R	yes	`charAt()`

* probabilistic
† fixed-length W keys

Q. What if strings do not have same length?

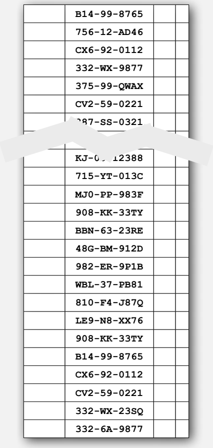
String sorting challenge 1

Problem. Sort a huge commercial database on a fixed-length key field.
Ex. Account number, date, SS number, ...

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD string sort.

256 (or 65,536) counters;
Fixed-length strings sort in W passes.



String sorting challenge 2a

Problem. Sort 1 million 32-bit integers.
Ex. Google interview (or presidential interview).

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.

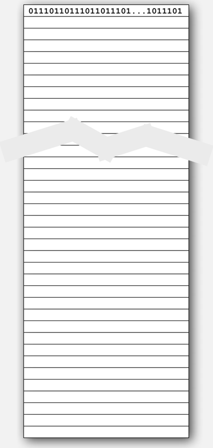
Google CEO Eric Schmidt interviews Barack Obama

String sorting challenge 2b

Problem. Sort huge array of random 128-bit numbers.
Ex. Supercomputer sort, internet router.

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.



LSD string sort: a moment in history (1960s)


card punch

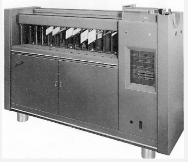

punched cards


card reader

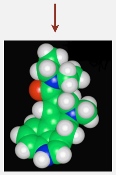

mainframe


line printer

To sort a card deck
- start on right column
- put cards into hopper
- machine distributes into bins
- pick up cards (stable)
- move left one column
- continue until sorted


card sorter

not related to sorting

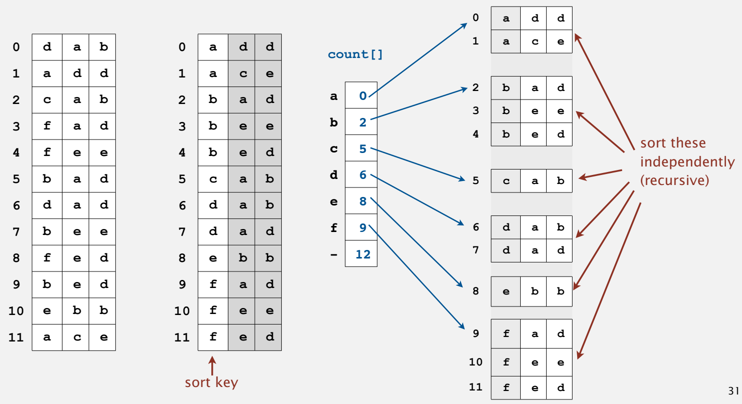

Lysergic Acid Diethylamide
(Lucy in the Sky with Diamonds)

- key-indexed counting
- LSD string sort
- **MSD string sort**
- 3-way string quicksort
- suffix arrays

Most-significant-digit-first string sort

MSD string sort.

- Partition file into R pieces according to first character (use key-indexed counting).
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).



0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

sort key

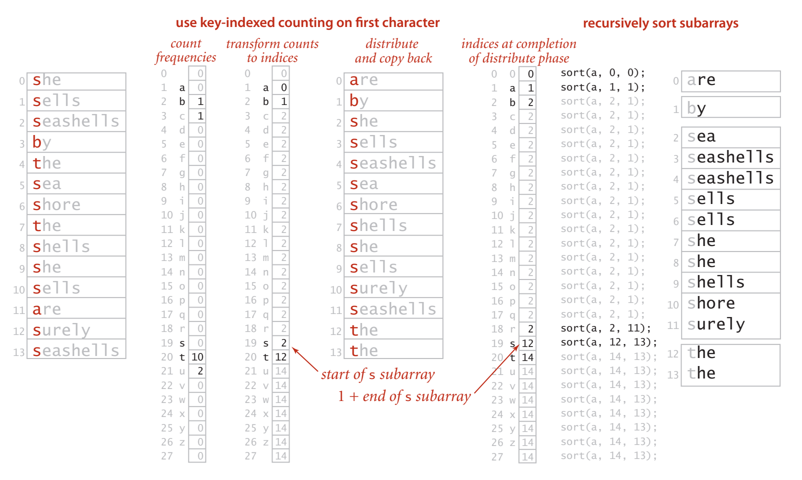
`count[]`

a	0
b	2
c	5
d	6
e	8
f	9
–	12

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

sort these independently (recursive)

MSD string sort: top-level trace



use key-indexed counting on first character

0	she
1	sells
2	seashells
3	by
4	the
5	sea
6	shore
7	the
8	shells
9	she
10	sells
11	are
12	surely
13	seashells

		count frequencies			transform counts to indices
0		0	0		0
1	a	0	1	a	0
2	b	1	2	b	1
3	c	1	3	c	2
4	d	0	4	d	2
5	e	0	5	e	2
6	f	0	6	f	2
7	g	0	7	g	2
8	h	0	8	h	2
9	i	0	9	i	2
10	j	0	10	j	2
11	k	0	11	k	2
12	l	0	12	l	2
13	m	0	13	m	2
14	n	0	14	n	2
15	o	0	15	o	2
16	p	0	16	p	2
17	q	0	17	q	2
18	r	0	18	r	2
19	s	0	19	s	2
20	t	10	20	t	12
21	u	2	21	u	14
22	v	0	22	v	14
23	w	0	23	w	14
24	x	0	24	x	14
25	y	0	25	y	14
26	z	0	26	z	14
27		0	27		14

	distribute and copy back
0	are
1	by
2	she
3	sells
4	seashells
5	sea
6	shore
7	shells
8	she
9	sells
10	surely
11	seashells
12	the
13	the

start of s subarray

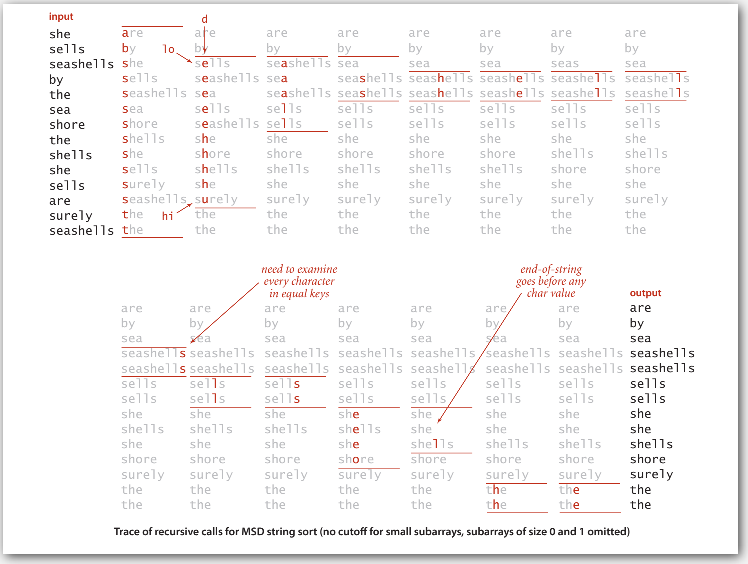
1 + end of s subarray

recursively sort subarrays

		indices at completion of distribute phase	
0	0	0	`sort(a, 0, 0);`
1	a	1	`sort(a, 1, 1);`
2	b	2	`sort(a, 2, 1);`
3	c	2	`sort(a, 2, 1);`
4	d	2	`sort(a, 2, 1);`
5	e	2	`sort(a, 2, 1);`
6	f	2	`sort(a, 2, 1);`
7	g	2	`sort(a, 2, 1);`
8	h	2	`sort(a, 2, 1);`
9	i	2	`sort(a, 2, 1);`
10	j	2	`sort(a, 2, 1);`
11	k	2	`sort(a, 2, 1);`
12	l	2	`sort(a, 2, 1);`
13	m	2	`sort(a, 2, 1);`
14	n	2	`sort(a, 2, 1);`
15	o	2	`sort(a, 2, 1);`
16	p	2	`sort(a, 2, 1);`
17	q	2	`sort(a, 2, 1);`
18	r	2	`sort(a, 2, 11);`
19	s	12	`sort(a, 12, 13);`
20	t	14	`sort(a, 14, 13);`
21	u	14	`sort(a, 14, 13);`
22	v	14	`sort(a, 14, 13);`
23	w	14	`sort(a, 14, 13);`
24	x	14	`sort(a, 14, 13);`
25	y	14	`sort(a, 14, 13);`
26	z	14	`sort(a, 14, 13);`
27		14	`sort(a, 14, 13);`

0	are
1	by
2	sea
3	seashells
4	seashells
5	sells
6	sells
7	she
8	she
9	shells
10	shore
11	surely
12	the
13	the

MSD string sort: example



input								
she	are	are	are	are	are	are	are	are
sells	by	by	by	by	by	by	by	by
seashells	she	sells	seashells	sea	sea	sea	seas	sea
by	sells	seashells	sea	seashells	seashells	seashells	seashells	seashells
the	seashells	sea	seashells	seashells	seashells	seashells	seashells	seashells
sea	sea	sells	sells	sells	sells	sells	sells	sells
shore	shore	seashells	sells	sells	sells	sells	sells	sells
the	shells	she	she	she	she	she	she	she
shells	she	shore	shore	shore	shore	shore	shells	shells
she	sells	shells	shells	shells	shells	shells	shore	shore
sells	surely	she	she	she	she	she	she	she
are	seashells	surely	surely	surely	surely	surely	surely	surely
surely	the	the	the	the	the	the	the	the
seashells	the	the	the	the	the	the	the	the

need to examine every character in equal keys

end-of-string goes before any char value

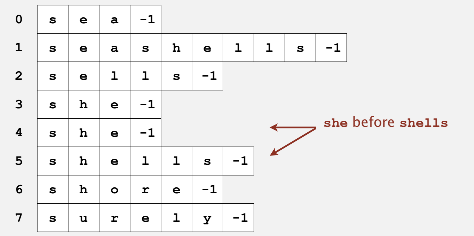
							output
are	are	are	are	are	are	are	are
by	by	by	by	by	by	by	by
sea	sea	sea	sea	sea	sea	sea	sea
seashells	seashells	seashells	seashells	seashells	seashells	seashells	seashells
seashells	seashells	seashells	seashells	seashells	seashells	seashells	seashells
sells	sells	sells	sells	sells	sells	sells	sells
sells	sells	sells	sells	sells	sells	sells	sells
she	she	she	she	she	she	she	she
shells	shells	shells	shells	she	she	she	she
she	she	she	she	shells	shells	shells	shells
shore	shore	shore	shore	shore	shore	shore	shore
surely	surely	surely	surely	surely	surely	surely	surely
the	the	the	the	the	the	the	the
the	the	the	the	the	the	the	the

Trace of recursive calls for MSD string sort (no cutoff for small subarrays, subarrays of size 0 and 1 omitted)

Variable-length strings

Treat strings as if they had an extra char at end (smaller than any char).

why smaller?



0	s	e	a	-1						
1	s	e	a	s	h	e	l	l	s	-1
2	s	e	l	l	s	-1				
3	s	h	e	-1						
4	s	h	e	-1						
5	s	h	e	l	l	s	-1			
6	s	h	o	r	e	-1				
7	s	u	r	e	l	y	-1			

`she` before `shells`

```
private static int charAt(String s, int d)
{
   if (d < s.length()) return s.charAt(d);
   else return -1;
}
```

C strings. Have extra char `'\0'` at end ⇒ no extra work needed.

MSD string sort: Java implementation

```
public static void sort(String[] a)
{
   aux = new String[a.length];
   sort(a, aux, 0, a.length, 0);
}

private static void sort(String[] a, String[] aux, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int[] count = new int[R+2];
   for (int i = lo; i <= hi; i++)
      count[charAt(a[i], d) + 2]++;
   for (int r = 0; r < R+1; r++)
      count[r+1] += count[r];
   for (int i = lo; i <= hi; i++)
      aux[count[charAt(a[i], d) + 1]++] = a[i];
   for (int i = lo; i <= hi; i++)
      a[i] = aux[i - lo];

   for (int r = 0; r < R; r++)
      sort(a, aux, lo + count[r], lo + count[r+1] - 1, d+1);
}
```

can recycle `aux[]` but not `count[]`

key-indexed counting

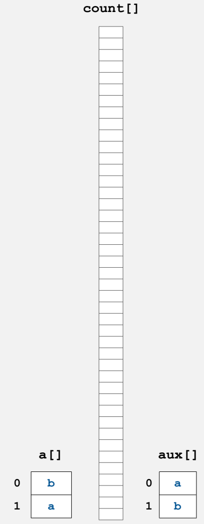
recursively sort subarrays

MSD string sort: potential for disastrous performance

Observation 1. Much too slow for small subarrays.

- Each function call needs its own `count[]` array.
- ASCII (256 counts): 100x slower than copy pass for $N = 2$.
- Unicode (65,536 counts): 32,000x slower for $N = 2$.

Observation 2. Huge number of small subarrays because of recursion.



Cutoff to insertion sort

Solution. Cutoff to insertion sort for small N.

- Insertion sort, but start at d^{th} character.
- Implement `less()` so that it compares starting at d^{th} character.

```
public static void sort(String[] a, int lo, int hi, int d)
{
   for (int i = lo; i <= hi; i++)
      for (int j = i; j > lo && less(a[j], a[j-1], d); j--)
         exch(a, j, j-1);
}
```

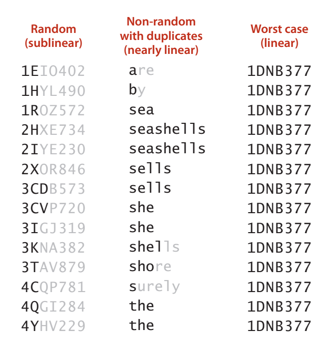
```
private static boolean less(String v, String w, int d)
{  return v.substring(d).compareTo(w.substring(d)) < 0;  }
```

in Java, forming and comparing substrings is faster than directly comparing chars with charAt()

MSD string sort: performance

Number of characters examined.

- MSD examines just enough characters to sort the keys.
- Number of characters examined depends on keys.
- Can be sublinear!



Random (sublinear)	Non-random with duplicates (nearly linear)	Worst case (linear)
1EIO402	are	1DNB377
1HYL490	by	1DNB377
1ROZ572	sea	1DNB377
2HXE734	seashells	1DNB377
2IYE230	seashells	1DNB377
2XOR846	sells	1DNB377
3CDB573	sells	1DNB377
3CVP720	she	1DNB377
3IGJ319	she	1DNB377
3KNA382	shells	1DNB377
3TAV879	shore	1DNB377
4CQP781	surely	1DNB377
4QGI284	the	1DNB377
4YHV229	the	1DNB377

Characters examined by MSD string sort

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2/2$	$N^2/4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 N W	2 N W	N + R	yes	`charAt()`
MSD ‡	2 N W	$N \log_R N$	N + D R	yes	`charAt()`

stack depth D = length of longest prefix match

* probabilistic

† fixed-length W keys

‡ average-length W keys

MSD string sort vs. quicksort for strings

Disadvantages of MSD string sort.

- Accesses memory "randomly" (cache inefficient).
- Inner loop has a lot of instructions.
- Extra space for `count[]`.
- Extra space for `aux[]`.

Disadvantage of quicksort.

- Linearithmic number of string compares (not linear).
- Has to rescan long keys for compares.

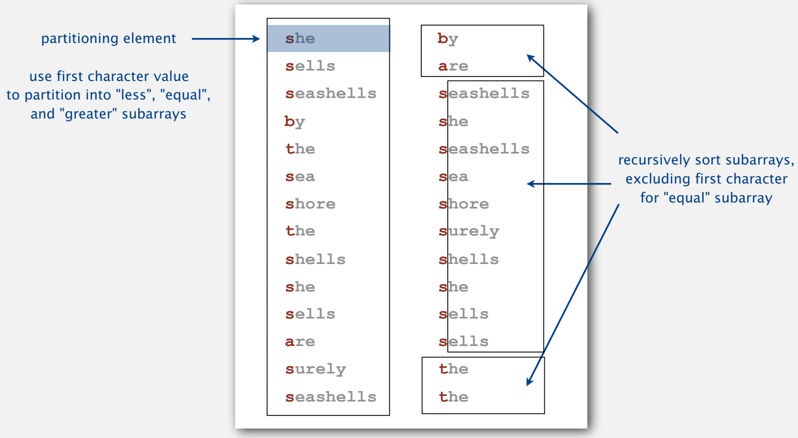
Goal. Combine advantages of MSD and quicksort.

- key-indexed counting
- LSD string sort
- MSD string sort
- **3-way string quicksort**
- suffix arrays

3-way string quicksort (Bentley and Sedgewick, 1997)

Overview. Do 3-way partitioning on the d^{th} character.

- Cheaper than R-way partitioning of MSD string sort.
- Need not examine again characters equal to the partitioning char.

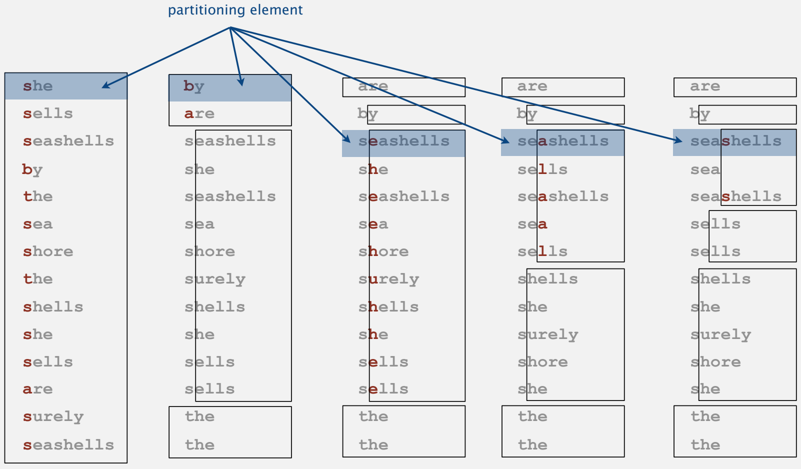

partitioning element

use first character value to partition into "less", "equal", and "greater" subarrays

recursively sort subarrays, excluding first character for "equal" subarray

---

3-way string quicksort: trace of recursive calls


partitioning element

Trace of first few recursive calls for 3-way string quicksort (subarrays of size 1 not shown)

3-way string quicksort: Java implementation

```
private static void sort(String[] a)
{  sort(a, 0, a.length - 1, 0);  }

private static void sort(String[] a, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   int v = charAt(a[lo], d);
   int i = lo + 1;
   while (i <= gt)
   {
      int t = charAt(a[i], d);
      if      (t < v) exch(a, lt++, i++);
      else if (t > v) exch(a, i, gt--);
      else            i++;
   }

   sort(a, lo, lt-1, d);
   if (v >= 0) sort(a, lt, gt, d+1);
   sort(a, gt+1, hi, d);
}
```

3-way partitioning (using d^{th} character)

to handle variable-length strings

sort 3 pieces recursively

3-way string quicksort vs. standard quicksort

Standard quicksort.
- Uses $2N \ln N$ string compares on average.
- Costly for long keys that differ only at the end (and this is a common case!)

3-way string quicksort.
- Uses $2N \ln N$ character compares on average for random strings.
- Avoids recomparing initial parts of the string.
- Adapts to data: uses just "enough" characters to resolve order.
- Sublinear when strings are long.

Proposition. 3-way string quicksort is optimal (to within a constant factor); no sorting algorithm can (asymptotically) examine fewer chars.

Pf. Ties cost to entropy. Beyond scope of 226.

3-way string quicksort vs. MSD string sort

MSD string sort.
- Has a long inner loop.
- Is cache-inefficient.
- Too much overhead reinitializing `count[]` and `aux[]`.

3-way string quicksort.
- Has a short inner loop.
- Is cache-friendly.
- Is in-place.

library call numbers

```
WUS-------10706-----7---10
WUS-------12692-----4---27
WLSOC------2542----30
LTK--6015-P-63-1988
LDS---361-H-4
...
```

Bottom line. 3-way string quicksort is the method of choice for sorting strings.

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2 /2$	$N^2 /4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 N W	2 N W	N + R	yes	`charAt()`
MSD ‡	2 N W	$N \log_R N$	N + D R	yes	`charAt()`
3-way string quicksort	1.39 W N lg N *	1.39 N lg N	log N + W	no	`charAt()`

* probabilistic
† fixed-length W keys
‡ average-length W keys

- key-indexed counting
- LSD string sort
- MSD string sort
- 3-way string quicksort
- **suffix arrays**

Warmup: longest common prefix

LCP. Given two strings, find the longest substring that is a prefix of both.

p	r	e	f	e	t	c	h
0	1	2	3	4	5	6	7
p	r	e	f	i	x		

```
public static String lcp(String s, String t)
{
   int n = Math.min(s.length(), t.length());
   for (int i = 0; i < n; i++)
   {
      if (s.charAt(i) != t.charAt(i))
         return s.substring(0, i);
   }
   return s.substring(0, n);
}
```

Running time. Linear-time in length of prefix match.

Space. Constant extra space.

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

Ex.

```
a a c a a g t t t a c a a g c a t g a t g c t g t a c t a
g g a g a g t t a t a c t g g t c g t c a a a c c t g a a
c c t a a t c c t t g t g t g t a c a c a c a c t a c t a
c t g t c g t c g t c a t a t a t c g a g a t c a t c g a
a c c g g a a g g c c g g a c a a g g c g g g g g g t a t
a g a t a g a t a g a c c c c t a g a t a c a c a t a c a
t a g a t c t a g c t a g c t a g c t c a t c g a t a c a
c a c t c t c a c a c t c a a g a g t t a t a c t g g t c
a a c a c a c t a c t a c g a c a g a c g a c c a a c c a
g a c a g a a a a a a a a c t c t a t a t c t a t a a a a
```

Applications. Bioinformatics, cryptanalysis, data compression, ...

Longest repeated substring: a musical application

Visualize repetitions in music. `http://www.bewitched.com`

Mary Had a Little Lamb

Bach's Goldberg Variations

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

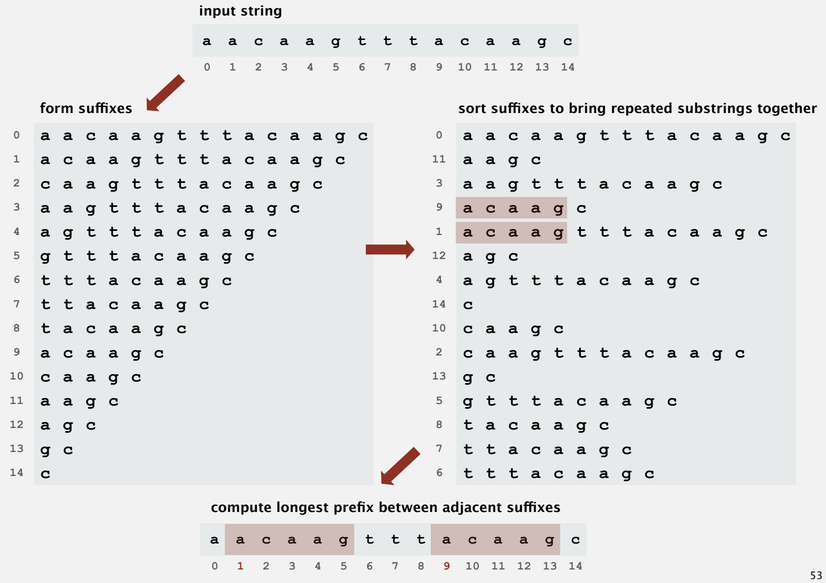
Brute-force algorithm.

- Try all indices i and j for start of possible match.
- Compute longest common prefix (LCP) for each pair.

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c

Analysis. Running time ≤ $M N^2$, where M is length of longest match.

Longest repeated substring: a sorting solution



input string

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

form suffixes

0	a a c a a g t t t a c a a g c
1	a c a a g t t t a c a a g c
2	c a a g t t t a c a a g c
3	a a g t t t a c a a g c
4	a g t t t a c a a g c
5	g t t t a c a a g c
6	t t t a c a a g c
7	t t a c a a g c
8	t a c a a g c
9	a c a a g c
10	c a a g c
11	a a g c
12	a g c
13	g c
14	c

sort suffixes to bring repeated substrings together

0	a a c a a g t t t a c a a g c
11	a a g c
3	a a g t t t a c a a g c
9	a c a a g c
1	a c a a g t t t a c a a g c
12	a g c
4	a g t t t a c a a g c
14	c
10	c a a g c
2	c a a g t t t a c a a g c
13	g c
5	g t t t a c a a g c
8	t a c a a g c
7	t t a c a a g c
6	t t t a c a a g c

compute longest prefix between adjacent suffixes

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Longest repeated substring: Java implementation

```
public String lrs(String s)
{
   int N = s.length();

   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = s.substring(i, N);

   Arrays.sort(suffixes);

   String lrs = "";
   for (int i = 0; i < N-1; i++)
   {
      String x = lcp(suffixes[i], suffixes[i+1]);
      if (x.length() > lrs.length()) lrs = x;
   }
   return lrs;
}
```

create suffixes (linear time and space)

sort suffixes

find LCP between suffixes that are adjacent after sorting

```
% java LRS < mobydick.txt
,- Such a funny, sporty, gamy, jesty, joky, hoky-poky lad, is the Ocean, oh! Th
```

Sorting challenge

Problem. Five scientists A, B, C, D, and E are looking for long repeated substring in a genome with over 1 billion nucleotides.

- A has a grad student do it by hand.
- B uses brute force (check all pairs).
- C uses suffix sorting solution with insertion sort.
- D uses suffix sorting solution with LSD string sort.
- ✓ E uses suffix sorting solution with 3-way string quicksort.

only if LRS is not long (!)

Q. Which one is more likely to lead to a cure cancer?

Longest repeated substring: empirical analysis

input file	characters	brute	suffix sort	length of LRS
LRS.java	2,162	0.6 sec	0.14 sec	73
amendments.txt	18,369	37 sec	0.25 sec	216
aesop.txt	191,945	1.2 hours	1.0 sec	58
mobydick.txt	1.2 million	43 hours †	7.6 sec	79
chromosome11.txt	7.1 million	2 months †	61 sec	12,567
pi.txt	10 million	4 months †	84 sec	14

† estimated

Suffix sorting: worst-case input

Longest repeated substring not long. Hard to beat 3-way string quicksort.

Longest repeated substring very long.

- String sorts are quadratic in the length of the longest match.
- Ex: two copies of Aesop's fables.

```
% more abcdefg2.txt
abcdefg
abcdefgabcdefg
bcdefg
bcdefghabcdefg
cdefg
cdefgabcdefg
defg
efgabcdefg
efg
fgabcdefg
fg
gabcdefg
g
```

	time to suffix sort (seconds)	
algorithm	`mobydick.txt`	`aesop2.txt`
brute-force	36,000 †	4000 †
quicksort	9.5	167
LSD	not fixed length	not fixed length
MSD	395	out of memory
MSD with cutoff	6.8	162
3-way string quicksort	2.8	400

† estimated

Suffix sorting challenge

Problem. Suffix sort an arbitrary string of length N.

Q. What is worst-case running time of best algorithm for problem?

- Quadratic.
- ✓ Linearithmic. ← Manber's algorithm
- ✓ Linear. ← suffix trees (see COS 423)
- Nobody knows.

Suffix sorting in linearithmic time

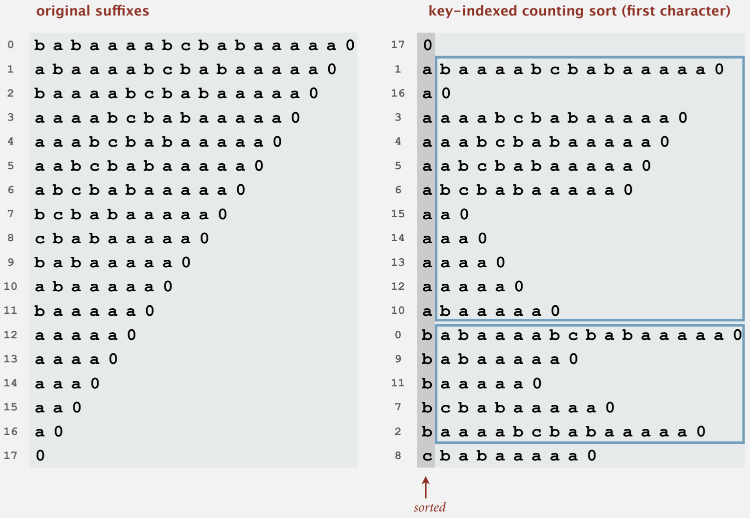
Manber's MSD algorithm overview.

- Phase 0: sort on first character using key-indexed counting sort.
- Phase i: given array of suffixes sorted on first 2^{i-1} characters, create array of suffixes sorted on first 2^i characters.

Worst-case running time. $N \lg N$.

- Finishes after $\lg N$ phases.
- Can perform a phase in linear time. (!) [ahead]

Linearithmic suffix sort example: phase 0



original suffixes

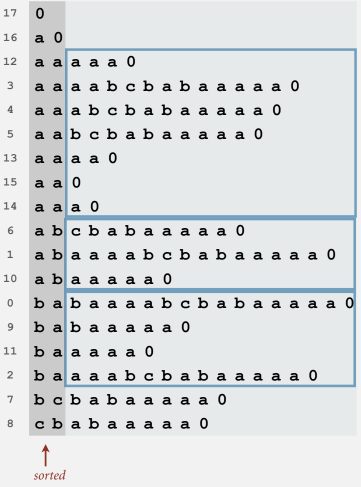
```
 0  b a b a a a a b c b a b a a a a a 0
 1  a b a a a a b c b a b a a a a a 0
 2  b a a a a b c b a b a a a a a 0
 3  a a a a b c b a b a a a a a 0
 4  a a a b c b a b a a a a a 0
 5  a a b c b a b a a a a a 0
 6  a b c b a b a a a a a 0
 7  b c b a b a a a a a 0
 8  c b a b a a a a a 0
 9  b a b a a a a a 0
10  a b a a a a a 0
11  b a a a a a 0
12  a a a a a 0
13  a a a a 0
14  a a a 0
15  a a 0
16  a 0
17  0
```

key-indexed counting sort (first character)

```
17  0
 1  a b a a a a b c b a b a a a a a 0
16  a 0
 3  a a a a b c b a b a a a a a 0
 4  a a a b c b a b a a a a a 0
 5  a a b c b a b a a a a a 0
 6  a b c b a b a a a a a 0
15  a a 0
14  a a a 0
13  a a a a 0
12  a a a a a 0
10  a b a a a a a 0
 0  b a b a a a a b c b a b a a a a a 0
 9  b a b a a a a a 0
11  b a a a a a 0
 7  b c b a b a a a a a 0
 2  b a a a a b c b a b a a a a a 0
 8  c b a b a a a a a 0
```

↑ *sorted*

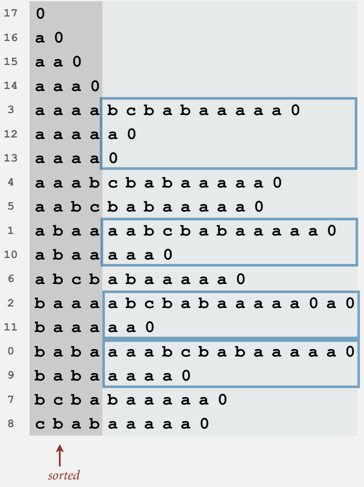
Linearithmic suffix sort example: phase 1



	original suffixes		index sort (first two characters)
0	b a b a a a a b c b a b a a a a a 0	17	0
1	a b a a a a b c b a b a a a a a 0	16	a 0
2	b a a a a b c b a b a a a a a 0	12	a a a a a 0
3	a a a a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0
4	a a a b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0
5	a a b c b a b a a a a a 0	5	a a b c b a b a a a a a 0
6	a b c b a b a a a a a 0	13	a a a a 0
7	b c b a b a a a a a 0	15	a a 0
8	c b a b a a a a a 0	14	a a a 0
9	b a b a a a a a 0	6	a b c b a b a a a a a 0
10	a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0
11	b a a a a a 0	10	a b a a a a a 0
12	a a a a a 0	0	b a b a a a a b c b a b a a a a a 0
13	a a a a 0	9	b a b a a a a a 0
14	a a a 0	11	b a a a a a 0
15	a a 0	2	b a a a a b c b a b a a a a a 0
16	a 0	7	b c b a b a a a a a 0
17	0	8	c b a b a a a a a 0

sorted

Linearithmic suffix sort example: phase 2



	original suffixes		index sort (first four characters)
0	b a b a a a a b c b a b a a a a a 0	17	0
1	a b a a a a b c b a b a a a a a 0	16	a 0
2	b a a a a b c b a b a a a a a 0	15	a a 0
3	a a a a b c b a b a a a a a 0	14	a a a 0
4	a a a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0
5	a a b c b a b a a a a a 0	12	a a a a a 0
6	a b c b a b a a a a a 0	13	a a a a 0
7	b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0
8	c b a b a a a a a 0	5	a a b c b a b a a a a a 0
9	b a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0
10	a b a a a a a 0	10	a b a a a a a 0
11	b a a a a a 0	6	a b c b a b a a a a a 0
12	a a a a a 0	2	b a a a a b c b a b a a a a a 0 a 0
13	a a a a 0	11	b a a a a a 0
14	a a a 0	0	b a b a a a a b c b a b a a a a a 0
15	a a 0	9	b a b a a a a a 0
16	a 0	7	b c b a b a a a a a 0
17	0	8	c b a b a a a a a 0

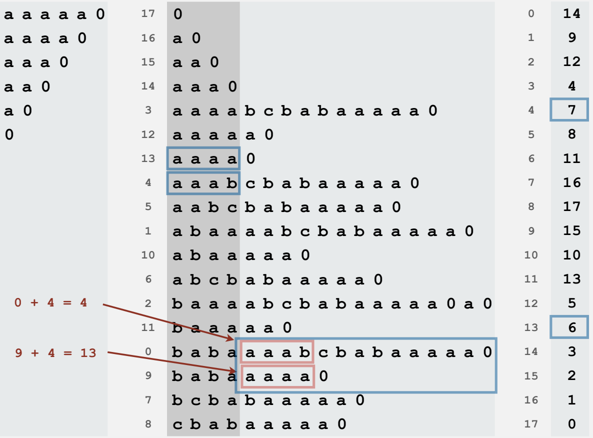
sorted

Linearithmic suffix sort example: phase 3

	original suffixes		index sort (first eight characters)
0	b a b a a a a b c b a b a a a a a 0	17	0
1	a b a a a a b c b a b a a a a a 0	16	a 0
2	b a a a a b c b a b a a a a a 0	15	a a 0
3	a a a a b c b a b a a a a a 0	14	a a a 0
4	a a a b c b a b a a a a a 0	13	a a a a 0
5	a a b c b a b a a a a a 0	12	a a a a a 0
6	a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0
7	b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0
8	c b a b a a a a a 0	5	a a b c b a b a a a a a 0
9	b a b a a a a a 0	10	a b a a a a a 0
10	a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0
11	b a a a a a 0	6	a b c b a b a a a a a 0
12	a a a a a 0	11	b a a a a a 0
13	a a a a 0	2	b a a a a b c b a b a a a a a 0 a 0
14	a a a 0	9	b a b a a a a a 0
15	a a 0	0	b a b a a a a b c b a b a a a a a 0
16	a 0	7	b c b a b a a a a a 0
17	0	8	c b a b a a a a a 0

finished (no equal keys)

Achieve constant-time string compare by indexing into inverse



	original suffixes		index sort (first four characters)		inverse
0	b a b a a a a b c b a b a a a a a 0	17	0	0	14
1	a b a a a a b c b a b a a a a a 0	16	a 0	1	9
2	b a a a a b c b a b a a a a a 0	15	a a 0	2	12
3	a a a a b c b a b a a a a a 0	14	a a a 0	3	4
4	a a a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0	4	7
5	a a b c b a b a a a a a 0	12	a a a a a 0	5	8
6	a b c b a b a a a a a 0	13	a a a a 0	6	11
7	b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0	7	16
8	c b a b a a a a a 0	5	a a b c b a b a a a a a 0	8	17
9	b a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0	9	15
10	a b a a a a a 0	10	a b a a a a a 0	10	10
11	b a a a a a 0	6	a b c b a b a a a a a 0	11	13
12	a a a a a 0	2	b a a a a b c b a b a a a a a 0 a 0	12	5
13	a a a a 0	11	b a a a a a 0	13	6
14	a a a 0	0	b a b a a a a b c b a b a a a a a 0	14	3
15	a a 0	9	b a b a a a a a 0	15	2
16	a 0	7	b c b a b a a a a a 0	16	1
17	0	8	c b a b a a a a a 0	17	0

0 + 4 = 4

9 + 4 = 13

`suffixes`$_4$`[13]` ≤ `suffixes`$_4$`[4]` (*because* `inverse[13]` < `inverse[4]`)

so `suffixes`$_8$`[9]` ≤ `suffixes`$_8$`[0]`

Suffix sort: experimental results

	time to suffix sort (seconds)	
algorithm	`mobydick.txt`	`aesopaesop.txt`
brute-force	36.000 †	4000 †
quicksort	9.5	167
LSD	not fixed length	not fixed length
MSD	395	out of memory
MSD with cutoff	6.8	162
3-way string quicksort	2.8	400
Manber MSD	17	8.5

† estimated

String sorting summary

We can develop linear-time sorts.

- Key compares not necessary for string keys.
- Use characters as index in an array.

We can develop sublinear-time sorts.

- Should measure amount of data in keys, not number of keys.
- Not all of the data has to be examined.

3-way string quicksort is asymptotically optimal.

- $1.39 N \lg N$ chars for random data.

Long strings are rarely random in practice.

- Goal is often to learn the structure!
- May need specialized algorithms.