3.3 Balanced Search Trees

- 2-3 search trees
- red-black BSTs
- B-trees
**Symbol table review**

<table>
<thead>
<tr>
<th>Implementation</th>
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**Challenge.** Guarantee performance.

**This lecture.** 2-3 trees, left-leaning red-black BSTs, B-trees.

Introduced to the world in COS 226, Fall 2007
- 2-3 search trees
- red-black BSTs
- B-trees
2-3 tree

Allow 1 or 2 keys per node.
• 2-node: one key, two children.
• 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.
Perfect balance. Every path from root to null link has same length.
Search in a 2-3 tree

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).
**Insertion in a 2-3 tree**

**Case 1.** Insert into a 2-node at bottom.
- Search for key, as usual.
- Replace 2-node with 3-node.
**Case 2.** Insert into a 3-node at bottom.

- Add new key to 3-node to create **temporary 4-node**.
- Move middle key in 4-node into parent.

![Diagram showing insertion of Z into a 2-3 tree](image)
Insertion in a 2-3 tree

**Case 2.** Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
Insertion in a 2-3 tree

**Case 2.** Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

**Remark.** Splitting the root increases height by 1.
2-3 tree construction trace

Standard indexing client.
The same keys inserted in ascending order.
Local transformations in a 2-3 tree

Splitting a 4-node is a local transformation: constant number of operations.
Global properties in a 2-3 tree

**Invariants.** Maintains symmetric order and perfect balance.

**Pf.** Each transformation maintains symmetric order and perfect balance.
2-3 tree: performance

**Perfect balance.** Every path from root to null link has same length.

Tree height.
- **Worst case:**
- **Best case:**
2-3 tree: performance

**Perfect balance.** Every path from root to null link has same length.

**Tree height.**
- **Worst case:** $\lg N$.  [all 2-nodes]
- **Best case:** $\log_3 N \approx .631 \lg N$.  [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Guaranteed logarithmic performance for search and insert.
## ST implementations: summary

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Constants depend upon implementation.
2-3 tree: implementation?

Direct implementation is complicated, because:

• Maintaining multiple node types is cumbersome.
• Need multiple compares to move down tree.
• Need to move back up the tree to split 4-nodes.
• Large number of cases for splitting.

Bottom line. Could do it, but there's a better way.
- 2-3 search trees
- red-black BSTs
- B-trees
Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2-3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3-nodes.

![Diagram 1](image1)

![Diagram 2](image2)

![Diagram 3](image3)

![Diagram 4](image4)
An equivalent definition

A BST such that:
• No node has two red links connected to it.
• Every path from root to null link has the same number of black links.
• Red links lean left.

"perfect black balance"
Key property. 1-1 correspondence between 2-3 and LLRB.
**Search implementation for red-black BSTs**

**Observation.** Search is the same as for elementary BST (ignore color).

but runs faster because of better balance

```java
public Val get(Key key)
{
    Node x = root;
    while (x != null)
    {
        int cmp = key.compareTo(x.key);
        if (cmp < 0) x = x.left;
        else if (cmp > 0) x = x.right;
        else if (cmp == 0) return x.val;
    }
    return null;
}
```

**Remark.** Most other ops (e.g., ceiling, selection, iteration) are also identical.
Red-black BST representation

Each node is pointed to by precisely one link (from its parent) ⇒ can encode color of links in nodes.

```java
private static final boolean RED = true;
private static final boolean BLACK = false;

private class Node {
    Key key;
    Value val;
    Node left, right;
    boolean color; // color of parent link
}

private boolean isRed(Node x) {
    if (x == null) return false;
    return x.color == RED;
}
```
Elementary red-black BST operations

Left rotation. Orient a (temporarily) right--leaning red link to lean left.

```
private Node rotateLeft(Node h)
{
    assert (h != null) && isRed(h.right);
    Node x = h.right;
    h.right = x.left;
    x.left = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.
Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

```
private Node rotateRight(Node h)
{
    assert (h != null) && isRed(h.left);
    Node x = h.left;
    h.left = x.right;
    x.right = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.
Elementary red-black BST operations

**Color flip.** Recolor to split a (temporary) 4-node.

```
private void flipColors(Node h)
{
    assert !isRed(h) && isRed(h.left) && isRed(h.right);
    h.color = RED;
    h.left.color = BLACK;
    h.right.color = BLACK;
}
```

**Invariants.** Maintains symmetric order and perfect black balance.
**Basic strategy.** Maintain 1-1 correspondence with 2-3 trees by applying elementary red-black tree operations.
Warmup 1. Insert into a tree with exactly 1 node.

Insertion in a LLRB tree

- **Left Case:**
  - Insert into a single 2-node (two cases)
  - Search ends at this null link
  - Red link to new node containing a converts 2-node to 3-node
  - Rotated left to make a legal 3-node

- **Right Case:**
  - Insert into a single 2-node (two cases)
  - Search ends at this null link
  - Attached new node with red link
  - Rotated left to make a legal 3-node
Case 1. Insert into a 2-node at the bottom.
• Do standard BST insert; color new link red.
• If new red link is a right link, rotate left.
Insertion in a LLRB tree

**Warmup 2. Insert into a tree with exactly 2 nodes.**

- **Larger**:
  - Search ends at this null link
  - Attached new node with red link
  - Colors flipped to black

- **Smaller**:
  - Search ends at this null link
  - Attached new node with red link
  - Rotated right
  - Colors flipped to black

- **Between**:
  - Search ends at this null link
  - Attached new node with red link
  - Rotated
  - Colors flipped to black

**Case 2.** Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
**Case 2.** Insert into a 3-node at the bottom.
- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).
LLRB tree construction trace

Standard indexing client.

insert S

E

A

R

C

H

red black tree

S

E

A

R

C

H

S

E

A

R

C

H

corresponding 2–3 tree

S

E

A

R

C

H

S
Standard indexing client (continued).

LLRB tree construction trace

red black tree

corresponding 2–3 tree
Insertion in a LLRB tree: Java implementation

Same code for both cases.

- Right child red, left child black: \textit{rotate left}.
- Left child, left-left grandchild red: \textit{rotate right}.
- Both children red: \textit{flip colors}.

```java
private Node put(Node h, Key key, Value val) {
    if (h == null) return new Node(key, val, RED);
    int cmp = key.compareTo(h.key);
    if      (cmp  < 0) h.left  = put(h.left,  key, val);
    else if (cmp  > 0) h.right = put(h.right, key, val);
    else if (cmp == 0) h.val = val;
    if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
    if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
    if (isRed(h.left)  && isRed(h.right))     flipColors(h);
    return h;
}
```
Insertion in a LLRB tree: visualization

N = 255
max = 8
avg = 7.0
opt = 7.0

255 insertions in ascending order
Insertion in a LLRB tree: visualization

N = 255
max = 8
avg = 7.0
opt = 7.0

255 insertions in descending order
Insertion in a LLRB tree: visualization

N = 50

50 random insertions
Insertion in a LLRB tree: visualization

N = 255
max = 10
avg = 7.3
opt = 7.0

255 random insertions
Balance in LLRB trees

**Proposition.** Height of tree is $\leq 2 \lg N$ in the worst case.

**Pf.**

- Every path from root to null link has same number of black links.
- Never two red links in-a-row.

**Property.** Height of tree is $\sim 1.00 \lg N$ in typical applications.
ST implementations: frequency counter

Costs for java FrequencyCounter 8 < tale.txt using BST

Costs for java FrequencyCounter 8 < tale.txt using RedBlackBST
### ST implementations: summary

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<td>2 lg N</td>
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* exact value of coefficient unknown but extremely close to 1
Why left-leaning trees?

old code (that students had to learn in the past)

```java
private Node put(Node x, Key key, Value val, boolean sw) {
    if (x == null)
        return new Node(key, value, RED);
    int cmp = key.compareTo(x.key);
    if (isRed(x.left) && isRed(x.right))
        {
            x.color = RED;
            x.left.color = BLACK;
            x.right.color = BLACK;
        }
    if (cmp < 0)
    {
        x.left = put(x.left, key, val, false);
        if (isRed(x) && isRed(x.left) && sw)
            x = rotateRight(x);
        if (isRed(x.left) && isRed(x.left.left))
            x = rotateRight(x);
        x.color = BLACK; x.right.color = RED;
    }
    else if (cmp > 0)
    {
        x.right = put(x.right, key, val, true);
        if (isRed(h) && isRed(x.right) && !sw)
            h = rotateLeft(h);
        if (isRed(h.left) && isRed(h.left.left))
            h = rotateRight(h);
        if (isRed(h.left) && isRed(h.right))
            flipColors(h);
    }
    else x.val = val;
    return x;
}
```

new code (that you have to learn)

```java
public Node put(Node h, Key key, Value val) {
    if (h == null)
        return new Node(key, val, RED);
    int cmp = key.compareTo(h.key);
    if (cmp < 0)
        h.left = put(h.left, key, val);
    else if (cmp > 0)
        h.right = put(h.right, key, val);
    else h.val = val;
    if (isRed(h.right) && !isRed(h.left))
        h = rotateLeft(h);
    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);
    if (isRed(h.left) && isRed(h.right))
        flipColors(h);
    return h;
}
```

straightforward (if you’ve paid attention)

extremely tricky
Why left-leaning red-black BSTs?

Simplified code.
• Left-leaning restriction reduces number of cases.
• Short inner loop.

Same ideas simplify implementation of other operations.
• Delete min/max.
• Arbitrary delete.

Improves widely-used balanced search trees.
• AVL trees, splay trees, randomized BSTs, ...
• 2-3 trees, 2-3-4 trees.
• Red-black BSTs.

Bottom line. Left-leaning red-black BSTs are among the simplest balanced BSTs to implement and among the fastest in practice.
War story: why red-black?

**Xerox PARC innovations. [1970s]**

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- InterPress.
- Laser printing.
- Bitmaped display.
- WYSIWYG text editor.
- ...

---

**A Dichromatic Framework For Balanced Trees**

Leo J. Guibas  
*Xerox Palo Alto Research Center, Palo Alto, California, and Carnegie-Mellon University*

Robert Sedgewick  
*Program in Computer Science, Brown University, Providence, R. I.*

**ABSTRACT**

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this framework the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its
War story: red-black BSTs

Telephone company contracted with database provider to build real-time database to store customer information.

Database implementation.
• Red-black BST search and insert; Hibbard deletion.
• Exceeding height limit of 80 triggered error-recovery process.

Extended telephone service outage.
• Main cause = height bounded exceeded!
• Telephone company sues database provider.
• Legal testimony:

“If implemented properly, the height of a red-black BST with N keys is at most 2 lg N.” — expert witness
2-3 search trees
red-black BSTs
B-trees
File system model

**Page.** Contiguous block of data (e.g., a file or 4096-byte chunk).

**Probe.** First access to a page (e.g., from disk to memory).

**Property.** Time required for a probe is much larger than time to access data within a page.

**Cost model.** Number of probes.

**Goal.** Access data using minimum number of probes.
**B-trees (Bayer-McCreight, 1972)**

**B-tree.** Generalize 2-3 trees by allowing up to $M - 1$ key-link pairs per node.

- At least 2 key-link pairs at root.
- At least $M/2$ key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

Choose $M$ as large as possible so that $M$ links fit in a page, e.g., $M = 1024$.

*Anatomy of a B-tree set ($M = 6$)*

- All nodes except the root are 3-, 4- or 5-nodes.
- Each red key is a copy of the min key in the subtree.
- Client keys (black) are in external nodes.
Searching in a B-tree

- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.

![Diagram of searching in a B-tree](image)
Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with $M$ key-link pairs on the way up the tree.

Inserting a new key into a B-tree set
Balance in B-tree

**Proposition.** A search or an insertion in a B-tree of order $M$ with $N$ keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

**Pf.** All internal nodes (besides root) have between $M / 2$ and $M - 1$ links.

**In practice.** Number of probes is at most 4. $\Rightarrow \log_{M/2} N \leq 4$

**Optimization.** Always keep root page in memory.
Building a large B tree

full page, about to split

external nodes
(line segment of length proportional to number of keys in that node)
Red-black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: `map`, `multimap`, `multiset`.
- Linux kernel: `completely fair scheduler`, `linux/rbtree.h`.

B-tree variants. B+ tree, B*tree, B# tree, ...

B-trees (and variants) are widely used for file systems and databases.

- Windows: HPFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.
Red-black BSTs in the wild

Common sense. Sixth sense. Together they're the FBI's newest team.
FADE IN:

48    INT. FBI HQ - NIGHT

Antonio is at THE COMPUTER as Jess explains herself to Nicole and Pollock. The CONFERENCE TABLE is covered with OPEN REFERENCE BOOKS, TOURIST GUIDES, MAPS and REAMS OF PRINTOUTS.

JESS
It was the red door again.

POLLOCK
I thought the red door was the storage container.

JESS
But it wasn't red anymore. It was black.

ANTONIO
So red turning to black means...
what?

POLLOCK
Budget deficits? Red ink, black ink?

NICOLE
Yes. I'm sure that's what it is. But maybe we should come up with a couple other options, just in case.

Antonio refers to his COMPUTER SCREEN, which is filled with mathematical equations.

ANTONIO
It could be an algorithm from a binary search tree. A red-black tree tracks every simple path from a node to a descendant leaf with the same number of black nodes.

JESS
Does that help you with girls?