Linear algebra review for cos424

RECTANGULAR MATRICES

Let $\mathbf{A} = [a_{ij}]_{\substack{i=1\dots m \\ j=1\dots n}}$ be a matrix with m lines and n columns.

Transposition: $\mathbf{B} = \mathbf{A}^\top \quad \stackrel{\triangle}{\Longleftrightarrow} \ \forall i = 1\dots n \ \ \forall j = 1\dots m \ \ b_{ij} = a_{ji}$.

- $(\mathbf{A}^\top)^\top = \mathbf{A}$, $(\mathbf{A}+\mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$, $(\lambda\mathbf{A})^\top = \lambda\,\mathbf{A}^\top$, $(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top$.

Kernel (nullspace): $\mathcal{K}er(\mathbf{A}) \triangleq \{\mathbf{x} \ : \ \mathbf{Ax} = \mathbf{0}\}$.

Image (column space): $\mathcal{I}m(\mathbf{A})$: the vectorial space spanned by the columns of $\mathbf{A}$.

- $\dim(\mathcal{I}m(\mathbf{A})) + \dim(\mathcal{K}er(\mathbf{A})) = n$. — Hint: consider a basis of $\mathbb{R}^n$ whose first elements span $\mathcal{K}er(\mathbf{A})$.

Rank: $\operatorname{rank}(\mathbf{A}) \triangleq \dim(\mathcal{I}m(\mathbf{A}))$.

- $\operatorname{rank}(\mathbf{A}) \le \min(m,n)$. — Proof: the n columns are elements of $\mathbb{R}^m$.
- $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^\top)$. — Hint: first show that $\operatorname{rank}(\mathbf{A}^\top) \le \operatorname{rank}(\mathbf{A})$.

Orthogonal matrix: $\mathbf{A}$ is orthogonal $\quad \stackrel{\triangle}{\Longleftrightarrow} \ \mathbf{A}^\top\mathbf{A} = \mathbf{I}$.

- Equivalent statement: the columns of $\mathbf{A}$ are orthogonal unit vectors.
- $\mathbf{A}$ is orthogonal $\implies \operatorname{rank}(\mathbf{A}) = n \implies n \le m$.
- The columns of a square *orthogonal matrix* form an *orthonormal basis*.

QR decomposition: $\mathbf{A} = \mathbf{QR}$ with $\mathbf{Q}$: $m \times r$, orthogonal, and $\mathbf{R}$: $r \times n$, upper triangular, $r = \operatorname{rank}(\mathbf{A})$.

- *Gram-Schmidt orthogonalization algorithm*:
  let $\mathbf{u}_i$ be the columns of $\mathbf{A}$, $\mathbf{v}_k$ be the columns of $\mathbf{Q}$, r_{ki} the coefficients of $\mathbf{R}$.
  $k \leftarrow 0$; **for** $i = 1\dots n$: **for** $j = 1\dots k$: $r_{ij} = \mathbf{u}_i\mathbf{v}_j$; **endfor**; $\mathbf{x} \leftarrow \mathbf{u}_i - \sum_{j=1}^{k} r_{ki}\mathbf{v}_k$;
  if $\mathbf{x} \ne \mathbf{0}$: $k \leftarrow k+1$; $r_{ik} = \|\mathbf{x}\|$; $\mathbf{v}_k = r_{ik}^{-1}\mathbf{x}$; **endif**; **endfor**.

SQUARE MATRICES

Let $\mathbf{A} = [a_{ij}]_{\substack{i=1\dots n \\ j=1\dots n}}$ be a matrix with n lines and n columns.

Invertible matrix: $\mathbf{A}$ is invertible $\quad \stackrel{\triangle}{\Longleftrightarrow} \ \operatorname{rank}(\mathbf{A}) = n$

- $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$ and $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Triangular matrix: All coefficients below (or above) the diagonal are null.

- $\mathbf{A}$ is upper triangular $\quad \stackrel{\triangle}{\Longleftrightarrow} \ \forall\, j < i,\ a_{ij} = 0$.
- $\mathbf{A}$ is lower triangular $\quad \stackrel{\triangle}{\Longleftrightarrow} \ \forall\, i < j,\ a_{ij} = 0$.

Determinant:
Let S_n is the set of the $n!$ permutations of $\{1\dots n\}$.
Let $\mathcal{N}(\sigma)$ be the number of pairs of indices misordered pairs by permutation $\sigma \in S_n$.
Then $\det(\mathbf{A}) \triangleq \displaystyle\sum_{\sigma\in S_n} (-1)^{\mathcal{N}(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}$.

- $\det(\mathbf{I}) = 1$.
- $\det(\mathbf{A}) = 0 \iff \mathbf{A}$ is not invertible.

- $\det(\mathbf{A}) = \det(\mathbf{A}^\top)$.
- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.
- $|\det(\mathbf{A})|$ is proportional to the volume of the convex hull formed by the origin and the columns of $\mathbf{A}$.
- $\mathbf{A}$ orthogonal $\implies$ $\det(\mathbf{A}) = \pm 1$.
- $\mathbf{A}$ triangular $\implies$ $\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii}$.

Eigenspectrum: $\mathbf{x} \neq \mathbf{0}$ is an eigenvector of $\mathbf{A}$ for eigenvalue λ $\stackrel{\Delta}{\iff}$ $\mathbf{Ax} = \lambda\mathbf{x}$.

- spectrum($\mathbf{A}$) is the set of eigenvalues of $\mathbf{A}$.
- If $A - \lambda\mathbf{I}$ is invertible, $\lambda \notin$ spectrum($\mathbf{A}$). – Hint: then $\mathbf{A} = \lambda\mathbf{x}$ has only one solution: $\mathbf{x} = \mathbf{0}$.
- The eigenvalues of $\mathbf{A}$ are the roots of the *characteristic polynomial* $P(\lambda) = \det(A - \lambda\mathbf{I})$.

Trace: $\mathrm{trace}(\mathbf{A}) \triangleq \sum_{i=1}^{n} a_{ii}$.

- $\mathrm{trace}(\mathbf{AB}) = \mathrm{trace}(\mathbf{BA})$. — Proof: calculus.

SYMMETRIC MATRICES: $\mathbf{A}^\top = \mathbf{A}$

Spectral theorem: There is an orthogonal matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{Q}^\top\mathbf{D}\,\mathbf{Q}$.
— Hint: study the complex roots of the characteristic polynomial (tricky.)

- The columns of $\mathbf{Q}$ are eigenvectors.
- The diagonal matrix $\mathbf{D}$ contains the corresponding eigenvalues.

Positive definite symmetric matrix $\stackrel{\Delta}{\iff}$ $\forall\mathbf{x} \neq \mathbf{0},\ \mathbf{x}^\top A\,\mathbf{x} > 0$.

- All eigenvalues are > 0.
- Equation $\mathbf{x}^\top\mathbf{A}\,\mathbf{x} = 1$ defines an ellipsoid whose principal axes are the eigenvectors.
- The *Cholevsky decomposition algorithm* computes a lower triangular matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{U}\,\mathbf{U}^\top$.
- Weaker variant: *Positive symmetric matrix* $\stackrel{\Delta}{\iff}$ all eigenvalues are ≥ 0.

NUMERICALLY SOLVING $\mathbf{Ax} = \mathbf{b}$

Avoid

- Cramer formulas involving ratios of determinants.
- Algorithms to compute the inverse of a matrix.

When A is a triangular matrix: back-substitution

- Solve the equation with a single unknown
- Substitute into the remaining equations and repeat.
- Warning: when $\mathbf{A}$ is not invertible, aim for the equation with the least unknowns.

When A is an orthogonal matric: transpose

- You already know the inverse: $\mathbf{x} = \mathbf{A}^\top\mathbf{A}\,\mathbf{x} = \mathbf{A}^\top\mathbf{b}$.

General case: use a decomposition algorithm.

- *QR decomposition*: $\mathbf{A} = \mathbf{QR}$ with $\mathbf{Q}$ orthogonal and $\mathbf{R}$ upper triangular;
  solve $\mathbf{Rx} = \mathbf{Q}^\top\mathbf{b}$ with one round of back-substitution.
  Warning: when QR tells you that $\mathrm{rank}(\mathbf{A}) < n$, verify the solution!
  Understanding the QR decomposition is sufficient for most purposes.
- Cholevsky decomposition is faster when $\mathbf{A}$ is positive definite symmetric.
- See Trefethen and Bau for: Column pivoting, LU decomposition, and SVD.

REFERENCE

- Lloyd N. Trefethen and David Bau, III: *Numerical Linear Algebra*. SIAM, 1997.