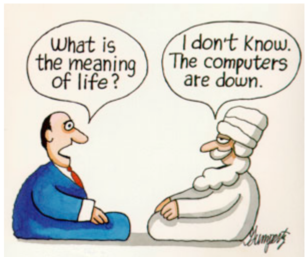
Universality and Computability



Fundamental Questions

Q. What is a general-purpose computer?
Q. Are there limits on the power of digital computers?
Q. Are there limits on the power of machines we can build?

Pioneering work in the 1930s.

- Princeton == center of universe.
- Automata, languages, computability, universality, complexity, logic.


David Hilbert


Kurt Gödel


Alan Turing


Alonzo Church


John von Neumann

7.4 Turing Machines


Alan Turing (1912-1954)

Turing Machine

Desiderata. Simple model of computation that is "as powerful" as conventional computers.

Intuition. Simulate how humans calculate.

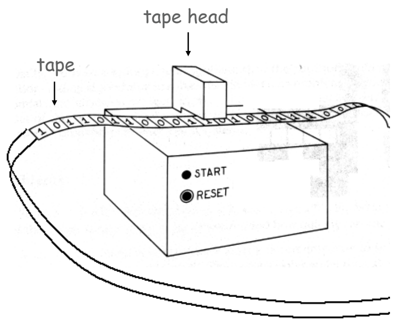
Ex. Addition.

			1	2	3	4	5	6		
		+	3	1	4	1	5	9		

Turing Machine: Tape

Tape.
- Stores input, output, and intermediate results.
- One arbitrarily long strip, divided into cells.
- Finite alphabet of symbols.

Tape head.
- Points to one cell of tape.
- Reads a symbol from active cell.
- Writes a symbol to active cell.
- Moves left or right one cell at a time.

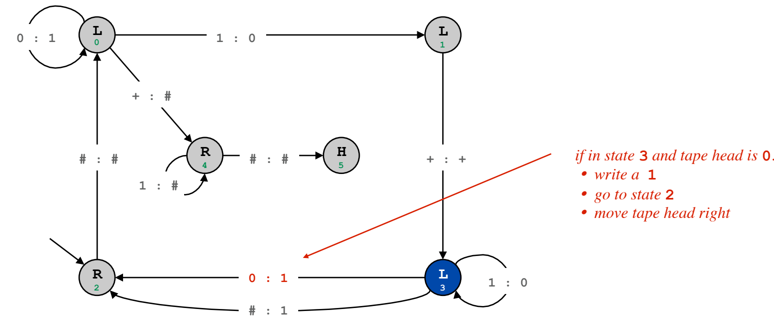


tape head

tape	...	#	1	1	0	0	+	1	0	1	1	#	...

Turing Machine: States

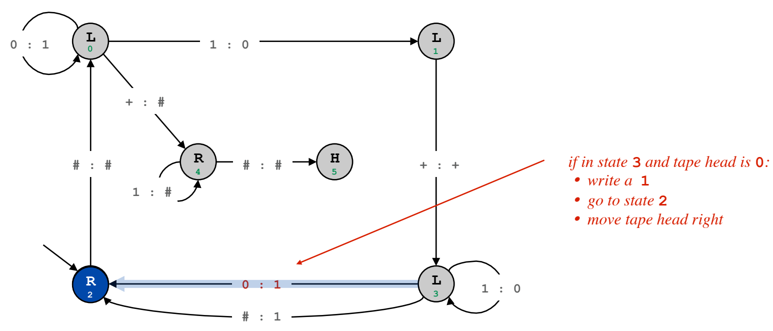
State. What machine remembers.
State transition diagram. Complete description of what machine will do.



tape (before)	...	#	1	1	0	0	+	1	0	1	1	#	...

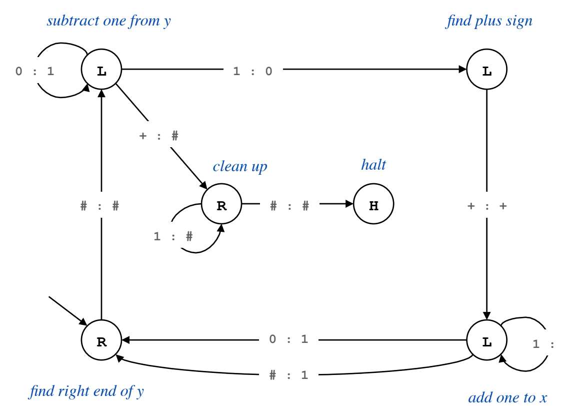
Turing Machine: States

State. What machine remembers.
State transition diagram. Complete description of what machine will do.



tape (after)	...	#	1	1	1	0	+	1	0	1	1	#	...

Binary Adder



...	#	1	0	1	0	+	1	1	1	1	#	...

$\underbrace{1\ 0\ 1\ 0}_{x}$ + $\underbrace{1\ 1\ 1\ 1}_{y}$

Turing Machine: Initialization and Termination

Initialization. Set input on some portion of tape; set tape head.

tape head

tape	...	#	1	0	1	0	+	1	1	1	1	#	...

Termination. Stop if enter `yes`, `no`, or `halt` state.

infinite loop possible

Program and Data

Program and Data

Data. Sequence of symbols (interpreted one way).
Program. Sequence of symbols (interpreted another way).

Ex 1. A compiler is a program that takes a program in one language as input and outputs a program in another language.

Java

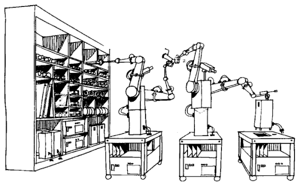
machine language

Program and Data

Data. Sequence of symbols (interpreted one way).
Program. Sequence of symbols (interpreted another way).

Ex 2. Self-replication. [von Neumann 1940s]

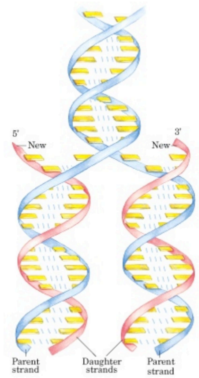
> Print the following statement twice, the second time in quotes.
> "Print the following statement twice, the second time in quotes."



Program and Data

Data. Sequence of symbols (interpreted one way).
Program. Sequence of symbols (interpreted another way).

Ex 3. Self-replication. [Watson-Crick 1953]

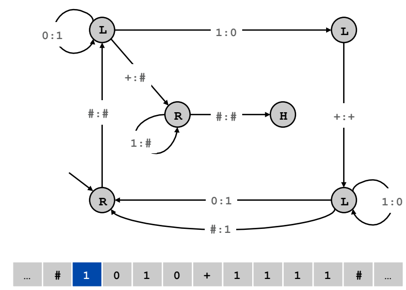


self-replicating DNA

Program and Data

Data. Sequence of symbols (interpreted one way).
Program. Sequence of symbols (interpreted another way).

Ex 4. Turing machine.



graphical representation

```
% more adder.tur
vertices
2 R
0 L
1 L
3 L
4 R
5 H

edges
0 0 0 1
0 1 1 0
0 4 + #
1 3 + +
2 0 # #
3 2 # 1
3 2 0 1
3 3 1 0
4 4 1 #
4 5 # #

tape
[1] 0 1 0 + 1 1 1 1
```

text representation

7.5 Universality

Universal Machines and Technologies

Dell PC

iMac

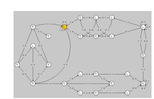
Diebold voting machine

iPod

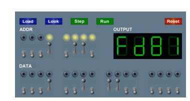
Printer

Xbox

Tivo

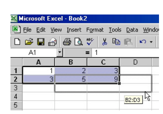


Turing machine



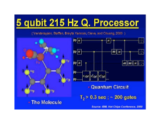
TOY

Java language

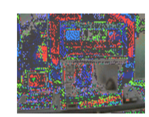


MS Excel

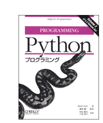
Blackberry



Quantum computer



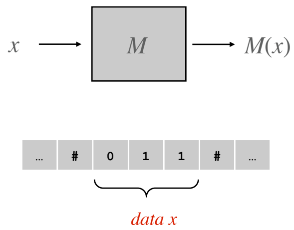
DNA computer



Python language

Universal Turing Machine

Turing machine M. Given input tape x, Turing machine M outputs $M(x)$.

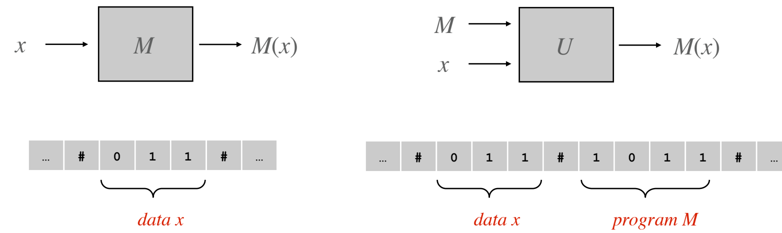


TM intuition. Software program that solves **one** particular problem.

Universal Turing Machine

Turing machine M. Given input tape x, Turing machine M outputs $M(x)$.

Universal Turing machine U. Given input tape with x and M,
universal Turing machine U outputs $M(x)$.



TM intuition. Software program that solves **one** particular problem.
UTM intuition. Hardware platform that can implement **any** algorithm.

Universal Turing Machine

Consequences. Your laptop (a UTM) can do **any** computational task.

- Java programming.
- Pictures, music, movies, games.
- Email, browsing, downloading files, telephony.
- Word-processing, finance, scientific computing.
- …

even tasks not yet contemplated
when laptop was purchased

Wenger Giant Swiss Army Knife

Universal Turing Machine

Consequences. Your laptop (a UTM) can do **any** computational task.

- Java programming.
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- …

even tasks not yet contemplated
when laptop was purchased

" Again, it [the Analytical Engine] might act upon other things besides numbers… the engine might compose elaborate and scientific pieces of music of any degree of complexity or extent. " — Ada Lovelace

Church-Turing Thesis

Church Turing thesis (1936). Turing machines can do anything that can be described by any physically harnessable process of this universe.

Remark. "Thesis" and not a mathematical theorem because it's a statement about the physical world and not subject to proof.

but can be falsified

Implications.

- No need to seek more powerful machines or languages.
- Enables rigorous study of computation (in this universe).

Bottom line. Turing machine is a simple and universal model of computation.

Church-Turing Thesis: Evidence

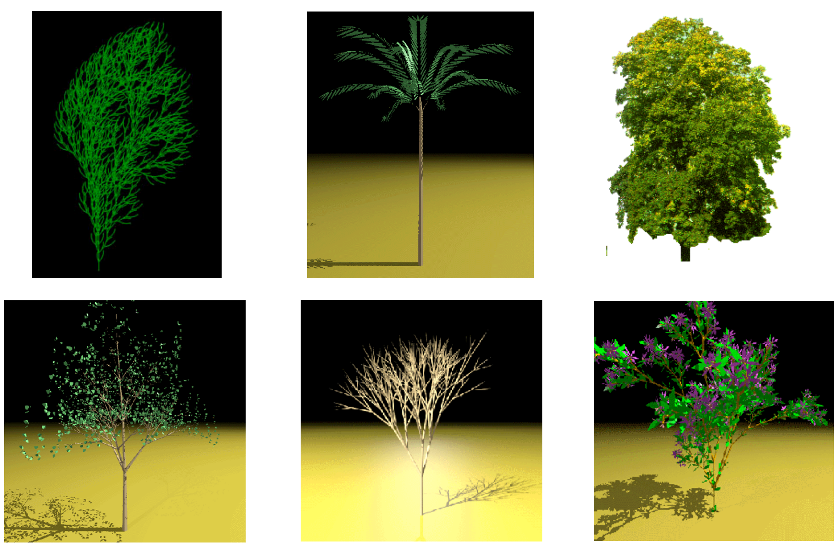
Evidence.

"universal"

- 7 decades without a counterexample.
- Many, many models of computation that turned out to be equivalent.

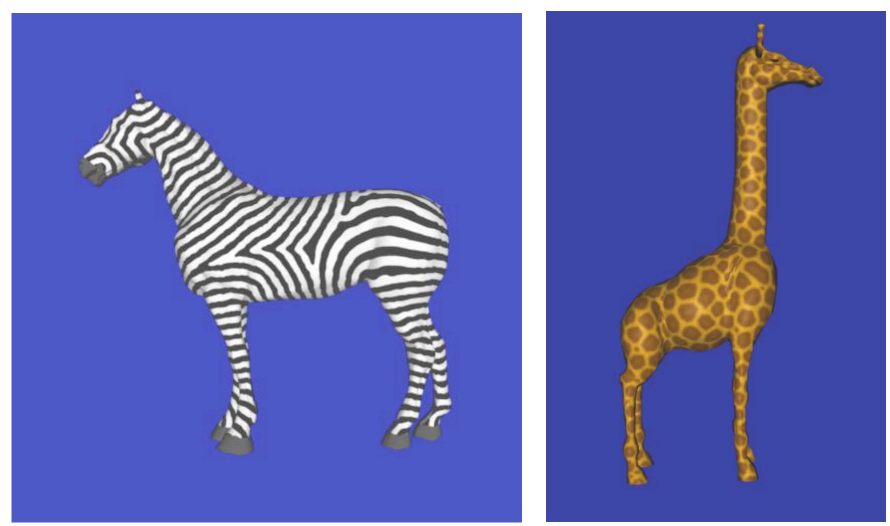
model of computation	description
enhanced Turing machines	multiple heads, multiple tapes, 2D tape, nondeterminism
untyped lambda calculus	method to define and manipulate functions
recursive functions	functions dealing with computation on integers
unrestricted grammars	iterative string replacement rules used by linguists
extended L-systems	parallel string replacement rules that model plant growth
programming languages	Java, C, C++, Perl, Python, PHP, Lisp, PostScript, Excel
random access machines	registers plus main memory, e.g., TOY, Pentium
cellular automata	cells which change state based on local interactions
quantum computer	compute using superposition of quantum states
DNA computer	compute using biological operations on DNA

Lindenmayer Systems: Synthetic Plants



http://astronomy.swin.edu.au/~pbourke/modelling/plants

Cellular Automata: Synthetic Zoo



Reference: *Generating textures on arbitrary surfaces using reaction-diffusion* by Greg Turk, SIGGRAPH, 1991.

History: *The chemical basis of morphogenesis* by Alan Turing, 1952.

7.6 Computability

Halting Problem

Halting problem. Write a Java function that reads in a Java function `f` and its input `x`, and decides whether `f(x)` results in an infinite loop.

relates to famous open math conjecture

Ex. Does `f(x)` terminate?

```
public void f(int x) {
   while (x != 1) {
      if  (x % 2 == 0)  x = x / 2;
      else              x = 3*x + 1;
   }
}
```

- `f(6)`: 6 3 10 5 16 8 4 2 1
- `f(27)`: 27 82 41 124 62 31 94 47 142 71 214 107 322 … 4 2 1
- `f(-17)`: -17 -50 -25 -74 -37 -110 -55 -164 -82 -41 -122 … -17 …

Undecidable Problem

A yes-no problem is undecidable if no Turing machine exists to solve it.

and (by universality) no Java program either

Theorem. [Turing 1937] The halting problem is undecidable.

Proof intuition: lying paradox.

- Divide all statements into two categories: truths and lies.
- How do we classify the statement: *I am lying.*

Key element of lying paradox and halting proof: self-reference.

Halting Problem Proof

Assume the existence of `halt(f,x)`:

- Input: a function `f` and its input `x`.
- Output: `true` if `f(x)` halts, and `false` otherwise.

Note. `halt(f,x)` does not go into infinite loop.

We prove by contradiction that `halt(f,x)` does not exist.

- *Reductio ad absurdum* : if any logical argument based on an assumption leads to an absurd statement, then assumption is false.

encode f and x as strings

```
public boolean halt(String f, String x) {
   if ( something terribly clever ) return true;
   else                             return false;
}
```

hypothetical halting function

Halting Problem Proof

Assume the existence of `halt(f,x)`:

- Input: a function `f` and its input `x`.
- Output: `true` if `f(x)` halts, and `false` otherwise.

Construct function `strange(f)` as follows:

- If `halt(f,f)` returns `true`, then `strange(f)` goes into an infinite loop.
- If `halt(f,f)` returns `false`, then `strange(f)` halts.

f is a string so legal (if perverse)
to use for second input

```
public void strange(String f) {
   if (halt(f, f)) {
      // an infinite loop
      while (true) { }
   }
}
```


Halting Problem Proof

Assume the existence of `halt(f,x)`:

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In other words:

- If `f(f)` halts, then `strange(f)` goes into an infinite loop.
- If `f(f)` does not halt, then `strange(f)` halts.

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- If `f(f)` halts, then `strange(f)` goes into an infinite loop.
- If `f(f)` does not halt, then `strange(f)` halts.

Call `strange()` with ITSELF as input.

- If `strange(strange)` halts then `strange(strange)` does not halt.
- If `strange(strange)` does not halt then `strange(strange)` halts.

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- If `strange(strange)` halts then `strange(strange)` does not halt.
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Either way, a contradiction. Hence `halt(f,x)` cannot exist.

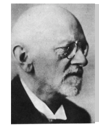
Consequences

Q. Why is debugging hard?
A. All problems below are undecidable.

Halting problem. Give a function f, does it halt on a given input x?
Totality problem. Give a function f, does it halt on every input x?
No-input halting problem. Give a function f with no input, does it halt?
Program equivalence. Do two functions f and always return same value?
Uninitialized variables. Is the variable x initialized before it's used?
Dead-code elimination. Does this statement ever get executed?

More Undecidable Problems

Hilbert's 10^{th} problem.



Devise a process according to which it can be determined by a finite number of operations whether a given multivariate polynomial has an integral root. — David Hilbert

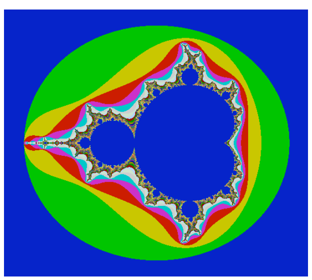
- $f(x, y, z) = 6x^3 y z^2 + 3xy^2 - x^3 - 10.$ yes : $f(5, 3, 0) = 0.$
- $f(x, y) = x^2 + y^2 - 3.$ no.

Definite integration. Given a rational function $f(x)$ composed of polynomial and trig functions, does $\int_{-\infty}^{+\infty} f(x)\,dx$ exist?

- $g(x) = \cos x\,(1 + x^2)^{-1}$ yes, $\int_{-\infty}^{+\infty} g(x)\,dx = \pi / e.$
- $h(x) = \cos x\,(1 - x^2)^{-1}$ no, $\int_{-\infty}^{+\infty} h(x)\,dx$ undefined.

More Undecidable Problems

Optimal data compression. Find the shortest program to produce a given string or picture.



Mandelbrot set (40 lines of code)

More Undecidable Problems

Virus identification. Is this program a virus?

```
Private Sub AutoOpen()
On Error Resume Next
If System.PrivateProfileString("", CURRENT_USER\Software\Microsoft\Office\9.0\Word\Security",
                                "Level") <> "" Then
CommandBars("Macro").Controls("Security...").Enabled = False
. . .
For oo = 1 To AddyBook.AddressEntries.Count
   Peep = AddyBook.AddressEntries(x)
   BreakUmOffASlice.Recipients.Add Peep
   x = x + 1
   If x > 50 Then oo = AddyBook.AddressEntries.Count
Next oo
. . .
BreakUmOffASlice.Subject = "Important Message From " & Application.UserName
BreakUmOffASlice.Body = "Here is that document you asked for ... don't show anyone else ;-)"
. . .
```

Can write programs in MS Word.
This statement disables security.

Melissa virus
March 28, 1999

Context: Mathematics and Logic

Mathematics. Any formal system powerful enough to express arithmetic.

Principia Mathematics
Peano arithmetic
Zermelo-Fraenkel set theory

Complete. Can prove truth or falsity of any arithmetic statement.
Consistent. Can't prove contradictions like 2 + 2 = 5.
Decidable. Algorithm exists to determine truth of every statement.

Q. [Hilbert, 1900] Is mathematics complete and consistent?
A. [Gödel's Incompleteness Theorem, 1931] No!!!

Q. [Hilbert's Entscheidungsproblem] Is mathematics decidable?
A. [Church 1936, Turing 1936] No!

Turing's Key Ideas

Turing machine.
formal model of computation

Program and data.
encode program and data as sequence of symbols

Universality.
concept of general-purpose, programmable computers

Church-Turing thesis.
computable at all == computable with a Turing machine

Computability.
inherent limits to computation

Hailed as one of top 10 science papers of 20th century.

Reference: *On Computable Numbers, With an Application to the Entscheidungsproblem* by A. M. Turing. In Proceedings of the London Mathematical Society, ser. 2. vol. 42 (1936-7), pp.230-265.

Alan Turing

Alan Turing (1912-1954).

- Father of computer science.
- Computer science's "Nobel Prize" is called the Turing Award.

> *It was not only a matter of abstract mathematics, not only a play of symbols, for it involved thinking about what people did in the physical world…. It was a play of imagination like that of Einstein or von Neumann, doubting the axioms rather than measuring effects…. What he had done was to combine such a naïve mechanistic picture of the mind with the precise logic of pure mathematics. His machines – soon to be called Turing machines – offered a bridge, a connection between abstract symbols, and the physical world.* — John Hodges

Alan Turing (left)
Elder brother (right)