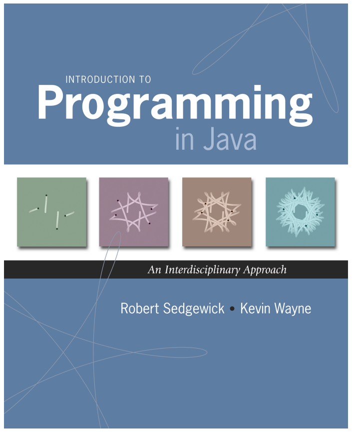
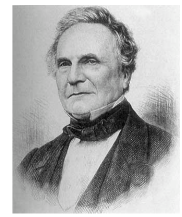
# 4.1 Performance



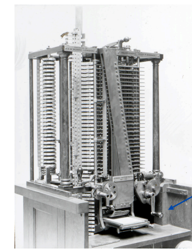



## Running Time

> *"As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time?"* – Charles Babbage



Charles Babbage (1864)



Analytic Engine

how many times do you have to turn the crank?



## The Challenge

Q. Will my program be able to solve a large practical problem?

compile | debug on test case | solve problems in practice

Key insight. [Knuth 1970s]
Use the scientific method to understand performance.



## Scientific Method

Scientific method.

- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.

- Experiments we design must be reproducible.
- Hypothesis must be falsifiable.

## Reasons to Analyze Algorithms

Predict performance.

- Will my program finish?
- When will my program finish?

Compare algorithms.

- Will this change make my program faster?
- How can I make my program faster?

Basis for inventing new ways to solve problems.

- Enables new technology.
- Enables new research.

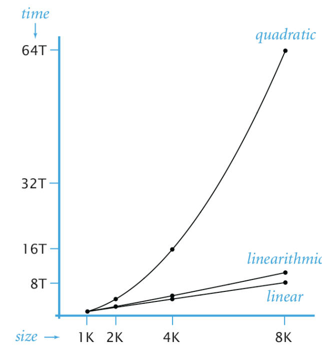


## Algorithmic Successes

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: $N^2$ steps.
- FFT algorithm: N log N steps, enables new technology.

Freidrich Gauss
1805

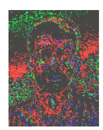




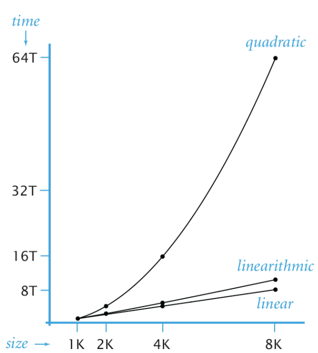

## Algorithmic Successes

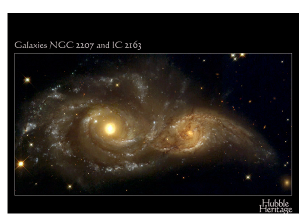
N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: $N^2$ steps.
- Barnes-Hut: N log N steps, enables new research.



Andrew Appel
PU '81






## Three-Sum Problem

Three-sum problem. Given $N$ integers, find triples that sum to $0$.
Context. Deeply related to problems in computational geometry.

```
% more 8ints.txt
30 -30 -20 -10 40 0 10 5

% java ThreeSum < 8ints.txt
 4
 30 -30   0
 30 -20 -10
-30 -10  40
-10   0  10
```

Q. How would you write a program to solve the problem?

## Three-Sum: Brute-Force Solution

```
public class ThreeSum {

   // return number of distinct triples (i, j, k)
   // such that (a[i] + a[j] + a[k] == 0)
   public static int count(int[] a) {
      int N = a.length;
      int cnt = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0) cnt++;
      return cnt;
   }

   public static void main(String[] args) {
      int[] a = StdArrayIO.readInt1D();
      StdOut.println(count(a));
   }
}
```

all possible triples i < j < k



# Empirical Analysis

## Empirical Analysis

Empirical analysis. Run the program for various input sizes.

| N | time † |
|---|---|
| 512 | 0.03 |
| 1024 | 0.26 |
| 2048 | 2.16 |
| 4096 | 17.18 |
| 8192 | 136.76 |

† Running Linux on Sun-Fire-X4100 with 16GB RAM



## Stopwatch

Q. How to time a program?
A. A stopwatch.

```
% java ThreeSum < 1Kints.txt
```


```
0
% java ThreeSum < 2Kints.txt
```


```
2
391930676 -763182495 371251819
-326747290 802431422 -475684132
```

# Stopwatch

Q. How to time a program?
A. A `Stopwatch` object.

| public class Stopwatch | |
|---|---|
| Stopwatch() | *create a new stopwatch and start it running* |
| double elapsedTime() | *return the elapsed time since creation, in seconds* |

```
public class Stopwatch {
   private final long start;

   public Stopwatch() {
      start = System.currentTimeMillis();
   }

   public double elapsedTime() {
      return (System.currentTimeMillis() - start) / 1000.0;
   }
}
```



# Stopwatch

Q. How to time a program?
A. A `Stopwatch` object.

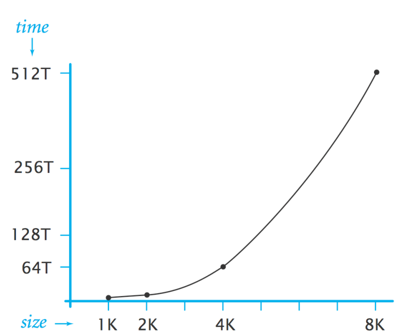
| public class Stopwatch | |
|---|---|
| Stopwatch() | *create a new stopwatch and start it running* |
| double elapsedTime() | *return the elapsed time since creation, in seconds* |

```
public static void main(String[] args) {
   int[] a = StdArrayIO.readInt1D();
   Stopwatch timer = new Stopwatch();
   StdOut.println(count(a));
   StdOut.println(timer.elapsedTime());
}
```



# Empirical Analysis

Data analysis. Plot running time vs. input size $N$.




Q. How fast does running time grow as a function of input size $N$?

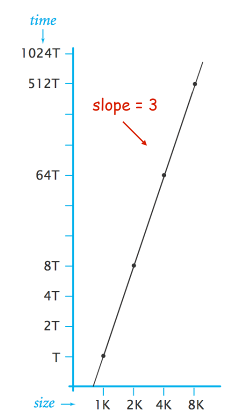


# Empirical Analysis

Initial hypothesis. Running time obeys power law $f(N) = a\,N^{b}$.

Data analysis. Plot running time vs. input size $N$ on a log-log scale.

Consequence. Power law yields straight line (slope = b).




slope

Refined hypothesis. Running time grows as cube of input size: $a\,N^{3}$.

## Doubling Hypothesis

Doubling hypothesis. Quick way to estimate b in a power law hypothesis.

Run program, doubling the size of the input?

| $N$ | time † | ratio |
|---|---|---|
| 512 | 0.033 | - |
| 1024 | 0.26 | 7.88 |
| 2048 | 2.16 | 8.43 |
| 4096 | 17.18 | 7.96 |
| 8192 | 136.76 | 7.96 |

seems to converge to a constant $c = 8$

Hypothesis. Running time is about $a N^b$ with $b = lg\ c$.



## Performance Challenge 1

Let F(N) be running time of `main()` as a function of input N.

```
public static void main(String[] args) {
   ...
   int N = Integer.parseInt(args[0]);
   ...
}
```

Scenario 1. F(2N) / F(N) converges to about 4.

Q. What is order of growth of the running time?



## Performance Challenge 2

Let F(N) be running time of `main()` as a function of input N.

```
public static void main(String[] args) {
   ...
   int N = Integer.parseInt(args[0]);
   ...
}
```

Scenario 2. F(2N) / F(N) converges to about 2.

Q. What is order of growth of the running time?



## Prediction and Validation

Hypothesis. Running time is about $a N^3$ for input of size $N$.

Q. How to estimate $a$?
A. Run the program!

| $N$ | time † |
|---|---|
| 4096 | 17.18 |
| 4096 | 17.15 |
| 4096 | 17.17 |

$17.17 = a\ 4096^3$
$\Rightarrow a = 2.5 \times 10^{-10}$

Refined hypothesis. Running time is about $2.5 \times 10^{-10} \times N^3$ seconds.

Prediction. 1,100 seconds for $N = 16{,}384$.

Observation.

| $N$ | time † |
|---|---|
| 16384 | 1118.86 |

← *validates hypothesis!*

# Mathematical Analysis

Donald Knuth
Turing award '74



Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0) count++;
```

| operation | frequency |
|---|---|
| variable declaration | $2$ |
| variable assignment | $2$ |
| less than comparison | $N + 1$ |
| equal to comparison | $N$ |
| array access | $N$ |
| increment | $\le 2N$ |

between N (no zeros)
and 2N (all zeros)





Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
int count = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         if (a[i] + a[j] == 0) count++;
```

| operation | frequency |
|---|---|
| variable declaration | $N + 2$ |
| variable assignment | $N + 2$ |
| less than comparison | $1/2\ (N + 1)\ (N + 2)$ |
| equal to comparison | $1/2\ N\ (N - 1)$ |
| array access | $N\ (N - 1)$ |
| increment | $\le N^2$ |

$0 + 1 + 2 + \ldots + (N-1) = 1/2\ \ N(N-1)$

becoming very tedious to count





Tilde notation.

- Estimate running time as a function of input size $N$.
- Ignore lower order terms.
  - when $N$ is large, terms are negligible
  - when $N$ is small, we don't care

Ex 1. $6N^3 + 17N^2 + 56 \quad \sim \quad 6N^3$

Ex 2. $6N^3 + 100N^{4/3} + 56 \quad \sim \quad 6N^3$

Ex 3. $6N^3 + \underbrace{17N^2 \log N}_{} \quad \sim \quad 6N^3$

discard lower-order terms
(e.g., N = 1000: 6 trillion vs. 169 million)

Technical definition. $f(N) \sim g(N)$ means $\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

## Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
{
   int N = a.length;
   int cnt = 0;                                      ← 1
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)                  ← N
         for (int k = j+1; k < N; k++)               ← ~N²/2
            if (a[i] + a[j] + a[k] == 0)             ← ~N³/6
               cnt++;
   return cnt;
}
```

inner loop

depends on input data

Inner loop. Focus on instructions in "inner loop."



## Constants in Power Law

Power law. Running time of a typical program is $\sim a\,N^b$.

Exponent $b$ depends on: algorithm.

Leading constant $a$ depends on:

- Algorithm.
- Input data.
- Caching.
- Machine.
- Compiler.
- Garbage collection.
- Just-in-time compilation.
- CPU use by other applications.

system independent effects

system dependent effects

Our approach. Use doubling hypothesis (or mathematical analysis) to estimate exponent $b$, run experiments to estimate $a$.



## Analysis: Empirical vs. Mathematical

Empirical analysis.

- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

Mathematical analysis.

- Analyze algorithm to estimate # ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

Critical difference. Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.



## Order of Growth Classifications

Observation. A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```
while (N > 1) {
   N = N / 2;
   ...
}
```

$\lg N$

$\lg N = \log_2 N$

```
for (int i = 0; i < N; i++)
   ...
```

$N$

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

$N^2$

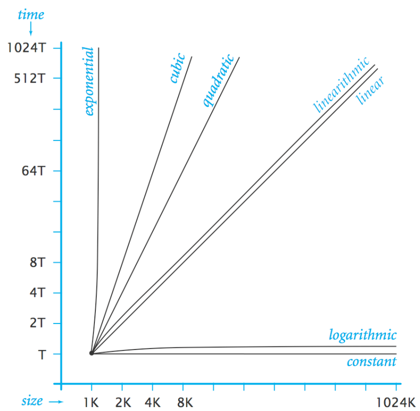
```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

$N \lg N$

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

$2^N$

## Order of Growth Classifications



*Orders of growth (log-log plot)*

| *order of growth* | | *factor for doubling hypothesis* |
|---|---|---|
| *description* | *function* | |
| constant | $1$ | 1 |
| logarithmic | $\log N$ | 1 |
| linear | $N$ | 2 |
| linearithmic | $N \log N$ | 2 |
| quadratic | $N^2$ | 4 |
| cubic | $N^3$ | 8 |
| exponential | $2^N$ | $2^N$ |



## Order of Growth: Consequences

| *order of growth* | *predicted running time if problem size is increased by a factor of 100* |
|---|---|
| linear | a few minutes |
| linearithmic | a few minutes |
| quadratic | several hours |
| cubic | a few weeks |
| exponential | forever |

*Effect of increasing problem size for a program that runs for a few seconds*

| *order of growth* | *predicted factor of problem size increase if computer speed is increased by a factor of 10* |
|---|---|
| linear | 10 |
| linearithmic | 10 |
| quadratic | 3-4 |
| cubic | 2-3 |
| exponential | 1 |

*Effect of increasing computer speed on problem size that can be solved in a fixed amount of time*

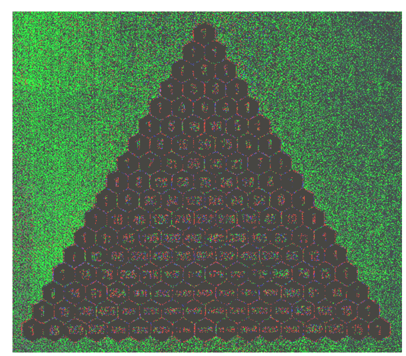


# Dynamic Programming

## Binomial Coefficients

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose $k$ of $n$ elements.

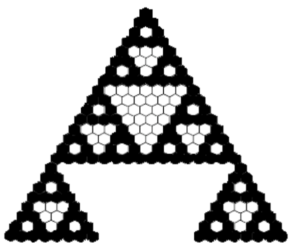
Pascal's identity. $$\binom{n}{k} = \underbrace{\binom{n-1}{k-1}}_{\text{contains first element}} + \underbrace{\binom{n-1}{k}}_{\text{excludes first element}}$$



## Binomial Coefficients: Sierpinski Triangle

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose $k$ of $n$ elements.

Sierpinski triangle. Color black the odd integers in Pascal's triangle.



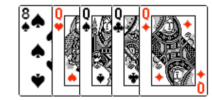


## Binomial Coefficients: Poker Odds

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose $k$ of $n$ elements.

Probability of "quads" in Texas hold 'em:

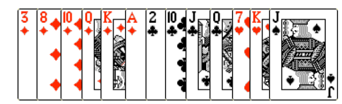
$$\frac{\binom{13}{1} \times \binom{48}{3}}{\binom{52}{7}} = \frac{224{,}848}{133{,}784{,}560} \quad (about\ \ 594:1)$$



Probability of 6-4-2-1 split in bridge:

$$\frac{\binom{4}{1} \times \binom{13}{6} \times \binom{3}{1} \times \binom{13}{4} \times \binom{2}{1} \times \binom{13}{2} \times \binom{1}{1} \times \binom{13}{1}}{\binom{52}{13}}$$

$$= \frac{29{,}858{,}811{,}840}{635{,}013{,}559{,}600} \quad (about\ \ 21:1)$$





## Binomial Coefficients: First Attempt

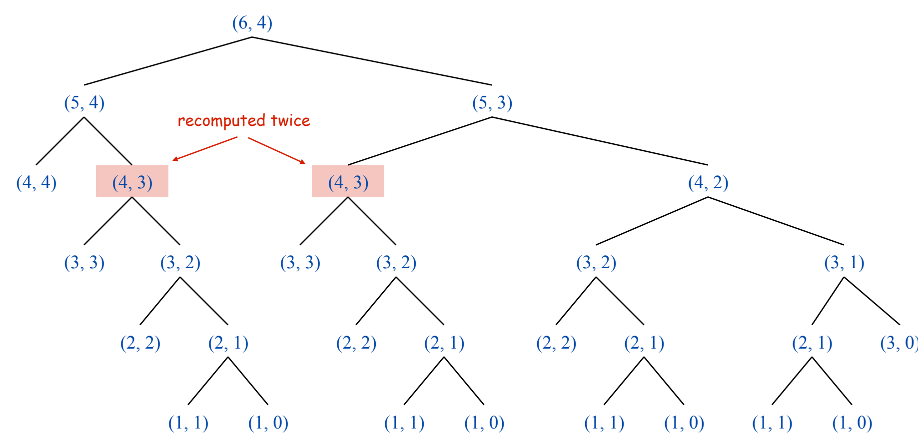
```java
public class SlowBinomial {

   // natural recursive implementation
   public static long binomial(long n, long k) {
      if (k == 0) return 1;
      if (n == 0) return 0;
      return binomial(n-1, k-1) + binomial(n-1, k);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int K = Integer.parseInt(args[1]);
      StdOut.println(binomial(N, K));
   }

}
```



## Performance Challenge 3

Q. Is this an efficient way to compute binomial coefficients?

A. No, no, no! [same essential recomputation flaw as naïve Fibonacci]



## Timing Experiments

Timing experiments: direct recursive solution.

| $(2n, n)$ | $time^{\dagger}$ |
|---|---|
| (26, 13) | 0.46 |
| (28, 14) | 1.27 |
| (30, 15) | 4.30 |
| (32, 16) | 15.69 |
| (34, 17) | 57.40 |
| (36, 18) | 230.42 |

increase n by 1, running time increases by about 4x

Q. Is running time linear, quadratic, cubic, exponential in n?



## Performance Challenge 4

Let F(N) be running time to compute `binomial(2N, N)`.

```
public static long binomial(long n, long k) {
   if (k == 0) return 1;
   if (n == 0) return 0;
   return binomial(n-1, k-1) + binomial(n-1, k);
}
```

Observation. F(N+1) / F(N) converges to about 4.

Q. What is order of growth of the running time?

A. Exponential: a $4^N$. ← will not finish unless N is small



## Dynamic Programming

Key idea. Save solutions to subproblems to avoid recomputation.

| *n* \ *k* | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 |
| 2 | 1 | 2 | 1 | 0 | 0 |
| 3 | 1 | 3 | 3 | 1 | 0 |
| 4 | 1 | 4 | 6 | 4 | 1 |
| 5 | 1 | 5 | 10 | 10 | 5 |
| 6 | 1 | 6 | 15 | 20 | 15 |

*binomial(n, k)*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

20 = 10 + 10

Tradeoff. Trade (a little) memory for (a huge amount of) time.



## Binomial Coefficients: Dynamic Programming

```
public class Binomial {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int K = Integer.parseInt(args[1]);
      long[][] bin = new long[N+1][K+1];

      // base cases
      for (int k = 1; k <= K; k++) bin[0][K] = 0;
      for (int n = 0; n <= N; n++) bin[N][0] = 1;

      // bottom-up dynamic programming
      for (int n = 1; n <= N; n++)
         for (int k = 1; k <= K; k++)
            bin[n][k] = bin[n-1][k-1] + bin[n-1][k];

      // print results
      StdOut.println(bin[N][K]);
   }
}
```

## Timing Experiments

Timing experiments for binomial coefficients via dynamic programming.

| $(2n, n)$ | *time* † |
|---|---|
| (26, 13) | instant |
| (28, 14) | instant |
| (30, 15) | instant |
| (32, 16) | instant |
| (34, 17) | instant |
| (36, 18) | instant |

Q. Is running time linear, quadratic, cubic, exponential in n?



## Performance Challenge 5

Let F(N) be running time to compute `binomial(2N, N)` using DP.

```
for (int n = 1; n <= N; n++)
   for (int k = 1; k <= K; k++)
      bin[n][k] = bin[n-1][k-1] + bin[n-1][k];
```

Q. What is order of growth of the running time?

A. Quadratic: a $N^2$. ← effectively instantaneous for small N

Remark. There is a profound difference between $4^N$ and $N^2$.

cannot solve a large problem

can solve a large problem



## Digression: Stirling's Approximation

Alternative: $\binom{n}{k} = \frac{n!}{n!\,(n-k)!}$

Caveat. 52! overflows a long, even though final result doesn't.

Instead of computing exact values, use Stirling's approximation:

$$\ln n! \approx n \ln n - n + \frac{\ln(2\pi n)}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}$$

Application. Probability of exact k heads in n flips with a biased coin.

$\binom{n}{k} p^k (1-p)^{n-k}$ (easy to compute approximate value with Stirling's formula)



# Memory

## Typical Memory Requirements for Java Data Types

Bit. 0 or 1.
Byte. 8 bits.
Megabyte (MB). 1 million bytes ~ $2^{10}$ bytes.
Gigabyte (GB). 1 billion bytes ~ $2^{20}$ bytes.

| type | bytes |
|---|---|
| boolean | 1 |
| byte | 1 |
| char | 2 |
| int | 4 |
| float | 4 |
| long | 8 |
| double | 8 |

| type | bytes |
|---|---|
| int[] | $4N + 16$ |
| double[] | $8N + 16$ |
| int[][] | $4N^2 + 20N + 16$ |
| double[][] | $8N^2 + 20N + 16$ |
| String | $2N + 40$ |

typical computer '10 has about 2GB memory

Q. What's the biggest `double[]` array you can store on your computer?



## Performance Challenge 6

Q. How much memory does this program require as a function of `N`?

```
public class RandomWalk {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int[][] count = new int[N][N];
      int x = N/2;
      int y = N/2;

      for (int i = 0; i < N; i++)  {
         // no new variable declared in loop
         ...
         count[x][y]++;
      }
   }
}
```

A.



## Summary

Q. How can I evaluate the performance of my program?
A. Computational experiments, mathematical analysis, scientific method.

Q. What if it's not fast enough? Not enough memory?
- Understand why.
- Buy a faster computer.
- Learn a better algorithm (COS 226, COS 423).
- Discover a new algorithm.

| attribute | better machine | better algorithm |
|---|---|---|
| cost | $$$ or more | $ or less |
| applicability | makes "everything" run faster | does not apply to some problems |
| improvement | quantitative improvements | dramatic qualitative improvements possible |