Note on a Spectral Theorem by Forster

Let $\mathfrak{X} = \{ x \mid x \in X \} \subseteq \mathbb{R}^k$ be a collection of unit vectors in $\mathbb{R}^k$ indexed by a set $X$ of cardinality at least $2k$. We assume that the vectors in $\mathfrak{X}$ are in general position, that is, every $k$-subset of $\mathfrak{X}$ is linearly independent. For $A \in \mathbb{R}^{k \times k}$, we define the following matrix

$$M(A) \equiv \sum_{x \in X} \frac{1}{\| Ax \|^2} (Ax \otimes Ax),$$

where $(Ax \otimes Ax)$ denotes the self-adjoint linear operator defined by $(Ax \otimes Ax)(y) = \langle Ax, y \rangle Ax$. We will show that there exists a matrix $A \in \mathbb{R}^{k \times k}$ such that

$$\lambda_{\min}(M(A)) = \frac{|X|}{k},$$

where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a symmetric matrix. Note that any matrix $A$ that satisfies (1) is necessarily invertible, for otherwise the matrix $M(A)$ has smallest eigenvalue 0. Also note that the matrix $M(A)$ has trace $|X|$, and hence the equation (1) implies that all eigenvalues are equal to $|X|/k$.

In the following claim, we gather a few facts about matrices of the form $M(A)$.

**Claim 1.** Let $A \in \mathbb{R}^{k \times k}$. Then,

1. $M(A)$ is symmetric and positive semidefinite,
2. $\text{Tr} M(A) = |X|$,
3. $\lambda_{\min}(M(A)) = \lambda_{\max}(M(A))$ if and only if $\lambda_{\min}(M(A)) = \frac{|X|}{k}$ (or $\lambda_{\max}(M(A)) = \frac{|X|}{k}$),
4. $M(\alpha A) = M(A)$ for any non-zero scalar $\alpha \in \mathbb{R}$,
5. $A \in \text{GL}(k)$ if and only if $M(A) \in \text{GL}(k)$.

**Proof.** Items 1–4 can be verified easily. 5.) The range of $M(A)$ is equal to the span of the set $\{ Ax \mid x \in X \}$. By the general position assumption for $\mathfrak{X}$, the span of the set $\{ Ax \mid x \in X \}$ is equal to $\mathbb{R}^k$ if $A$ is non-singular. On the other hand, if $A$ is singular, then $\{ Ax \mid x \in X \}$ cannot span $\mathbb{R}^k$. \hfill $\Box$

**Lemma 2.** For every non-singular matrix $B \in \mathbb{R}^{k \times k}$, there exists $\delta > 0$ such that

$$\lambda_{\min}(M(AB)) \geq \min \left\{ \frac{|X|}{k}, \lambda_{\min}(M(B)) + \delta \right\},$$

where $A = M(B)^{-1/2}$.

**Proof.** For $x \in X$, let $x' = \frac{1}{\| Bx \|} Bx$ be the unit vector in direction $Bx$. Note that $(\sum_{x \in X} (x' \otimes x')) = M(B) = A^{-2}$. We may assume $\lambda_{\min} \equiv \lambda_{\min}(\sum_{x \in X} (x' \otimes x')) < |X|/k$, for otherwise the lemma is trivially true. Also note that $\lambda_{\min} = 1/\lambda_{\max}(A^2) = 1/\lambda_{\max}(A)^2$. 

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In order to prove the lemma, it remains to show that the matrix $M(AB) - \lambda_{\text{min}}I$ has only positive eigenvalues. Since $A$ is symmetric, we have $(Ax' \otimes Ax') = A(x' \otimes x')A$ and thus $\sum_{x \in X} (Ax' \otimes Ax') = A(\sum_{x \in X} (x' \otimes x'))A = I$. Hence,

$$
M(A) - \lambda_{\text{min}}I = \sum_{x \in X} \frac{1}{\|Ax'\|^2} (Ax' \otimes Ax') - \lambda_{\text{min}}I
$$

$$
= \sum_{x \in X} \left( \frac{1}{\|Ax'\|^2} - \lambda_{\text{min}} \right) (Ax' \otimes Ax')
$$

$$
= \sum_{x \in X} \alpha_x (Ax' \otimes Ax')
$$

$$
\geq 0.
$$

(\text{using } \sum (Ax' \otimes Ax') = I)

(\text{using } \alpha_x \equiv 1/\|Ax'\|^2 - \lambda_{\text{min}})

Let $X_0$ denote the set of indices $x$ such that $\alpha_x = 0$. We claim that $X_0$ has cardinality at most $k$. Assuming this claim, we can finish the proof of the lemma as follows. If $|X_0| < k$, then there are at least $|X| - k \geq k$ indices such that $\alpha_x > 0$. By the general position assumption, the corresponding set of vectors $\{Ax' : x \in X \setminus X_0\}$ spans $\mathbb{R}^k$. Hence for every unit vector $y$, there exists an index $x_1 \in X \setminus X_0$ such that $\langle Ax'_1, y \rangle \neq 0$. Thus

$$
\langle \left( \sum_{x \in X} \alpha_x (Ax' \otimes Ax') \right) y, y \rangle = \langle \alpha_{x_1} (Ax'_1 \otimes Ax'_1), y, y \rangle
$$

$$
= \alpha_{x_1} (Ax'_1, y)^2 > 0.
$$

It follows that the matrix

$$
\sum_{x \in X} \alpha_x (Ax' \otimes Ax') = \sum_{x \in X} \frac{1}{\|Ax'\|^2} (Ax' \otimes Ax') - \lambda_{\text{min}}I = M(AB) - \lambda_{\text{min}}I
$$

has only positive eigenvalues, which proves the lemma.

It remains to prove the claim that $|X_0| < k$. For the sake of a contradiction, assume $|X_0| \geq k$. Then there are $k$ linearly independent vectors $x'$ such that $\|Ax'\| = \lambda_{\text{max}}(A)$. Thus the eigenspace of $A$ corresponding to $\lambda_{\text{max}}(A)$ has dimension $k$. It follows that the eigenspace of $A^{-2} = \sum_{x \in X} (x' \otimes x')$ corresponding to $\lambda_{\text{min}} = 1/\lambda_{\text{max}}(A)^2$ has dimension $k$. Hence $|X| = \text{Tr}(A^{-2}) = k\lambda_{\text{min}}$, which contradicts our assumption $\lambda_{\text{min}} < \frac{k}{|X|/k}$.

Lemma 3. Let $\mathcal{A} = \{A^{(\ell)} : \ell \in \mathbb{N}\} \subseteq \mathbb{R}^{k \times k}$ be any sequence of non-singular matrices with $\|A^{(\ell)}\| = 1$. Suppose $\mathcal{A}$ has a subsequence that converges to a singular matrix $A$. Then, for every $\varepsilon > 0$, there exists an $\ell \in \mathbb{N}$ such that

$$
\lambda_{\text{min}} \left( M(A^{(\ell)}) \right) < 1 + \varepsilon.
$$

Proof. Suppose that the kernel of $A$ has dimension $d > 0$. Then there exist $d$ ortho-normal vectors $e_1, \ldots, e_d$ such that $\langle Ax, e_i \rangle = 0$ for every $i \in [d]$ and $x \in X$. Let $X_0$ denote the set of indices $x$ such that $Ax = 0$. Since $\|A\| = \lim_{\ell \rightarrow \infty} \|A^{(\ell)}\| = 1$, the matrix $A$ cannot be 0 and hence $d < k$. Therefore, using the general positive assumption, $X_0$ has cardinality at most $d$. In the
following, we restrict \( \mathcal{A} \) to the subsequence that converges to \( A \). Then,

\[
\limsup_{\ell \to \infty} \sum_{i \in [d]} \langle M(A^{(\ell)}) e_i, e_i \rangle \\
= \limsup_{\ell \to \infty} \sum_{i \in [d]} \left( \sum_{x \in X} \frac{1}{\|A^{(\ell)} x\|^2} (A^{(\ell)} x \otimes A^{(\ell)} x) e_i, e_i \right) \\
= \limsup_{\ell \to \infty} \sum_{i \in [d]} \sum_{x \in X} \frac{1}{\|A^{(\ell)} x\|^2} \langle A^{(\ell)} x, e_i \rangle^2 \\
= \limsup_{\ell \to \infty} \sum_{x \in X_0} \sum_{i \in [d]} \langle \frac{1}{\|A^{(\ell)} x\|} A^{(\ell)} x, e_i \rangle^2 \quad \text{(using } \lim_{\ell \to \infty} \langle \frac{1}{\|A^{(\ell)} x\|} A^{(\ell)} x, e_i \rangle = 0 \text{ for } x \not\in X_0) \\
\leq \limsup_{\ell \to \infty} \sum_{x \in X_0} 1 \\
\leq d.
\]

It follows that for every \( \epsilon > 0 \), there exists \( i \in [d] \) and \( \ell \in \mathbb{N} \) such that

\[
\langle M(A^{(\ell)}) e_i, e_i \rangle < 1 + \epsilon,
\]

which proves the lemma. \( \square \)

**Proof of the Theorem**

We will need the following claim.

**Claim 4.** At every matrix \( A_0 \in \text{GL}(k) \), the following functions are continuous:

\[
g(A) = \lambda_{\min}(M(A)), \quad f(A) = \lambda_{\min}(M(M(A)^{-1/2} A)).
\]

**Proof.** Follows from the fact that the composition of continuos mappings is continuos. \( \square \)

**Theorem 5.** There exists a matrix \( A^* \in \mathbb{R}^{k \times k} \) such that

\[
\lambda_{\min}(M(A^*)) = \frac{|X|}{k}.
\]

**Proof.** We define a sequence \( \mathcal{A} = \{ A^{(\ell)} \mid \ell \in \mathbb{N} \} \subseteq \mathbb{R}^{k \times k} \) of non-singular matrices with \( \|A\| = 1 \) by the following recurrence

\[
A^{(\ell+1)} = \frac{1}{\|M(A^{(\ell)})^{-1/2} A^{(\ell)}\|} M(A^{(\ell)})^{-1/2} A^{(\ell)} ,
\]

where we choose \( A^{(1)} \) to be the linear operator that maps the first \( k \) vectors in \( X \) to the canonical (orthogonal) basis of \( \mathbb{R}^k \). Note that \( M(A^{(\ell)}) = M(M(A^{(\ell-1)})^{-1/2} A^{(\ell-1)}) \). Hence, by Lemma 2, the sequence \( \{ \lambda^{(\ell)}_{\min} \mid \ell \in \mathbb{N} \} \) defined by

\[
\lambda^{(\ell)}_{\min} = g(A^{(\ell)}) = \lambda_{\min}(M(A^{(\ell)}))
\]

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is strictly increasing in \( \ell \) until it possibly reaches \( |X|/k \). Furthermore, we have \( \lambda_{\min}^{(i)} \geq 1 \), because

\[
M(A^{(1)}) = \sum_{x \in X} \frac{1}{\|A^{(1)}x\|^2} (A^{(1)} x \otimes A^{(1)} x) \geq \sum_{x \in \{x^1, \ldots, x^k\}} \frac{1}{\|A^{(1)}x\|^2} (A^{(1)} x \otimes A^{(1)} x) = \sum_{i=1}^k (e_i \otimes e_i) = I,
\]

where \( x^1, \ldots, x^k \) are the first \( k \) indices of \( X \), and \( e_1, \ldots, e_k \) is the canonical basis of \( \mathbb{R}^k \). It follows that \( \lambda_{\min}^{(2)} \geq 1 + \epsilon \) for some \( \epsilon > 0 \).

Let \( \mathcal{A}' = \{ A^{(\ell)}(t) \mid t \in \mathbb{N} \} \) denote a converging subsequence of \( \mathcal{A} \). Note that \( \mathcal{A} \) has a converging subsequence, because it is contained in the bounded set \( \{ A \in \mathbb{R}^{k \times k} \mid \|A\| \leq 1 \} \). By Lemma 3 and the observation \( \lambda_{\min}^{(\ell)} \geq 1 + \epsilon \) for \( \ell > 1 \), the limit of \( \mathcal{A}' \) is a non-singular matrix \( A^* \). By the continuity of the function \( f \) at non-singular matrices,

\[
f(A^*) = \lim_{t \to \infty} f(A^{(\ell)(t)}) \quad \text{(using continuity of } f \text{ at } A^*)
\]

\[
= \lim_{t \to \infty} \lambda_{\min}(M(M(A^{(\ell)(t)})^{-1/2} A^{(\ell)(t)}))
\]

\[
= \lim_{t \to \infty} \lambda_{\min}(M(A^{(\ell)(t+1)})) \quad \text{(using } M(M(A^{(\ell)})^{-1/2} A^{(\ell)} = M(A^{(\ell+1)}) \text{ for } \ell \in \mathbb{N})
\]

\[
= \lim_{\ell \to \infty} \lambda_{\min}^{(\ell)}
\]

\[
= \lim_{t \to \infty} \lambda_{\min}^{(\ell)(t)}
\]

\[
= \lim_{t \to \infty} g(A^{(\ell)(t)})
\]

\[
= g(A^*) \quad \text{(using continuity of } g \text{ at } A^*)
\]

By Lemma 2, the condition \( f(A^*) = g(A^*) \) implies that \( \lambda_{\min}(M(A^*)) = \frac{|X|}{k} \), which proves the theorem. \( \square \)