Eventually simple situation:

1 seller, 1 buyer

\[ \tilde{v}_1, \tilde{v}_2 \]

IPV \( \tilde{v}_i \) is distributed as density \( f_i(.) > 0 \) on \([0, 6] \)

\( f_2(.) > 0 \) on \([0, 4] \)

risk neutral

additively separable utility for money and object

Basic Question: Among all possible bargaining mechanisms, which have desirable economic efficiency properties?

Review of Resolution Principle:

A direct bargaining mechanism

![Diagram](image)

- direct mechanism: characterization \((p, x)\), where

\[ p(\tilde{v}_1, \tilde{v}_2) = \text{prob. transfer } 1 \rightarrow 2 \]

\[ x(\tilde{v}_1, \tilde{v}_2) = \text{expected payment } 1 \rightarrow 2 \]

A direct mechanism is (Bayesian) incentive-compatible if honest reporting forms a BNE.
Revelation Principle

For every equilibrium in a bargaining game, there exists an incentive-compatible direct mechanism.

Proof: if there were an incentive to lie when reporting, then there would be an incentive to lie to oneself in the original game.

...we can restrict attention to incentive-compatible direct mechanisms.

So we have \((p, x)\).

Some basic quantities:

\[
\bar{X}_i(v_i) = \int_{a_i}^{b_i} x(v_i, t_2) f_2(t_2) dt_2 = E[\text{revenue of seller}]
\]

\[
\bar{P}_i(v_i) = \int_{a_i}^{b_i} p(v_i, t_2) f_2(t_2) dt_2 = \text{prob. of 1 selling to 2}
\]

\[
U_i(v_i) = \bar{X}_i(v_i) - v_i \cdot \bar{P}_i(v_i) = E[\text{profit of seller, 1}]
\]

\& similarly,

\[
U_2(v_2) = v_2 \cdot \bar{P}_2(v_2) - \bar{X}_2(v_2) = E[\text{profit of buyer, 2}]
\]

In this notation, incentive-compatible means

\[
U_i(v_i) \geq \bar{X}_i(v_i) - v_i \cdot \bar{P}_i(v_i) \quad \forall v_i, \hat{v}_i \in [a_i, b_i]
\]

\[
U_2(v_2) \geq v_2 \cdot \bar{P}_2(v_2) - \bar{X}_2(v_2) \quad \forall v_2, \hat{v}_2 \in [a_2, b_2]
\]
Another desirable property:
A mechanism is individually rational iff

\[ U_1(v_1) \geq 0 \quad \forall v_1 \in [a_1, b_1], \quad U_2(v_2) \geq 0 \quad \forall v_2 \in [a_2, b_2] \]

And another desirable property:
A mechanism is ex post efficient iff

\[ p(v_1, v_2) = \begin{cases} 1 & \text{if } v_1 < v_2 \\ 0 & \text{if } v_1 > v_2 \end{cases} \]

I.e. object is sold iff buyer values it more highly.

Main Result (Corollary 1)

If \([a_1, b_1] \cap [a_2, b_2] \neq \emptyset\) (overlap)

Then no incentive-compatible individually rational trading mechanism can be ex post efficient!
Some details of proof technique:

**First part (Theorem 1)**

Incentive-compatible & individually rational \[ \Rightarrow U_1(b_1) + U(a_2) \geq 0 \]

\[ \uparrow \quad \uparrow \]

min. expected profit of seller max. expected profit of buyer

Follows from definitions and crude mugging.

**Second part**

Ex post efficient \[ \Rightarrow U_1(b_1) + U_2(a_2) \]

\[ = - \int_a^b [1 - F_2(t)] F_1(t) dt < 0 \]

\[ \text{seller} \quad [\ldots] \quad \text{buyer} \]

\[ a_1 \quad [\ldots] \quad b_1 \quad [\ldots] \quad b_2 \]

This quantity, \[ \int_a^b [1 - F_2(t)] F_1(t) dt \] is the smallest lump-sum subsidy to create a mechanism that is

- Incentive compatible
- Individually rational
- Ex post efficient

[Of the beginning of Vickrey 61!]

Example 1. Shows that it is necessary that $f_i > 0$ on respective intervals.

Counterexample. Suppose

$$\Pr(\tilde{v}_1 = 1) = \Pr(\tilde{v}_1 = 4) = \Pr(\tilde{v}_2 = 0) = \Pr(\tilde{v}_2 = 3) = \frac{1}{2}$$

<table>
<thead>
<tr>
<th>Seller</th>
<th>Buyer 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/4</td>
</tr>
<tr>
<td>4</td>
<td>1/4</td>
</tr>
<tr>
<td>0</td>
<td>1/4</td>
</tr>
<tr>
<td>3</td>
<td>1/4</td>
</tr>
</tbody>
</table>

Claim: the mechanism "sell at price 2 if both are willing, else no trade" is incentive-compatible, individually rational, & ex post efficient.

- Incentive-compatible: Honest reporting is a BNE (check)
- Individually rational: Clearly $E[profit] > 0$
- Efficient trade occurs only when $v_i < v_i$. 
Example 2: \( \tilde{V}_1, \tilde{V}_2 \) both Uniform \([0,1]\)

Theorem becomes

\[
\mathbb{E}_1 (V_1, V_2) + \mathbb{E}_2 (V_2, V_2) = \int \int \left[ V_2 - \frac{1}{c_1} \right] - \left[ V_1 + \frac{c_1}{c_2} \right] p(V_1, V_2) \, dV_1 \, dV_2 
- \frac{1}{c_1} \int \int (V_2 - V_2) p(V_1, V_2) \, dV_1 \, dV_2 
\]

\[ = \int \int \left[ V_2 - \frac{1}{c_1} \right] - \left[ V_1 + \frac{c_1}{c_2} \right] p(V_1, V_2) \, dV_1 \, dV_2 \]

\[ = \int \int (V_2 - V_2) p(V_1, V_2) \, dV_1 \, dV_2 - \frac{1}{c_1} \int \int p(V_1, V_2) \, dV_1 \, dV_2 \geq 0 \]

\[ \Rightarrow \mathbb{E}\left[ \frac{V_2 - V_1}{V_2} \right] \geq \frac{1}{2} \]

But in general

\[ \mathbb{E}\left[ \frac{V_2 - V_1}{V_2} \right] = \frac{1}{3} \]

\[ \mathbb{E}_2 \left[ \text{subsidy} \right] \text{required for ex-post efficiency} = \frac{1}{6} \]

Check: (1) of M&S 83

\[
\int_0^1 [1 - F_2(t)] F_1(t) \, dt
\]

\[ = \int_0^1 (1 - t) \, dt = \frac{1}{6} \]