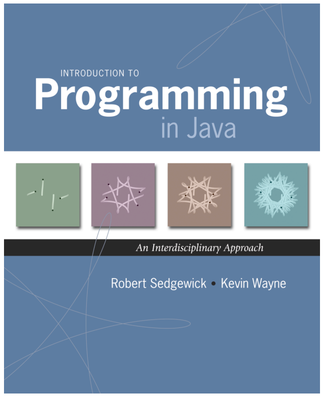
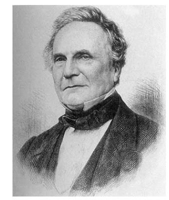
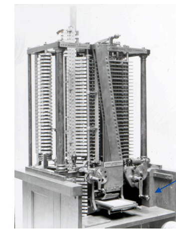
4.1 Performance



Running Time

> *"As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time?"* – Charles Babbage


Charles Babbage (1864)


how many times do you have to turn the crank?

Analytic Engine

Scientific Method

Analysis of algorithms. Framework for comparing algorithms and predicting performance.

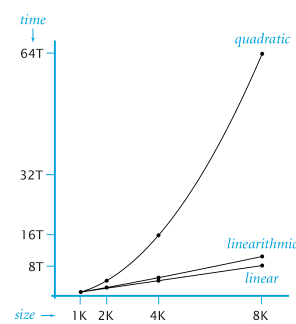
Scientific method.
- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles. Experiments we design must be reproducible; hypothesis must be falsifiable.

Algorithmic Successes

Discrete Fourier transform.
- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: N^2 steps.
- FFT algorithm: N log N steps, enables new technology.


Freidrich Gauss
1805

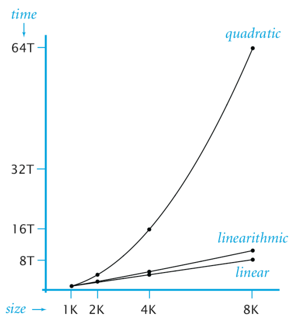


Algorithmic Successes

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut: N log N steps, enables new research.

Andrew Appel
PU '81



Three-Sum Problem

Three-sum problem. Given N integers, find triples that sum to 0.
Application. Deeply related to problems in computational geometry.

```
% more 8ints.txt
30 -30 -20 -10 40 0 10 5

% java ThreeSum < 8ints.txt
 4
 30 -30   0
 30 -20 -10
-30 -10  40
-10   0  10
```

Q. How would you write a program to solve the problem?

Three-Sum

```
public class ThreeSum {

   // return number of distinct triples (i, j, k)
   // such that (a[i] + a[j] + a[k] == 0)
   public static int count(int[] a) {
      int N = a.length;
      int cnt = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0) cnt++;
      return cnt;
   }

   public static void main(String[] args) {
      int[] a = StdArrayIO.readInt1D();
      StdOut.println(count(a));
   }
}
```


Empirical Analysis

Empirical Analysis

Empirical analysis. Run the program for various input sizes.

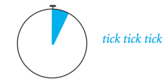
N	*time* †
512	0.03
1024	0.26
2048	2.16
4096	17.18
8192	136.76

† Running Linux on Sun-Fire-X4100 with 16GB RAM

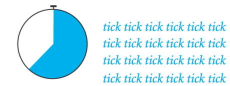
Stopwatch

Q. How to time a program?
A. A stopwatch.

```
% java ThreeSum < 1Kints.txt
```




```
0
% java ThreeSum < 2Kints.txt
```




```
2
391930676 -763182495 371251819
-326747290 802431422 -475684132
```


Stopwatch

Q. How to time a program?
A. A `Stopwatch` object.

public class Stopwatch

	`Stopwatch()`	*create a new stopwatch and start it running*
`double`	`elapsedTime()`	*return the elapsed time since creation, in seconds*

```java
public class Stopwatch {
   private final long start;

   public Stopwatch() {
      start = System.currentTimeMillis();
   }

   public double elapsedTime() {
      return (System.currentTimeMillis() - start) / 1000.0;
   }
}
```


Stopwatch

Q. How to time a program?
A. A `Stopwatch` object.

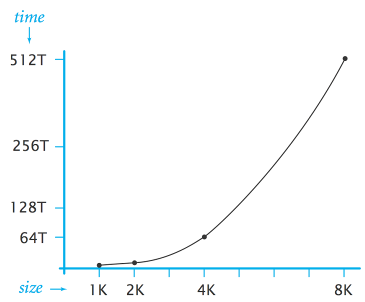
public class Stopwatch

	`Stopwatch()`	*create a new stopwatch and start it running*
`double`	`elapsedTime()`	*return the elapsed time since creation, in seconds*

```java
public static void main(String[] args) {
   int[] a = StdArrayIO.readInt1D();
   Stopwatch timer = new Stopwatch();
   StdOut.println(count(a));
   StdOut.println(timer.elapsedTime());
}
```

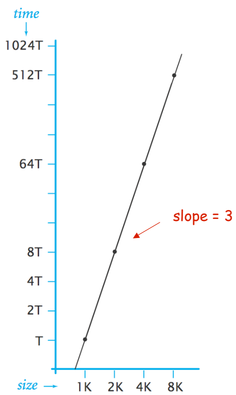
Empirical Analysis

Data analysis. Plot running time vs. input size N.



Empirical Analysis

Data analysis. Plot running time vs. input size N on log-log scale.



power law

slope

Regression. Fit line through data points: $a\,N^{b}$.

Hypothesis. Running time grows cubically with input size: $a\,N^{3}$.

Prediction and Verification

Hypothesis. $2.5 \times 10^{-10} \times N^{3}$ seconds for input of size N.

Prediction. 17.18 seconds for $N = 4{,}096$.

Observations.

N	time †
4096	17.18
4096	17.15
4096	17.17

agrees

Prediction. 1100 seconds for $N = 16{,}384$.

Observation.

N	time †
16384	1118.86

agrees

Doubling Hypothesis

Doubling hypothesis. Quick way to formulate a power law hypothesis.

Q. What is effect on the running time of doubling the size of the input?

N	time †	ratio
512	0.033	-
1024	0.26	7.88
2048	2.16	8.43
4096	17.18	7.96
8192	136.76	7.96

† Running Linux on Sun-Fire-X4100 with 16GB RAM

lg of ratio is
exponent in power law
($\lg 8 \approx 3$)

Mathematical Analysis

Donald Knuth
Turing award '74

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0) count++;
```

operation	frequency
variable declaration	2
variable assignment	2
less than comparison	$N + 1$
equal to comparison	N
array access	N
increment	$\le 2N$

between N + 1 (no zeros)
and 2N + 1 (all zeros)

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
int count = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         if (a[i] + a[j] == 0) count++;
```

operation	frequency
variable declaration	$N + 2$
variable assignment	$N + 2$
less than comparison	$1/2\,(N + 1)\,(N + 2)$
equal to comparison	$1/2\,N\,(N - 1)$
array access	$N\,(N - 1)$
increment	$\le N^2$

$$0 + 1 + 2 + \ldots + (N-1) = \frac{N(N-1)}{2}$$

becoming very tedious to count

Tilde Notation

Tilde notation.

- Estimate running time as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

Ex 1. $6N^3 + 17N^2 + 56 \sim 6N^3$

Ex 2. $6N^3 + 100N^{4/3} + 56 \sim 6N^3$

Ex 3. $6N^3 + \underbrace{17N^2 \log N}_{} \sim 6N^3$

discard lower-order terms
(e.g., N = 1000: 6 trillion vs. 169 million)

Technical definition. $f(N) \sim g(N)$ means $\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
{
   int N = a.length;
   int cnt = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
               cnt++;
   return cnt;
}
```

1

N

$\sim N^2/2$

$\sim N^3/6$

inner loop

depends on input data

Inner loop. Focus on instructions in "inner loop."

Constants in Power Law

Power law. Running time of a typical program is $\sim c\,N^a$.

Exponent a depends on: algorithm.

Constant c depends on:

- algorithm
- input data

system independent effects

- caching
- machine
- compiler
- garbage collection
- just-in-time compilation
- CPU use by other applications

system dependent effects

Our approach. Use doubling hypothesis (or mathematical analysis) to estimate exponent a, run experiments to estimate c.

Analysis: Empirical vs. Mathematical

Empirical analysis.

- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

Mathematical analysis.

- Analyze algorithm to estimate # ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

Critical difference. Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.

Order of Growth Classifications

Observation. A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```
while (N > 1) {
   N = N / 2;
   ...
}
```

$\lg N$

$\lg N = \log_2 N$

```
for (int i = 0; i < N; i++)
   ...
```

N

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

N^2

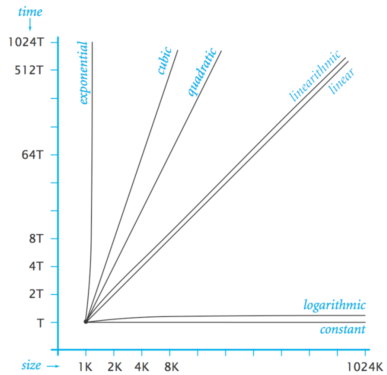
```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

$N \lg N$

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

2^N

Order of Growth Classifications



Orders of growth (log-log plot)

order of growth		factor for doubling hypothesis
description	*function*	
constant	1	1
logarithmic	$\log N$	1
linear	N	2
linearithmic	$N \log N$	2
quadratic	N^2	4
cubic	N^3	8
exponential	2^N	2^N

Order of Growth: Consequences

order of growth	*predicted running time if problem size is increased by a factor of 100*
linear	a few minutes
linearithmic	a few minutes
quadratic	several hours
cubic	a few weeks
exponential	forever

Effect of increasing problem size for a program that runs for a few seconds

order of growth	*predicted factor of problem size increase if computer speed is increased by a factor of 10*
linear	10
linearithmic	10
quadratic	3-4
cubic	2-3
exponential	1

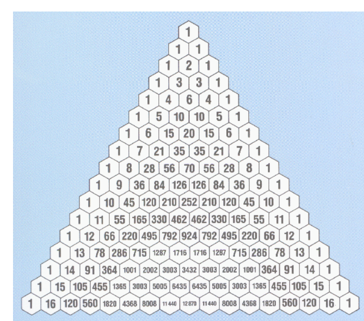
Effect of increasing computer speed on problem size that can be solved in a fixed amount of time

Dynamic Programming

Binomial Coefficients

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

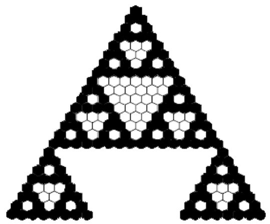
Pascal's identity. $$\binom{n}{k} = \underbrace{\binom{n-1}{k-1}}_{\text{contains first element}} + \underbrace{\binom{n-1}{k}}_{\text{excludes first element}}$$



Binomial Coefficients: Sierpinski Triangle

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

Sierpinski triangle. Color black the odd integers in Pascal's triangle.

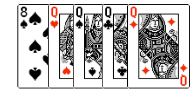


Binomial Coefficients: Poker Odds

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

Probability of "quads" in Texas hold 'em:

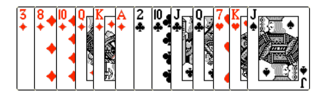
$$\frac{\binom{13}{1} \times \binom{48}{3}}{\binom{52}{7}} = \frac{224{,}848}{133{,}784{,}560} \quad (\textit{about} \;\; 594:1)$$



Probability of 6-4-2-1 split in bridge:

$$\frac{\binom{4}{1} \times \binom{13}{6} \times \binom{3}{1} \times \binom{13}{4} \times \binom{2}{1} \times \binom{13}{2} \times \binom{1}{1} \times \binom{13}{1}}{\binom{52}{13}}$$

$$= \frac{29{,}858{,}811{,}840}{635{,}013{,}559{,}600} \quad (\textit{about} \;\; 21:1)$$



Binomial Coefficients: First Attempt

```
public class SlowBinomial {

   // natural recursive implementation
   public static long binomial(long n, long k) {
      if (k == 0) return 1;
      if (n == 0) return 0;
      return binomial(n-1, k-1) + binomial(n-1, k);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int K = Integer.parseInt(args[1]);
      StdOut.println(binomial(N, K));
   }

}
```


Timing Experiments

Timing experiments: direct recursive solution.

$(2n, n)$	time †
(26, 13)	0.46
(28, 14)	1.27
(30, 15)	4.30
(32, 16)	15.69
(34, 17)	57.40
(36, 18)	230.42

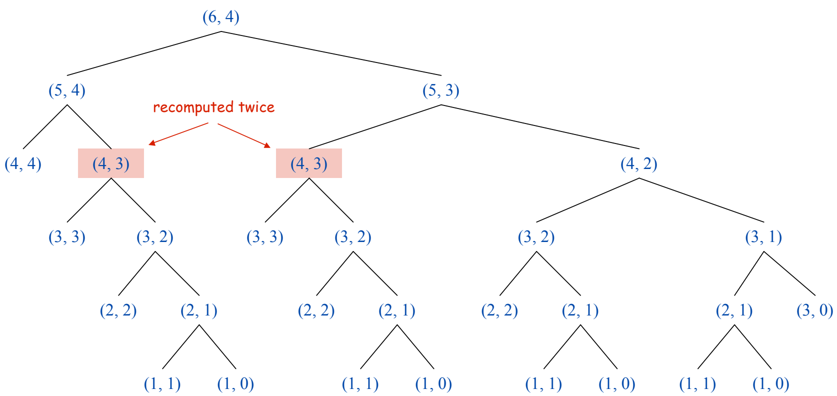
increase n by 1, running time increases by about 4x

† Running Linux on Sun-Fire-X4100 with 16GB RAM

Q. Is running time linear, quadratic, cubic, exponential in n?

Why So Slow?

Function call tree.



Dynamic Programming

Key idea. Save solutions to subproblems to avoid recomputation.

n \ k	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	2	1	0	0
3	1	3	3	1	0
4	1	4	6	4	1
5	1	5	10	10	5
6	1	6	15	20	15

binomial(n, k)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

20 = 10 + 10

Tradeoff. Trade memory for time.

Binomial Coefficients: Dynamic Programming

```java
public class Binomial {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int K = Integer.parseInt(args[1]);
      long[][] bin = new long[N+1][K+1];

      // base cases
      for (int k = 1; k <= K; k++) bin[0][K] = 0;
      for (int n = 0; n <= N; n++) bin[N][0] = 1;

      // bottom-up dynamic programming
      for (int n = 1; n <= N; n++)
         for (int k = 1; k <= K; k++)
            bin[n][k] = bin[n-1][k-1] + bin[n-1][k];

      // print results
      StdOut.println(bin[N][K]);
   }
}
```


Timing Experiments

Timing experiments: dynamic programming.

(2n, n)	*time* †
(26, 13)	instant
(28, 14)	instant
(30, 15)	instant
(32, 16)	instant
(34, 17)	instant
(36, 18)	instant

† Running Linux on Sun-Fire-X4100 with 16GB RAM

Q. Is running running time linear, quadratic, cubic, exponential in n?

Stirling's Approximation

Alternative: $\binom{n}{k} = \frac{n!}{n!\,(n-k)!}$

Caveat. 52! overflows a long, even though final result doesn't.

Stirling's approximation:

$$\ln n! \approx n \ln n - n + \frac{\ln(2\pi n)}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}$$

Application. Probability of exact k heads in n flips with a biased coin.

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Memory

Typical Memory Requirements for Java Data Types

Bit. 0 or 1.
Byte. 8 bits.
Megabyte (MB). 2^{10} bytes ~ 1 million bytes.
Gigabyte (GB). 2^{20} bytes ~ 1 billion bytes.

type	bytes
boolean	1
byte	1
char	2
int	4
float	4
long	8
double	8

type	bytes
int[]	$4N + 16$
double[]	$8N + 16$
int[][]	$4N^2 + 20N + 16$
double[][]	$8N^2 + 20N + 16$
String	$2N + 40$

typical computer '08 has about 1GB memory

Q. What's the biggest `double` array you can store on your computer?

An Example

Q. How much memory does this program use as a function of `N`?

```
public class RandomWalk {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int[][] count = new int[N][N];
      int x = N/2;
      int y = N/2;

      for (int i = 0; i < N; i++)  {
         // no new variable declared in loop
         ...
         count[x][y]++;
      }
   }
}
```

A.

Summary

Q. How can I evaluate the performance of my program?
A. Computational experiments, mathematical analysis.

Q. What if it's not fast enough? Not enough memory?

- Understand why.
- Buy a faster computer.
- Find a better algorithm in a textbook.
- Discover a new algorithm.

attribute	better machine	better algorithm
cost	$$$ or more.	$ or less.
applicability	makes "everything" run faster	does not apply to some problems
improvement	quantitative improvements	dramatic qualitative improvements possible